Tetrahedra and Physics

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Astract

E8 Physics (viXra 1405.0030) at high energies has Octonionic 8-dim Spacetime
that is fundamentally a superposition of E8 Lattices
each of which has vertices surrounded by the 240-vertex E8 Root Vector Polytope.

At lower energies Octonionic symmetry is broken to Quaternionic symmetry
in accord with E8 = H4 + H4 so that the 240-vertex E8 Polytope is decomposed
into two copies of the Quaternionic 4-dim 120-vertex 600-cell whose relative
size is the Golden Ratio. If you give one copy a rational number size,
then the size of the other will be in a Golden Ratio Algebraic Extension space.

Let the Rational Number 600-cell be
the Vertex Polytope for 4-dim M4 Physical Spacetime of M4 x CP2 Kaluza-Klein
and
the Algebraic Extension 600-cell be
the Vertex Polytope for 4-dim CP2 Internal Symmetry Space of M4 x CP2 Kaluza-Klein

Look at the 4-dim Physical Spacetime 600-cell. It has 120 vertices and 600 tetrahedra.
20 x 24 = 480 of the 600 tetrahedra are in 24 icosahedra within the 600-cell.
5 x 24 = 120 of the 600 tetrahedra are, 5 in each,
connected to each of the 24 icosahedra to form 24 octahedra.
The 24 octahedra form a 4-dim 24-cell,
the Vertex Polytope of the 4-dim Feynman Checkerboard.
24 of the 120 vertices correspond to vertices of the 24-cell and
96 of the 120 vertices correspond to Golden Ratio points, arranged in one of the two
possible consistent ways, on the 96 edges of the 24-cell dual to the original 24-cell.

Even though 3-dim simplex tetrahedra cannot tile flat 3-dim space,
they can combine to form curved 3-dim subspaces in 4-dim space, so that
3-dim simplex tetrahedra can be used as building blocks to construct E8 Physics
by taking 1200 of them to make two 600-cells, each in its own 4-dim space,
and then combining the two 600-cells and their two 4-dim spaces
to make 8-dim E8 Root Vector Polytopes, and then to make E8 Lattices
whose E8 Lie Algebra lives in Cl(16) Clifford Algebras whose completion of union of all
tensor products form a generalized hyperfinite II1 von Neumann factor AQFT
(Algebraic Quantum Field Theory) based on the realistic E8 Physics Lagrangian
and corresponding to a realistic 4-dim Feynman Checkerboard.

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8-dim E8 and 4-dim 600-cell

E8 Physics (viXra 1405.0030) at high energies has Octonionic 8-dim Spacetime
that is fundamentally a superposition of E8 Lattices
each of which has vertices surrounded by the 240-vertex E8 Root Vector Polytope.

E8 Lattice = D8 Lattice + ([1] + D8 Lattice)

There are 7 independent E8 Integral Domain Lattices.

Physically, the D8 Lattice represents SpaceTime and Gauge Bosons
while the ([1] + D8 Lattice) represents Fermions.
At high energies (for example, during Inflation) E8 Physics is Octonionic and
there is only one generation of fermions, so the first generation is the only generation.
Therefore, each charged Dirac fermion particle, and its antiparticle, correspond to one
imaginary Octonion, to one associative triangle, and to one E8 lattice
so each Fermion propagates in its own **E8 8D Feynman Checkerboard Lattice**:

```
red Down Quark                 red Up Quark          .
   green Down Quark    Electron     green Up Quark
          blue Down Quark                 blue Up Quark

      rD     gD     bD       E      rU     gU     bU

      I      J      K       E      i      j      k

                            j
                           / \
                          i---k

      J      j      J              I      J      K
     / \    / \    / \            / \    / \    / \
    i---K  I---K  I---k          E---i  E---j  E---k

     3E8    6E8    4E8     7E8    1E8    2E8    5E8
```

Since all the E8 lattices have in common the vertices { ±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke }, all the charged Dirac fermions can interact with each other. Composite particles, such as Quark-AntiQuark mesons and 3-Quark hadrons, propagate on the common parts of the E8 lattices involved. The uncharged neutrino fermion, which corresponds to the Octonion real axis with basis {1}, propagates on the 8th Kirmse E8 Lattice that is not an independent Octonion Integral Domain.

At lower energies Octonionic symmetry is broken to Quaternionic symmetry in accord with E8 = H4 + H4 so that the 240-vertex E8 Polytope is decomposed into two copies of the Quaternionic 4-dim 120-vertex 600-cell whose relative size is the Golden Ratio. If you give one copy a rational number size, then the size of the other will be in a Golden Ratio Algebraic Extension space.

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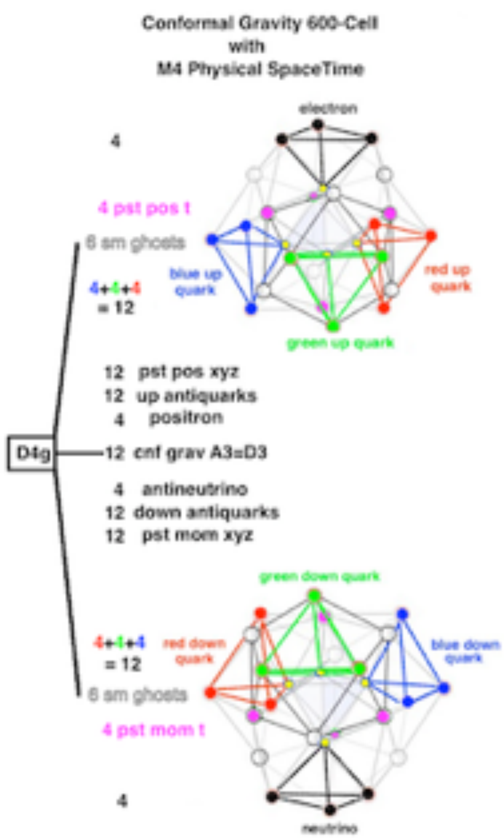

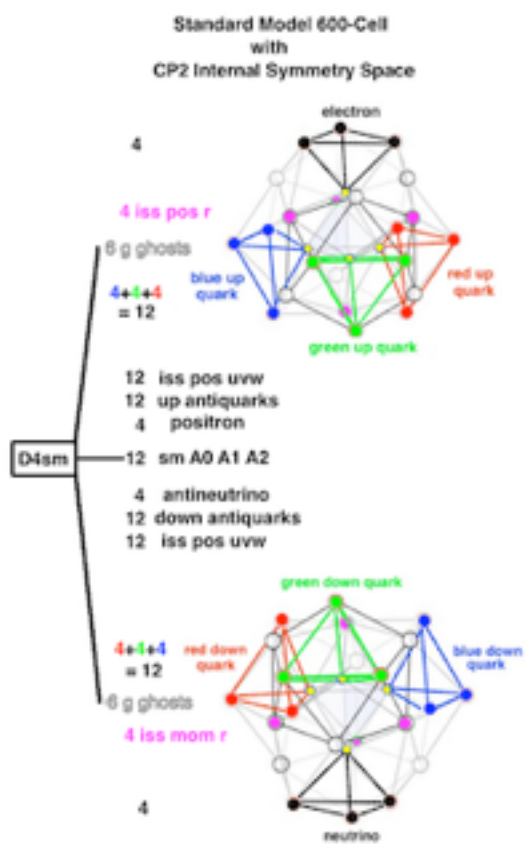

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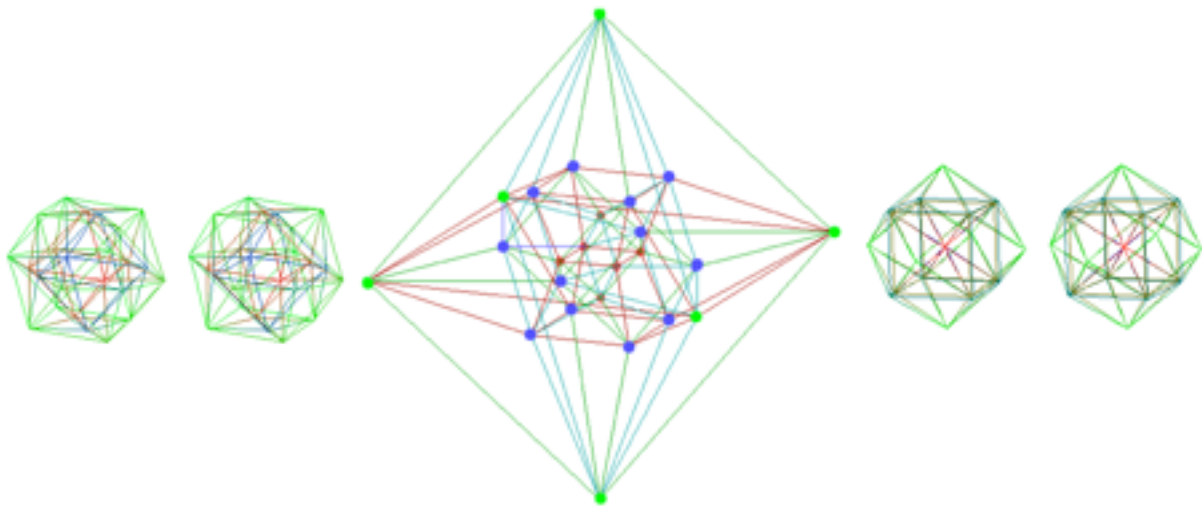
The 24 octahedra form a 4-dim 24-cell (center image from Frans Marcelis web site)



the Vertex Polytope of the 4-dim Feynman Checkerboard.
24 of the 120 vertices correspond to vertices of the 24-cell and
96 of the 120 vertices correspond to Golden Ratio points, arranged in one of the two possible consistent ways, on the 96 edges of the 24-cell dual to the original 24-cell.

Each of the 24 Octahedra that fill up the volume of the 24-cell contains an Icosahedron

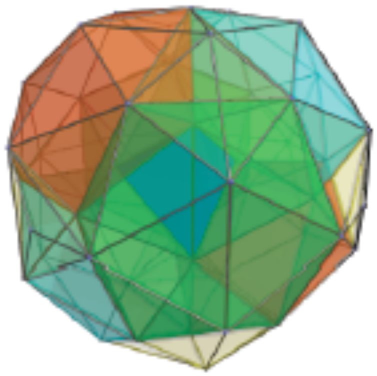
(image from wolfram mathworld)

plus some extra volume in each Octahedron.

The extra volume for all 24 Octahedra is made up of
24 vertex Tetrahedra (6 Octahedra meet at a 24-cell vertex, 6 x 24/6 = 24)
+
96 edge Tetrahedra (3 Octahedra meet at a 24-cell edge, 12 x 24/3 = 96)

so

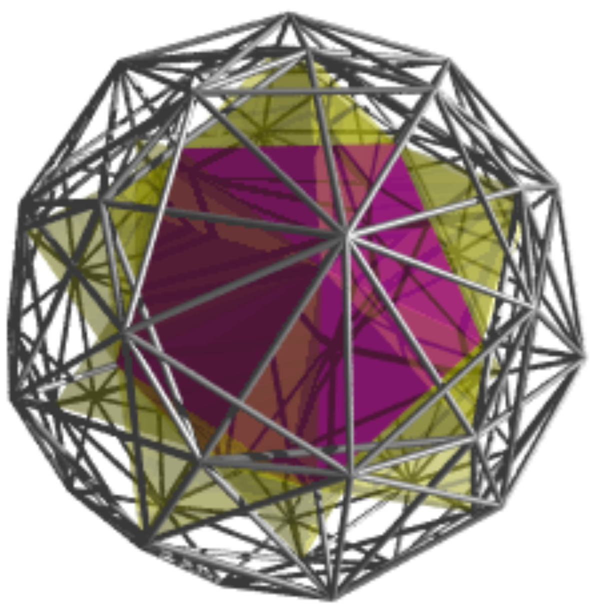
the 24-cell becomes a **Snub 24-cell = 24 Icosahedral and 120 Tetrahedral cells**



(image from eusebia.dyndns.org)

Each of the 24 Icosahedra contains 20 Tetrahedra for a total of 480 Tetrahedra
which when added to the 24+96 = 120 Tetrahedra outside the Icosahedra
give you the **480+120 = 600 Tetrahedra of the 600-cel**l.

According to Wikipedia: "...



... This image shows a vertex-first perspective projection of the 600-cell into 3D. The 600-cell is scaled to a vertex-center radius of 1, and the 4D viewpoint is placed 5 units away. Then the following enhancements are applied: The 20 tetrahedra meeting at the vertex closest to the 4D viewpoint are rendered in solid color. Their icosahedral arrangement is clearly shown. The tetrahedra immediately adjoining these 20 cells are rendered in transparent yellow. The remaining cells are rendered in edge-outline. Cells facing away from the 4D viewpoint (those lying on the "far side" of the 600-cell) have been culled, to reduce visual clutter in the final image. ...".

Sections of 600-cell

Sadoc and Mosseri in their book “Geometrical Frustration” (Cambridge 1999, 2006), say: “...

250 *A5 Polytope {3, 3, 5}*

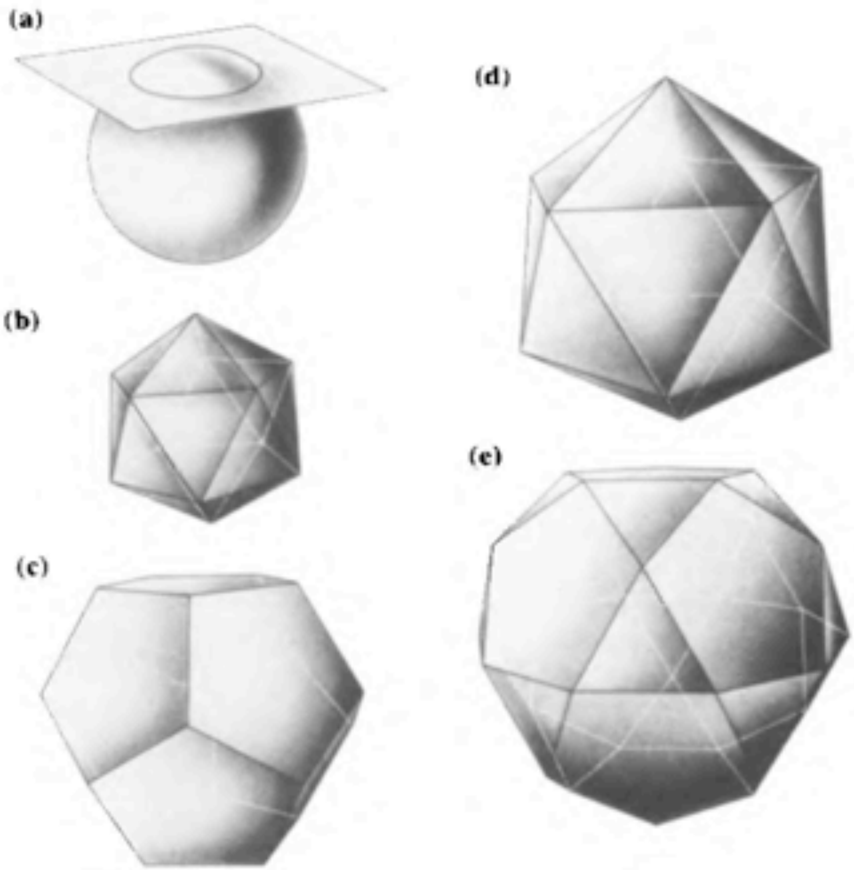


Fig. A5.1. The $\{3, 3, 5\}$ polytope. Different flat sections in S^3 (with one site on top) give the following successive shells; (a) an icosahedral shell formed by the first 12 neighbours, (b) a dodecahedral shell, (c) a second and larger icosahedral shell, (d) an icosidodecahedral shell on the equatorial sphere. Then other shells are symmetrically disposed in the second ‘south’ hemi-hypersphere, relative to the equatorial sphere (e).

$\omega = \pi/2$: the ‘equatorial’ sphere is tiled by 30 vertices which form a regular icosidodecahedron. For larger values of ω, the situation is then symmetrical with respect to the equatorial sphere.
$\omega = 3\pi/5$: an icosahedron.
$\omega = 2\pi/3$: a dodecahedron.
$\omega = 4\pi/5$: an icosahedron.
$\omega = \pi$: one vertex at the south pole $x_0 = -R$, $x_1 = x_2 = x_3 = 0$.

...

...

Table A5.1. *Sections of the {3, 3, 5} polytope (with an edge length equal to $2\tau^{-1}$) beginning with a vertex*

Section	x_0	(x_1, x_2, x_3)[1]	Vertex number	Shape
0	2	(0, 0, 0)	1	point
1	τ	$(1, 0, \tau^{-1})$	12	icosahedron
2	1	(1, 1, 1) $(\tau, \tau^{-1}, 0)$	20	dodecahedron
3	τ^{-1}	$(\tau, 0, 1)$	12	icosahedron
4	0	(2, 0, 0) $(\tau, 1, \tau^{-1})$	30	icosidodecahedron
5	$-\tau^{-1}$	$(\tau, 0, 1)$	12	icosahedron
6	-1	(1, 1, 1) $(\tau, \tau^{-1}, 0)$	20	dodecahedron
7	$-\tau$	$(1, 0, \tau^{-1})$	12	icosahedron
8	-2	(0, 0, 0)	1	point

[1]Cyclic permutation with all possible changes of signs. $\tau = (1 + \sqrt{5})/2$.

... **Another ... description ... fixing a polytope cell center at the north pole** ...

Table A5.2. *Section of the {3, 3, 5} polytope (edge length $2\tau^{-1}\sqrt{2}$) beginning with a cell*

Section	x_0	(x_1, x_2, x_3)	Vertex number	Shape
0	τ^2	$(\tau^{-1}, \tau - 1, \tau^{-1})$[1]	4	tetrahedron
1	$\sqrt{5}$	$(-1, 1, 1)$	4	tetrahedron
2	2	(2, 0, 0)	6	octahedron
3	τ	(τ, τ, τ^{-2})	12	distorted
4	1	$(\sqrt{5}, 1, 1)$	12	cubo-octahedron
5	τ^{-1}	$(\tau^2, \tau^{-1}, \tau^{-1})$	12	
6	τ^{-2}	(τ, τ, τ)	4	tetrahedron
7	0	(2, 2, 0)	12	cubo-octahedron
8	$-\tau^{-2}$	$(-\tau, \tau, \tau)$	4	tetrahedron
—	—	—	—	
14	$-\tau^2$	$(-\tau^{-1}, \tau^{-1}, \tau^{-1})$	4	tetrahedron

[1]Permutation with an even number of sign changes. $\tau = (1 + \sqrt{5})/2$. Distorted cubo-octahedra are such that their square faces are changed into golden rectangles.

... ". **At the north pole and its antipodal south pole are Maximal Contact Groupings (57G) with 4+4+6+12 = 26 vertices**.

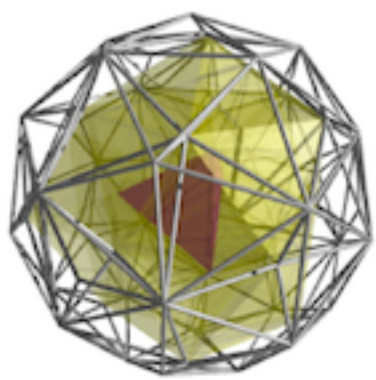
The Wikipedia entry on the 600-cell says:
“... the **600-cell** ... is the convex regular polytope ...]{3,3,5}. Its boundary is composed of 600 tetrahedral cells with 20 meeting at each vertex ... they form 1200 triangular faces, 720 edges, and 120 vertices. The edges form 72 flat regular decagons. Each vertex of the 600-cell is a vertex of six such decagons. ... Its vertex figure is an icosahedron ... It has a dihedral angle of 164.48 degrees. ... **Each cell touches, in some manner, 56 other cell**s.
[4+1 = 5] One cell contacts each of the four faces;

[
2x6 +5 = 17] two cells contact each of the six edges, but not a face;

[
10x4 +17 = 57] and ten cells contact each of the four vertices, but not a face or edge.

[] ...

This image shows the 600-cell in cell-first perspective projection into 3D. ...



... The nearest cell to the 4d viewpoint is rendered in solid color, lying at the center of the projection image. The cells surrounding it (sharing at least 1 vertex) are rendered in transparent yellow. [**They are a 57G Maximal Contact Grouping**]
The remaining cells are rendered in edge-outline.
Cells facing away from the 4D viewpoint have been culled for clarity. ...”.

240 vertices of Two 600-cells and E8

Sadoc and Mosseri in that book also say:
"... {3,3,5} vertices ... as a set of 120 unit quaternions, form the binary icosahedral group ... the 120 polytope vertices can be grouped into four symmetry related sets of 30 sites, whose local order is linear arrangement of tetrahedra resembling Coxeter's 'simplicial helix' ...  ... In R3, the simplicial helix has pseudo-periods ... every 30 tetrahedra, the structure almost repeats itself ... In the polytope ... {3,3,5} ...the set of 30 tetrahedra perfectly closes on itself on a great circle ...
It is possible to describe the {3,3,5} polytope using the ... spherical torus which is a two-dimensional surface embedded in the spherical space S3 ...[that]... can be built from a square sheet, whose opposite sides are joined together. ...
Any line parallel to a diagonal of the square corresponds to a great circle of the 3-sphere ... For the spherical torus ... the two 'axes' of the torus ... are great circles ... the {3,3,5} polytope has two sets of 10 vertices ... on the two opposite axes of the torus foliation. The remaining 100 vertices belong to two sets of 50 vertices, forming triangular tilings on two tori ... placed symmetrically ... to the spherical torus.

We can represent one such torus by a cylinder ...  ...
if we decompose further the sets of 50 vertices on each torus into five sets of 10 vertices, we ... get ... 12 sets of 10 vertices belonging to 12 great circles, which is .. the discretized Hopf fibration ... The Hopf mapping of this discrete set onto S2 gives 12 points which form a regular icosahedron on the base
...
The E8 lattice is ... the densest sphere packing in eight dimensions ...
[its] first ... shell is a 240-vertex ... Gosset polytope ...
split its 240 vertices into ten ... subsets ... each ... belonging ... to a sphere S3 ...
This is ... a discrete version of the Hopf fibration of S7 with S3 fibres and a S4 base
...
On each fibre, the 24 points form a [24-cell] polytope {3,4,3} ...
each fibre ... generates a four-dimensional sublattice {3,3,4,3} of the E8 lattice.
There are ten ... sublattices through the origin, associated with the ten points on the base S4 ... a 'cross' polytope on S4 ... images of the fibres under the Hopf map.
Let P be ... (1/sqrt(5))(1,1,1,1,1) ... It ... defines a four-dimensional space E.
The mapping of the Gossett polytope onto E produces two sets of five (3,4,3) ... form[ing] ... two concentric {3,3,5} on E ... which differ by a factor [of the Golden Ratio] ...".

E8 Physics and 8D Feynman Checkerboard

E8 Physics is described in http://vixra.org/pdf/1405.0030vG.pdf
in which the 240 vertices of the Gosset polytope are given physical interpretations that produce a Local Classical Lagrangian for Gravity and the Standard Model. Embedding E8 in the Real Clifford Algebra Cl(16) = Cl(8)xCl(8) and taking the completion of the union of all tensor products of Cl(16) gives a realistic Algebraic Quantum Field Theory (AQFT).

An equivalent Quantum Field Theory can be constructed using Tetrahedra, 600-cells, and the E8 Gossett polytope along with a generalized Feynman Checkerboard in 4 SpaceTime dimensions.

Conway and Sloane, in their book Sphere Packings, Lattices, and Groups (3rd edition, Springer, 1999), in chapter 4, section 7.3, pages 119-120) define a packing [where the glue vector [1] = (1/2, ... , 1/2)]

D+n = Dn u ([1] + Dn)

and say:

"... D+n is a lattice packing if and only if n is even.
D+3 is the tetrahedral or diamond packing ... and
D+4 = Z4.
When n = 8 this construction is especially important,
the lattice **D+8 being known as E8** ...".

Therefore

E8 Lattice = D8 Lattice + ([1] + D8 Lattice)

There are 7 independent E8 Integral Domain Lattices.

Physically, the D8 Lattice represents SpaceTime and Gauge Bosons while the ([1] + D8 Lattice) represents Fermions.

At high energies (for example, during Inflation) E8 Physics is Octonionic and there is only one generation of fermions, so the first generation is the only generation.

Therefore, each charged Dirac fermion particle, and its antiparticle, correspond to one imaginary Octonion, to one associative triangle, and to one E8 lattice so each Fermion propagates in its own **E8 8D Feynman Checkerboard Lattice**:

```
red Down Quark                 red Up Quark            .
   green Down Quark   Electron     green Up Quark
          blue Down Quark                 blue Up Quark

           rD     gD     bD       E      rU     gU     bU

           I      J      K       E      i      j      k

                                j
                               / \
                              i---k

           J      j      J              I      J      K
          / \    / \    / \            / \    / \    / \
         i---K  I---K  I---k          E---i  E---j  E---k

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Since all the E8 lattices have in common the vertices { ±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke }, all the charged Dirac fermions can interact with each other. Composite particles, such as Quark-AntiQuark mesons and 3-Quark hadrons, propagate on the common parts of the E8 lattices involved. The uncharged neutrino fermion, which corresponds to the Octonion real axis with basis {1}, propagates on the 8th Kirmse E8 Lattice that is not an independent Octonion Integral Domain.

If a preferred Quaternionic Structure is introduced into an Octonionic E8 Lattice then the Octonionic E8 Lattice is transformed into Quaternionic Lattice structure. The Quaternionic Integral Domain Lattice is the D4 Lattice.

D8 Lattice is transformed to D4g + D4sm

([1] + D8 Lattice) is transformed to ([1] + D4g) + ([1] + D4sm)

so

E8 is transformed to { D4g + ([1] + D4g) } + { D4 sm + ([1] + D4sm) }

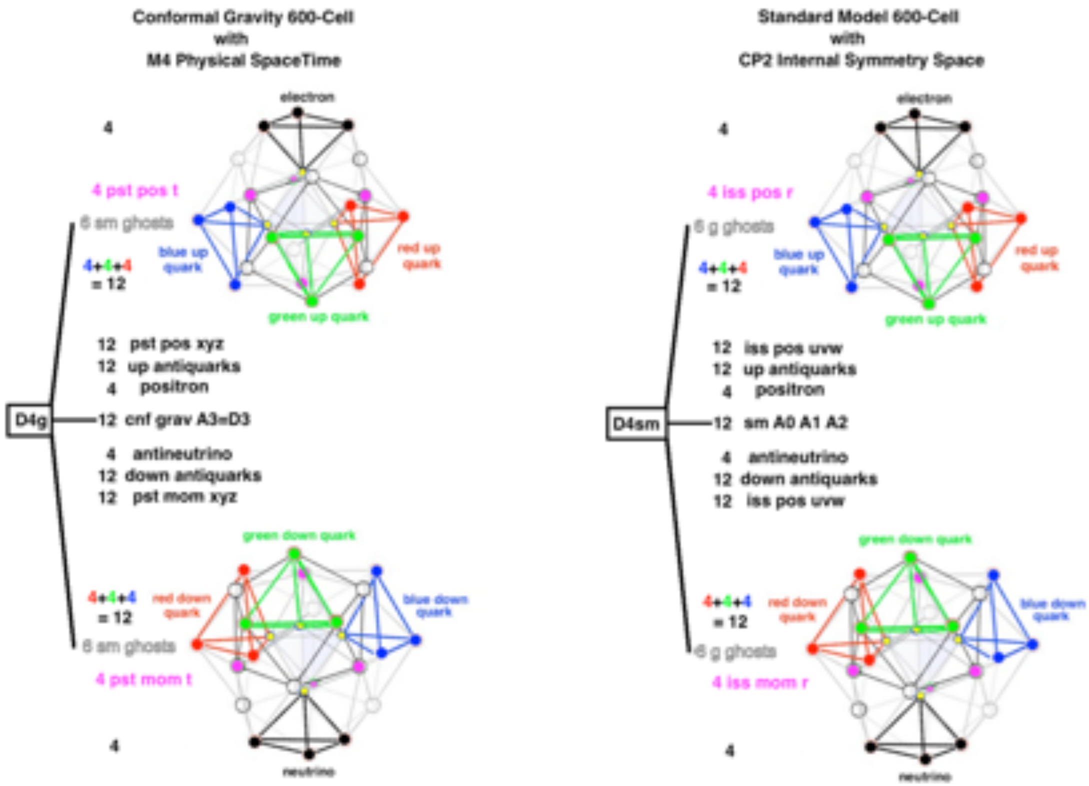
E8 = D+4g + D+4sm

D+4g corresponds to the 600-cell containing D4g

D+4sm corresponds to the 600-cell containing D4sm

To begin, consider
the two 600-cells underlying the Gosset polytope at each vertex of an E8 Lattice.

Split 8-dim Kaluza-Klein E8 SpaceTime into its two 4-dimensional components:
M4 Physical SpaceTime and CP2 = SU(3 / SU(2)xU(1) Internal Symmetry Space
Let one 600-cell represent Gravity and physics of Physical SpaceTime.
Let the other 600-cell represent the Standard Model and its Internal Symmetry Space.



The 120 vertices of the D4g 600-cell and the 120 vertices of the D4sm 600-cell combined form the 240 vertices of the E8 Root Vectors of E8 Physics:

E8 lives inside the Real Clifford Algebra Cl(16) as E8 = D8 + Cl(16) half-spinors
so

240 E8 Root Vectors = 112 D8 Root Vectors + 128 Cl(16) half-spinors

E8 Lattice = D8 Lattice + ([1] + D8 Lattice)
where the lattice shifting glue vector [1] = (1/2, ... , 1/2)

4D Feynman Checkerboard Quantum Theory

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      I      J      K       E      i      j      k


                            j
                           / \
                          i---k

      J      j      J              I      J      K
     / \    / \    / \            / \    / \    / \
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so
E8 is transformed to { D4g + ([1] + D4g) } + { D4 sm + ([1] + D4sm) }

E8 = D+4g + D+4sm
D+4g corresponds to the 600-cell containing D4g
D+4sm corresponds to the 600-cell containing D4sm

Conway and Sloane (Sphere Packings, Lattices, and Groups - Springer) (Chapter 4, eq. 49) give equations for the number of vertices N(m) in the m-th layer of the D+4 HyperDiamond lattice where d is a divisor (including 1 and m) of m:

for m odd: N(m) = 8 SUM(d|m) d   for m even: N(m) = 24 SUM(d|m, d odd) d

Here are the numbers of vertices in some of the layers of the D4+ lattice.
The even-numbered layers correspond to the even D4 sublattice:

m=norm of layer	N(m)=no. vert.		
0	1		
1	8	=	1 x 8
2	24	=	1 x 24
3	32	=	(1 + 3) x 8
4	24	=	1 x 24
5	48	=	(1 + 5) x 8
6	96	=	(1 + 3) x 24
7	64	=	(1 + 7) x 8
8	24	=	1 x 24
9	104	=	(1 + 3 + 9) x 8
10	144	=	(1 + 5) x 24
11	96	=	(1 + 11) x 8
12	96	=	(1 + 3) x 24
13	112	=	(1 + 13) x 8
14	192	=	(1 + 7) x 24
15	192	=	(1 + 3 + 5 + 15) x 8
16	24	=	1 x 24
17	144	=	(1 + 17) x 8

First Stage of 4D Feynman Checkerboard:

D+4g vertices have HyperOctahedron 8 nearest-neighbors {+/-1,+/-i,+/-j,+/-k}
where 4-dim 1,i,j,k are descendants of 8-dim 1,i,j,k
to be used as 4D Feynman Checkerboard Primary Links representing the 4-dim M4 Physical SpaceTime of the Kaluza-Klein of E8 Physics whose 4 basis elements are {1,i,j,k} each of which has 8 momentum components with respect to 8-dim SpaceTime to represent 4x8 = 32 of 600-cell vertices.

D+4g vertices have 24-cell 24 next-nearest neighbors representing the
12 Conformal Gravitons (Root Vectors of U(2,2) and
12 Ghosts of Standard Model Gauge Bosons
that live on the nearest-neighbor links and represent 24 of 600-cell vertices.

D+4g vertices have 6-semi-HyperCube 32 next-next-nearest neighbors representing 4 M4 Physical SpaceTime components of 8 First-Generation Fermion Particles.
Fermion AntiParticles are represented by Particles moving backward in Time for representation of 2x32 = 64 of 600-cell vertices.

D+4g odd (1 and 3) layers correspond to Vectors and Fermion Spinors which are related by Triality.
D+4g even (2) layers correspond to BiVectors.

From each vertex of the 4D Feynman Checkerboard the First Stage uses a Triad of Quantum Choice Vectors.

Second Stage of 4D Feynman Checkerboard:

D+4sm vertices have HyperOctahedron 8 nearest-neighbors {+/-1,+/-i,+/-j,+/-k}
where 4-dim 1,i,j,k are descendants of 8-dim E,I,J,K
to be used as 4D Feynman Checkerboard Secondary Links representing the 4-dim CP2 Internal Symmetry Space of the Kaluza-Klein of E8 Physics whose 4 basis elements are {1,i,j,k} each of which has 8 momentum components with respect to 8-dim SpaceTime to represent 4x8 = 32 of 600-cell vertices.

D+4sm vertices have 24-cell 24 next-nearest neighbors representing the
12 Standard Model Gauge Bosons and
12 Ghosts of Conformal Gravitons (Root Vectors of U(2,2)
that live on the nearest-neighbor links and represent 24 of 600-cell vertices.

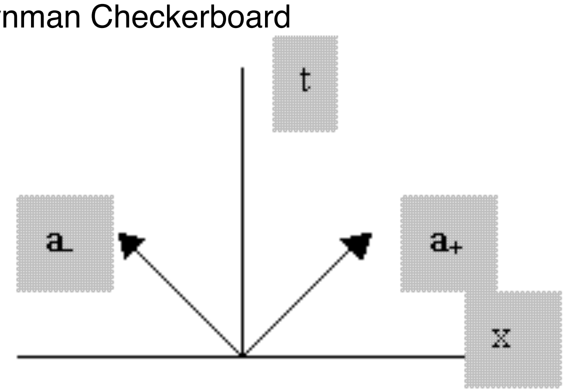
D+4sm vertices have 6-semi-HyperCube 32 next-next-nearest neighbors representing 4 CP2 Internal Symmetry Space components of 8 First-Generation Fermion Particles. Fermion AntiParticles are represented by Particles moving backward in Time for representation of 2x32 = 64 of 600-cell vertices.

D+4g odd (1 and 3) layers correspond to Vectors and Fermion Spinors which are related by Triality.
D+4g even (2) layers correspond to BiVectors.

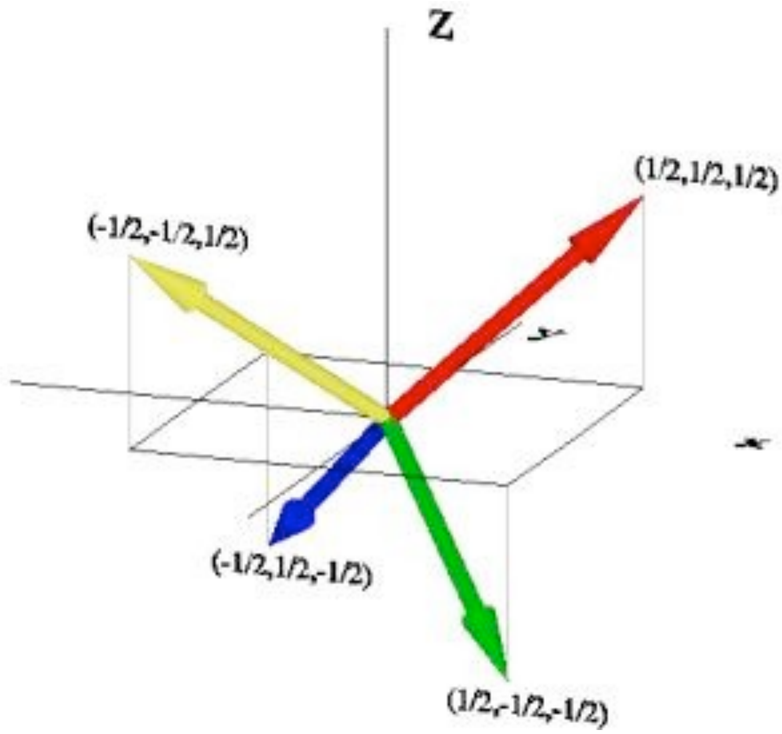
From each vertex of the 4D Feynman Checkerboard the Second Stage uses a second Triad of Quantum Choice Vectors.

A significant consequence of using two Triads of Quantum Choice Vectors is the emergence of Second and Third Generation Fermions.

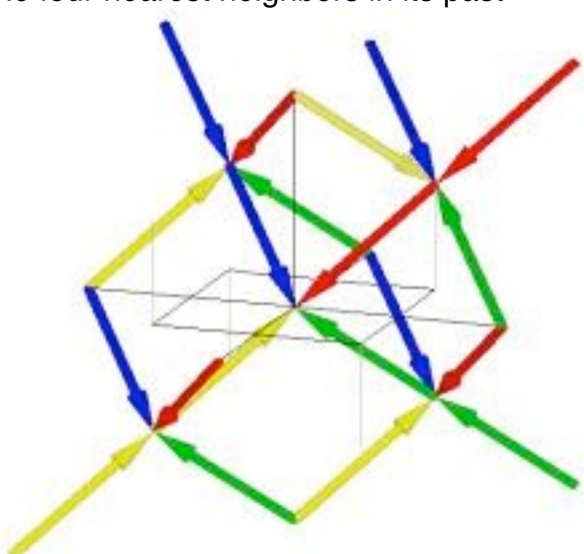
In my earlier paper (arXiv quant-ph/9503015) I used **a simpler version of 4D Feynman Checkerboard which is useful for showing consistency with the Dirac equation** using the following approach: The Feynman Checkerboard in 1+3 SpaceTime dimensions reproduces the Dirac equation, using work of Urs Schreiber and George Raetz. (See my paper at CERN-CDS-EXT-2004-030) A very nice feature of the George Raetz web site is its illustrations, which include an image of a vertex of a 1+1 dimensional Feynman Checkerboard



and an image of a projection into three dimensions of a vertex of a 1+3 dimensional Feynman Checkerboard



and an image of flow contributions to a vertex in a HyperDiamond Random Walk from the four nearest neighbors in its past



Urs Schreiber wrote on the subject:

Re: Physically understanding the Dirac equation and 4D

in the newsgroup sci.physics.research on 2002-04-03 19:44:31 PST (including an appended forwarded copy of an earlier post)
and again on 2002-04-10 19:03:09 PST as found on the web page
http://www-stud.uni-essen.de/~sb0264/spinors-Dirac-checkerboard.html

and the following are excerpts from those posts:

"... I know ... the ... lanl paper ...[http://xxx.lanl.gov/abs/quant-ph/9503015]...
and
I know that Tony Smith does give
a generalization of Feynman's summing prescription from 1+1 to 1+3 dimensions.

But I have to say that I fail to see that this generalization reproduces the Dirac propagator in 1+3 dimensions,
and that I did not find any proof that it does.

Actually, I seem to have convinced myself that it does not,
but
I may of course be quite wrong.

I therefore take this opportunity to state my understanding of these matters.

First, I very briefly summarize (my understanding of) Tony Smith's construction: The starting point is the observation that the left |-> and right |+> going states of the 1+1 dim checkerboard model can be labeled by complex numbers

|-> ---> (1 + i)
|+> ---> (1 - i)

(up to a factor) so that multiplication by the negative imaginary unit swaps components:

(-i) (1 + i)/2 = (1 - i)/2
(-i) (1 - i)/2 = (1 + i)/2 .

Since the path-sum of the 1+1 dim model reads
phi = sum over all possible paths of (-i eps m)^(number of bends of path) =
= sum over all possible paths of product over all steps of one path of -i eps m
(if change of direction after this step generated by i) 1 (otherwise)
this makes it look very natural to identify the imaginary unit appearing in the sum over paths with the "generator" of kinks in the path.
To generalize this to higher dimensions, more square roots of -1 are added,
which gives the quaternion algebra in 1+3 dimensions.
The two states |+> and |-> from above, which were identified with complex numbers,
are now generalized to four states identified with the following quaternions
(which can be identified with vectors in M^4 indicating the direction in which a given path is heading at one instant of time):
(1 + i + j + k) (1 + i - j - k) (1 - i + j - k) (1 - i - j + k) ,
which again constitute a (minimal) left ideal of the algebra
(meaning that applying i,j, or k from the left on any linear combination of these four states gives another linear combination of these four states).
Hence,
now i,j,k are considered as "generators" of kinks in three spatial dimensions
and the above summing prescription naturally generalizes to
phi = sum over all possible paths of product over all steps of one path of
-i eps m (if change of direction after this step generated by i)
-j eps m (if change of direction after this step generated by j)
-k eps m (if change of direction after this step generated by k)
1 (otherwise)
The physical amplitude is taken to be
A * e^(i alpha)
where A is the norm of phi and alpha the angle it makes with the x0 axis.

As I said, this is merely my paraphrase of Tony Smith's proposal as I understand it.

I fully appreciate that the above construction is a nice (very "natural") generalization of the summing prescription of the 1+1 dim checkerboard model.

But if it is to describe real fermions propagating in physical spacetime, this generalized path-sum has to reproduce the propagator obtained from the Dirac equation in 1+3 dimensions, which we know to correctly describe these fermions. Does it do that?
...
Hence I have taken a look at the material [that] ... George Raetz ... present[s] ... titled "The HyperDiamond Random Walk", found at
http://www.pcisys.net/~bestwork.1/QRW/the_flow_quaternions.htm ,
which is mostly new to me. ...

I am posting this in order to make a suggestion for a more radical modification
...
[The]... equation ... DQ = (iE)Q ... is not covariant.
That is because of that quaternion E sitting on the left of the spinor Q
in the rhs of [the] equation ... .
The Dirac operator D is covariant,
but the unit quaternion E on the rhs refers to a specific frame.
Under a Lorentz transformation L one finds
L DQ = iE LQ = L E' Q <=> DQ = E'Q now with E' = L~ E L instead of E.
This problem disappears
when the unit quaternion E is brought to the *right* of the spinor Q.
What we would want is an equation of the form DQ = Q(iE) .
In fact, demanding that the spinor Q be an element of the minimal left ideal
generated by the primitive projector P = (1+y0)(1+E)/4 ,
so that Q = Q' P ,
one sees that DQ = Q(iE) almost looks like the the *Dirac-Lanczos equation*.
(See hep-ph/0112317, equation (5) or ... equation (9.36) [of]... W. Baylis, Clifford (Geometric) Algebras, Birkhaeuser (1996) ...).
To be equivalent to the Dirac-Lanczos equation, and hence to be correct,
we need to require that D = y0 @0 + y1 @1 + y2 @2 + y3 @3
instead of ... = @0 + e1 @1 + e2 @2 + e3 @3 .
All this amounts to sorting out
in which particular representation we are actually working here.
In an attempt to address these issues, I now redo the steps presented on
http://www.pcisys.net/~bestwork.1/QRW/the_flow_quaternions.htm
with some suitable modifications to arrive at the correct Dirac-Lanczos equation
(this is supposed to be a suggestion subjected to discussion):

So consider a lattice in Minkoswki space
generated by a unit cell spanned by the four (Clifford) vectors

r = (y0 + y1 + y2 + y3)/2 g = (y0 + y1 - y2 - y3)/2 b =
= (y0 - y1 + y2 - y3)/2 y = (y0 - y1 - y2 + y3)/2 .

(yi are the generators of the Dirac algebra {yi,yj} = diag(+1,-1,-1,-1)_ij.)

This is Tony Smith's "hyper diamond".

(Note that I use Clifford vectors instead of quaternions.)
Now consider a "Clifford algebra-weighted" random walk along the edges of this lattice,
which is described by four Clifford valued "amplitudes": Kr, Kg, Kb, Ky and such that
@r Kr = k (Kg y2 y3 + Kb y3 y1 + Ky y1 y2)
@b Kb = k (Ky y2 y3 + Kr y3 y1 + Kg y1 y2) @g Kg = k (Kr y2 y3 + Ky y3 y1 + Kb y1 y2) @y Ky = k (Kb y2 y3 + Kg y3 y1 + Kr y1 y2) .

(This is geometrically motivated. The generators on the rhs are those that rotate the unit vectors corresponding to the amplitudes into each other. "k" is some constant.)

Note that I multiply the amplitudes from the *right* by the generators of rotation, instead of multiplying them from the left.

Next, assume that this coupled system of differential equations is solved by a spinor Q
Q = Q' (1+y0)(1+iE)/4
E = (y2 y3 + y3 y1 + y1 y2)/sqrt(3) with
Kr = r Q Kg = g Q Kb = b Q Ky = y Q .

This ansatz for solving the above system by means of a single spinor Q is,
as I understand it, the central idea.
But note that I have here modified it on the technical side:

Q is explicitly an algebraic Clifford spinor in a definite minial left ideal,

E squares to -1, not to +1,

and the Ki are obtained from Q by premultiplying with the Clifford basis vectors defined above.

Substituting this ansatz into the above coupled system of differential equations one can form one covariant expression by summing up all four equations:

(r @r + g @g + b @b + y @y) Q = k sqrt(3) Q E
The left hand side is immediate.
To see that the right hand side comes out as indicated
simply note that r + g + b + y = y0 and that Q y0 = Q by construction.

The above equation is the Dirac-Lanczos-Hestenes-Guersey equation,
the algebraic version of the equation describing the free relativistic electron.

The left hand side is the flat Dirac operator r @r + g @g + b @b + y @y = ym @m
and
the right hand side, with k = mc / (hbar sqrt(3)) ,
is equal to the mass term i mc / hbar Q.
As usual, there are a multitude of ways to rewrite this.
If one wants to emphasize biquaternions then
premultiplying everything with y0 and
splitting off the projector P on the right of Q to express everything in terms of the,
then also biquaternionic, Q' (compare the definitions given above)
gives Lanczos' version (also used by Baylis and others).

I think this presentation improves a little on that given on George Raetz's web site:

The factor E on the right hand side of the equation is no longer a nuisance
but a necessity.

Everything is manifestly covariant (if one recalls that algebraic spinors are manifestly covariant when nothing non-covariand stands on their *left* side). The role of the quaternionic structure is clarified,
the construction itself does not depend on it.
Also, it is obvious how to generalize to arbitrary dimensions.
In fact, one may easily check that for 1+1 dimensions the above scheme reproduces the Feynman model.

While I enjoy this, there is still some scepticism in order
as long as a central questions remains to be clarified:

How much of the Ansatz K(r,g,b,y) = (r,g,b,y) Q is whishful thinking?

For sure,
every Q that solves the system of coupled differential equations that describe the amplitude of the random walk on the hyper diamond lattice
also solves the Dirac equation.

But what about the other way round?
Does every Q that solves the Dirac equation also describe such a random walk. ...".

My proposal to answer the question raised by Urs Schreiber

Does every solution of the Dirac equation
also describe a HyperDiamond Feynman Checkerboard random walk?

uses symmetry.

The hyperdiamond random walk transformations
include the transformations of the Conformal Group:

rotations and boosts (to the accuracy of lattice spacing);
translations (to the accuracy of lattice spacing);
scale dilatations (to the accuracy of lattice spacing): and
special conformal transformations (to the accuracy of lattice spacing).

Therefore, to the accuracy of lattice spacing,
the hyperdiamond random walks give you all the conformal group Dirac solutions,
and since the full symmetry group of the Dirac equation is the conformal group,
the answer to the question is "Yes".

Thanks to the work of Urs Schreiber:

The HyperDiamond Feynman Checkerboard in 1+3 dimensions does reproduce the correct Dirac equation.

Here are some references to the **conformal symmetry of the Dirac equation**:

R. S. Krausshar and John Ryan in their paper Some Conformally Flat Spin Manifolds, Dirac Operators and Automorphic Forms at math.AP/022086 say:
"... In this paper we study Clifford and harmonic analysis on some conformal flat spin manifolds. ... manifolds treated here include RPn and S1 x S(n-1).
Special kinds of Clifford-analytic automorphic forms associated to the different choices of are used to construct Cauchy kernels, Cauchy Integral formulas, Green's kernels and formulas together with Hardy spaces and Plemelj projection operators for Lp spaces of hypersurfaces lying in these manifolds. ...
Solutions to the Dirac equation are called Clifford holomorphic functions or monogenic functions.
Such functions are covariant under ... conformal or .... Mobius transformations acting over Rn u {oo}. ...".

Barut and Raczka, in their book Theory of Group Representations and Applications (World 1986), say, in section 21.3.E, at pages 616-617:
"... E. The Dynamical Group Interpretation of Wave Equations.
... Example 1. Let G = O(4,2).
Take U to be the 4-dimensional non-unitary representation in which the generators of G are given in terms of the 16 elements of the algebra of Dirac matrices as in exercise 13.6.4.1.
Because (1/2)L_56 = gamma_0 has eigenvalues n = +/-1,
taking the simplest mass relation mn = K, we can write
(m gamma_0 - K) PSI(dotp) = 0, where K is a fixed constant.
Transforming this equation with the Lorentz transformation of parameter E
PSI(p) = exp(i E N) PSI(p)
N = (1/2) gamma_0 gamma
gives
(gamma^u p_u - K) PSI(p) = 0
which is the Dirac equation ...".

P. A. M. Dirac, in his paper Wave Equations in Conformal Space, Ann. Math. 37 (1936) 429-442, reprinted in The Collected Works of P. A. M. Dirac: Volume 1: 1924-1948, by P. A. M. Dirac (author), Richard Henry Dalitz (editor), Cambridge University Press (1995), at pages 823-836, said:
"... by passing to a four-dimensional conformal space ...
a ... greater symmetry of ... equations of physics ... is shown up, and their invariance under a wider group is demonstrated. ...
The spin wave equation ... seems to be the only
simple conformally invariant wave equation involving the spin matrices. ... This equation is equivalent to the usual wave equation for the electron, except ...[that it is multiplied by]... the factor (1 + alpha_5) ,
which introduces a degeneracy. ...".

Here are some comments on **Lorentz Invariance based on D4 Lattice** properties:

The D4 lattice nearest neighbor vertex figure, the 24-cell,
is the 4HD HyperDiamond lattice next-to-nearest neighbor vertex figure.
Fermions move from vertex to vertex along links.
Gauge bosons are on links between two vertices, and so can also be considered as moving from vertex to vertex along links.
The only way a translation or rotation can be physically defined is by a series of movements of a particle along links.
A TRANSLATION is defined as a series of movements of a particle along links, each of which is
the CONTINUATION of the immediately preceding link IN THE SAME DIRECTION.
An APPROXIMATE rotation, within an APPROXIMATION LEVEL D,
is defined with respect to a given origin as a series of movements of a particle along links among vertices ALL of which
are in the SET OF LAYERS LYING WITHIN D of norm (distance^2) R from the origin, that is,
the SET OF LAYERS LYING BETWEEN norm R-D and norm R+D from the origin.
Conway and Sloane (Sphere Packings, Lattices, and Groups - Springer) pp. 118-119 and 108, is the reference that I have most used for studying lattices in detail.
(Conway and Sloane define the norm of a vector x to be its squared length xx.)
In the D4 lattice of integral quaternions,
layer 2 has the same number of vertices as layer 1, N(1) = N(2) = 24.
Also (this only holds for real, complex, quaternionic, or octonionic lattices),
K(m) = N(m)/24 is multiplicative,
meaning that, if p and q are relatively prime, K(pq) = K(p)K(q).
The multiplicative property implies that:
K(2^a) = K(2) = 1 (for a greater than 0) and
K(p^a) = 1 + p + p^2 + ... + p^a (for a greater than or equal to 0).
So,
for the D4 lattice,
there is always an arbitrarily large layer (norm xx = 2^a, for some large a)
with exactly 24 vertices, and
there is always an arbitrarily large layer(norm xx = P, for some large prime P)
with 24(P+1) vertices (note that Mersenne primes are adjacent to powers of 2), and
given a prime number P whose layer is within D of the origin,
which layer has N vertices,
there is a layer kP with at least N vertices within D of any other given layer in D4.
Some examples I have used are chosen so that
the 2^a layer adjoins the prime 2^a +/- 1 layer.

The notation in the following table is based on the minimal norm of the D4 Lattice being 1, in which case **the D4 lattice is the lattice of integral quaternions**.
This is the second definition (equation 90) of the D4 Lattice in
Chapter 4 of Sphere Packings, Lattices, and Groups, 3rd ed., by Conway and Sloane (Springer 1999)
who note that the Dn lattice is the checkerboard lattice in n dimensions.

m=norm of layer	N(m)=no. vert.	K(m)=N(m)/24
1	24	1
2	24	1
3	96	4
4	24	1
5	144	6
6	96	4
7	192	8
8	24	1
9	312	13
10	144	6
11	288	12
12	96	4
13	336	14
14	192	8
15	576	24
16	24	1
17	432	18
18	312	13
19	480	20
20	144	6
127	3,072	128
128	24	1
65,536=2^16	24	1
65,537	1,572,912	65,538
2,147,483,647	51,539,607,552	2,147,483,648
2,147,483,648=2^31	24	1

What about 3D ?

Single Tetrahedron:

Tetrahedra can be used as Josephson Junctions.

A very useful reference is the 2003 dissertation of Christopher Bell at St. John's College Cambridge entitled "Nanoscale Josephson devices", on the web at http://www.dspace.cam.ac.uk/bitstream/1810/34607/1/chris_bell_thesis.pdf

Feigelman, Ioffe, Geshkenbein, Dayal, and Blatter in cond-mat/0407663 say: "... Superconducting tetrahedral quantum bits ...

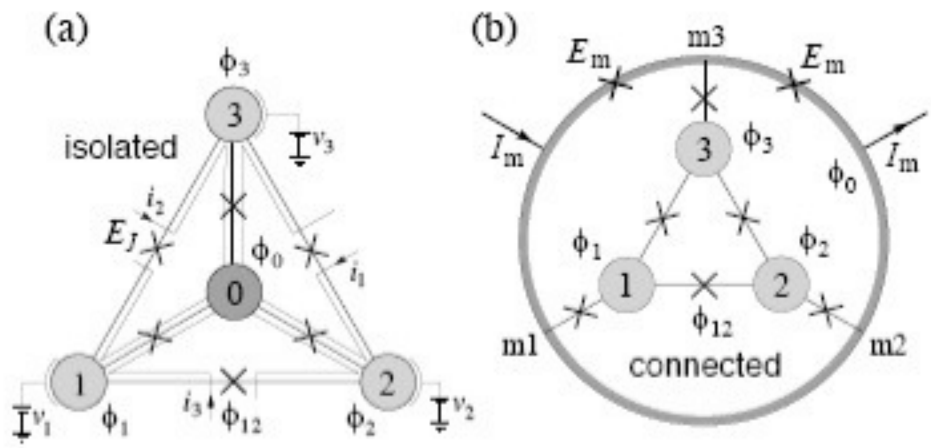


FIG. 1: (a) Tetrahedral superconducting qubit involving four islands and six junctions (with Josephson coupling E_J and charging energy E_C); all islands and junctions are assumed to be equal and arranged in a symmetric way. The islands are attributed phases ϕ_i, $i = 0, \ldots, 3$. The qubit is manipulated via bias voltages v_i and bias currents i_i. In order to measure the qubit's state it is convenient to invert the tetrahedron as shown in (b) — we refer to this version as the 'connected' tetrahedron with the inner dark-grey island in (a) transformed into the outer ring in (b). The measurement involves additional measurement junctions with couplings $E_m \gg E_J$ on the outer ring which are driven by external currents I_m (schematic, see Fig. 6 for details); the large coupling E_m effectively binds the ring segments into one island.

... The novel tetrahedral qubit design we propose below operates in the phase-dominated regime and exhibits two remarkable physical properties:

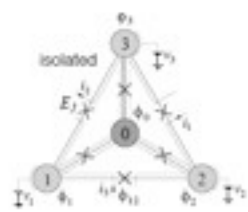
first, its non-Abelian symmetry group (the tetrahedral group Td) leads to the natural appearance of degenerate states and appropriate tuning of parameters provides us with a doubly degenerate groundstate. Our tetrahedral qubit then emulates a spin-1/2 system in a vanishing magnetic field, the ideal starting point for the construction of a qubit.
Manipulation of the tetrahedral qubit through external bias signals translates into application of magnetic fields on the spin;
the application of the bias to different elements of the tetrahedral qubit corresponds to rotated operations in spin space.
Furthermore, geometric quantum computation via Berry phases ... might be implemented through adiabatic change of external variables.
Going one step further, one may hope to make use of this type of systems in the future physical realization of non-Abelian anyons, thereby aiming at a new generation of topological devices ... which keep their protection even during operation ...

The second property we wish to exploit is geometric frustration:
In our tetrahedral qubit ... it appears in an extreme way by rendering the classical minimal states continuously degenerate along a line in parameter space. Semi-classical states then appear only through a fluctuation-induced potential, reminiscent of the Casimir effect ... and the concept of inducing
'order from disorder'...
The quantum-tunneling between these semi-classical states defines the operational energy scale of the qubit, which turns out to be unusually large due to the weakness of the fluctuation-induced potential. Hence the geometric frustration present in our tetrahedral qubit provides a natural boost for the quantum fluctuations without the stringent requirements on the smallness of the junction capacitances, thus avoiding the disadvantages of both the charge- and the phase- device:
The larger junctions reduce the demands on the fabrication process and the susceptibility to charge noise and mesoscopic effects, while the large operational energy scale due to the soft fluctuation-induced potential reduces the effects of flux noise. Both types of electromagnetic noise, charge- and flux noise, appear only in second order ...

in order to benefit from a protected degenerate ground state doublet, the qubit design requires a certain minimal complexity; it seems to us that the tetrahedron exhibits the minimal symmetry requirements necessary for this type of protection and thus the minimal complexity necessary for its implementation. ...".

Multiple Tetrahedra - try to build in flat 3D just a 57G part of the 600-cell:

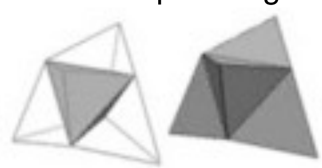
Eric A. Lord, Alan L. Mackay, and S. Ranganathan in their book “New Geometries for New Materials” (Cambridge 2006) said:
“... start ... from a single tetrahedron

[

]

Place four spheres in contact.

Then place a sphere over each face of the tetrahedral cluster.
The centres and bonds then form a stella quadrangula



built from five regular tetrahedra ...[a total of 1+4 = 5 tetrahedra]...

Six more spheres [vertices] placed over the edges of the original tetrahedron form an octagonal shell. In terms of the network of centres and bonds we now have added 12 [= 2x6] more tetrahedra ...

There are now five tetrahedra around each edge of the original tetrahedron. ...

...[we now have 1+4+12 = 17 tetrahedra]...

[The 12 newly added tetrahedra]... are not quite regular ...

[i.e., nonzero Fuller unzipping angles appear as described by Thomas Banchoff in his book “Beyond the Third Dimension” (Scientific American Library 1990) where he said:

“... in three-space .... we can fit five tetrahedra around an edge ...

[image from Conway and Torquato PNAS 103 (2006) 10612-10617

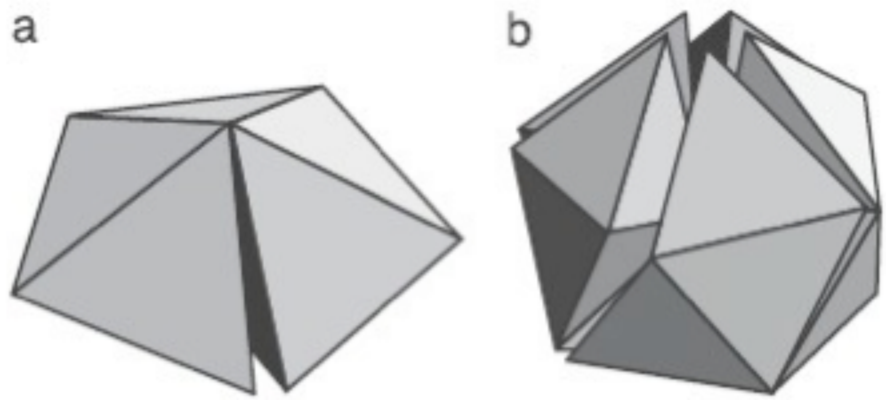


Fig. 1. Certain arrangements of tetrahedra. (*a*) Five regular tetrahedra about a shared edge. The angle of the gap is 7.36°. (*b*) Twenty regular tetrahedra about a shared vertex. The gaps amount to 1.54 steradians.

]

... with a ... small amount of room to spare,
which allows folding into 4-space ...[where the fit can be made exact]...”.

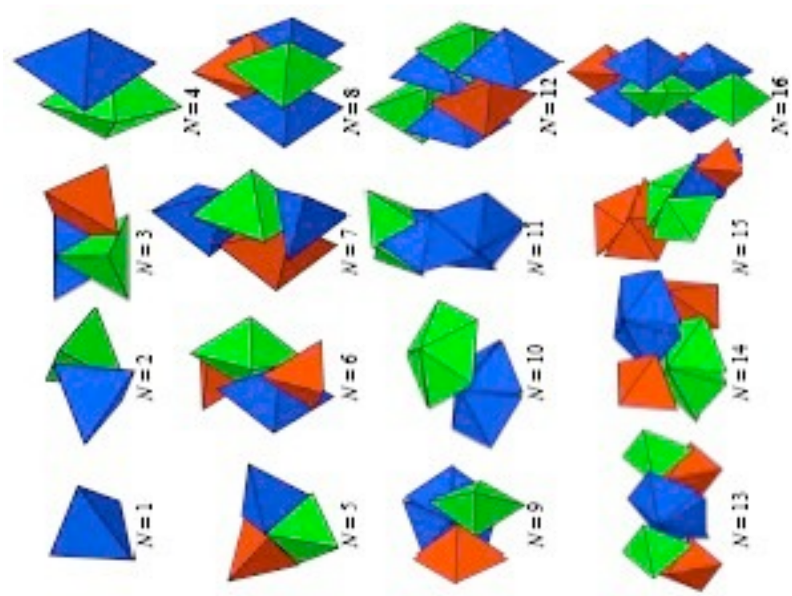
Add 4 half-Icosahedra (10 Tetrahedra each) to form a 40-Tetrahedron Outer Shell around the 17 Tetrahedra and so form a 57G

Like the 12 of 17, the Outer 40 do not exactly fit together in flat 3-dim space.

If you could force all 57 Tetrahedra to fit together exactly, you would be curving 3-dim space by a Dark Energy Conformal Transformation.

What happens if you require the 3-dim space to remain flat ?

If you construct with (exactly regular) tetrahedra in 3-dim space that remains flat that is like making a tetrahedral dense packing of flat 3-dim space.
The densest such packing now known is described by Chen, Engel, and Glotzer in arXiv 1001.0586 :
"... We present the densest known packing of regular tetrahedra with density Phi = 4000 / 4671 = 0.856347 ...



... The dimer structures are remarkable in the relative simplicity of the 4-tetrahedron unit cell as compared to the 82-tetrahedron unit cell of the quasicrystal approximant, whose density is only slightly less than that of the densest dimer packing.
The dodecagonal quasicrystal is the only ordered phase observed to form from random initial configurations of large collections of tetrahedra at moderate densities.
It is thus interesting to note that for some certain values of N, when the small systems do not form the dimer lattice packing, they instead prefer clusters (motifs) present in the quasicrystal and its approximant, predominantly pentagonal dipyramids. This suggests that the two types of packings - the dimer crystal and the quasicrystal/ approximant - may compete, raising interesting questions about
the relative stability of the two very different structures at finite pressure. ...".

If you regard a Tetrahedron as a pair of Binary Dipoles

then the Chen - Engel - Glotzer high (0.85+) density configurations have the same 8-periodicity property as the Real Clifford Algebras:

#Binary Dipoles M	Maximum Density Numerical, $\hat{\phi}$	Analytical, ϕ	Success Rate	Motifs, Structural Description
2	0.367346	18/49	100%	1 monomer [11]
4	0.719486	ϕ_2	100%	2 monomers, transitive [22]
6	0.666665	2/3	21%	3 monomers, three-fold symmetric
8	0.856347	4000/4671	80%	2 dimers (positive + negative)
10	0.748096	ϕ_5	22%	1 pentamer, asymmetric
12	0.764058	ϕ_6	11%	2 dimers + 2 monomers
14	0.749304	3500/4671	15%	2×2 dimers minus 1 monomer
16	0.856347	4000/4671	44%	2×2 dimers, identical to $N = 4$
18	0.766081		—	1 pentagonal dipyramid + 2 dimers
20	0.829282	ϕ_{10}	2%	2 pentagonal dipyramids
22	0.794604		—	1 nonamer + 2 monomers
24	0.856347	4000/4671	3%	3×2 dimers, identical to $N = 4$
26	0.788728		4%	1 pentagonal dipyramid + 4 dimers
28	0.816834		3%	2 pentagonal dipyramids + 2 dimers
30	0.788693		—	Disordered, non-optimal
32	0.856342	4000/4671	< 1%	4×2 dimers, identical to $N = 4$
⋮	⋮			⋮
164x8	0.850267			Quasicrystal approximant [21]

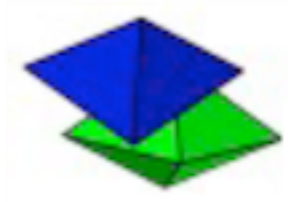
The Binary Pair of one Tetrahedron corresponds to the Cl(2) Real Clifford Algebra, isomorphic to the Quaternions, with graded strucure 1+2+1.
The 4 Binary Pairs of 4 Tetrahedra (2 Dimers) correspond to Cl(2x4) = Cl(8).

The Large N Limit of 4N Tetra Clusters =

= **Completion of Union of All 4N Tetra Clusters** would correspond to

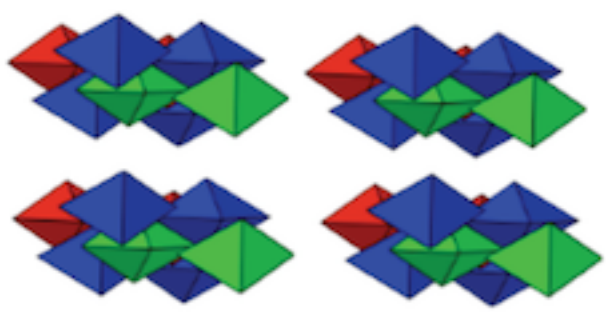
the same generalized Hyperfinite II1 von Neumann factor of Cl(16)-E8 Physics that gives a natural Algebraic Quantum Field Theory structure.

Geometrically:



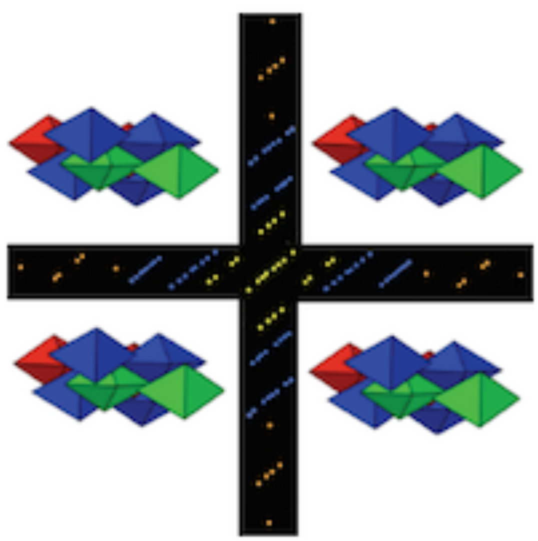
represents Cl(8) Clifford Algebra Vectors and Half-Spinors

represents Cl(16) Vectors



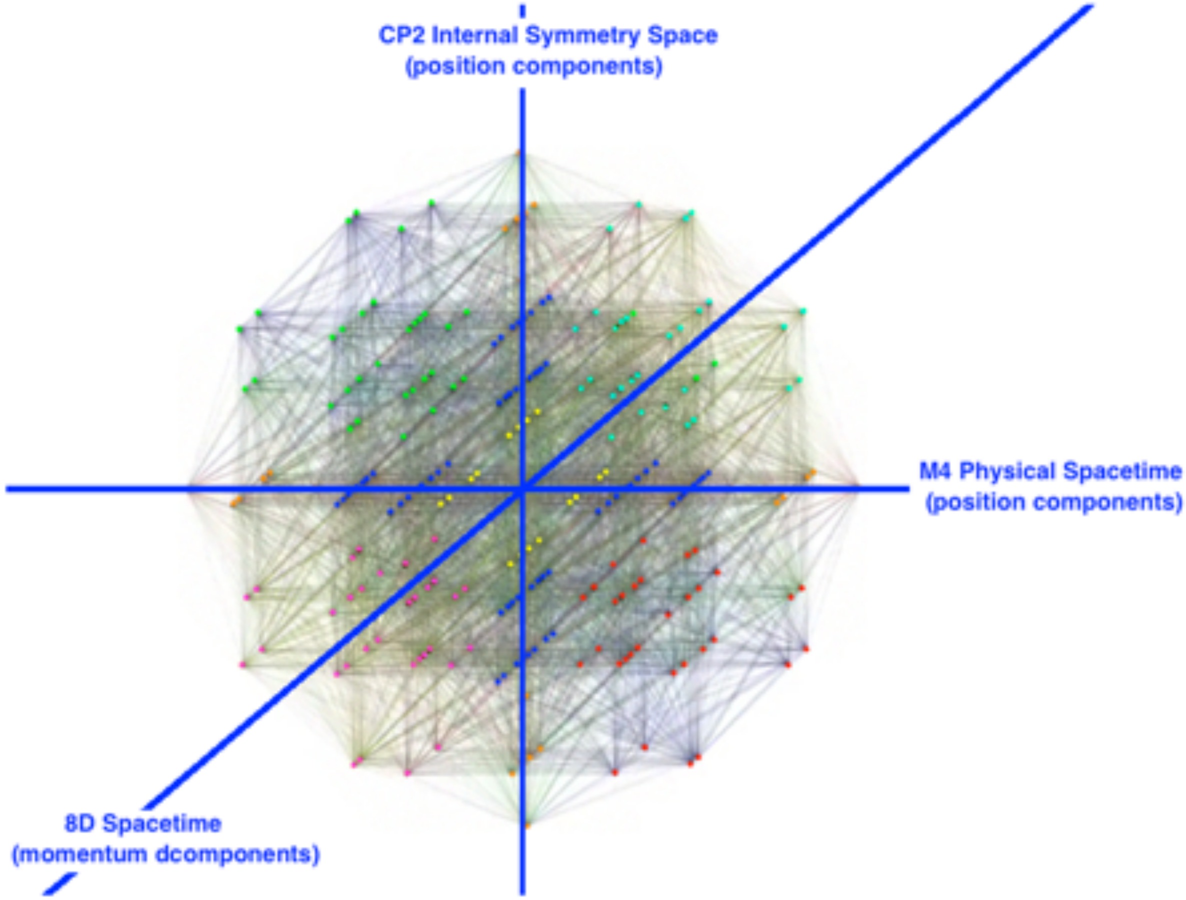
represents Cl(16) half-spinors

E8 = Cl(16) half-spinors + Cl(16) BiVectors



where the axes (central cross) corresponds to Cl(16) = Cl(8) x Cl(8) tensor product:
Cl(8) Vectors x Cl(8) Vectors (blue dots)
Cl(8) BiVectors x Cl(8) Scalar (yellow dots)
Cl(8) Scalar x Cl(8) Bivectors (orange dots)

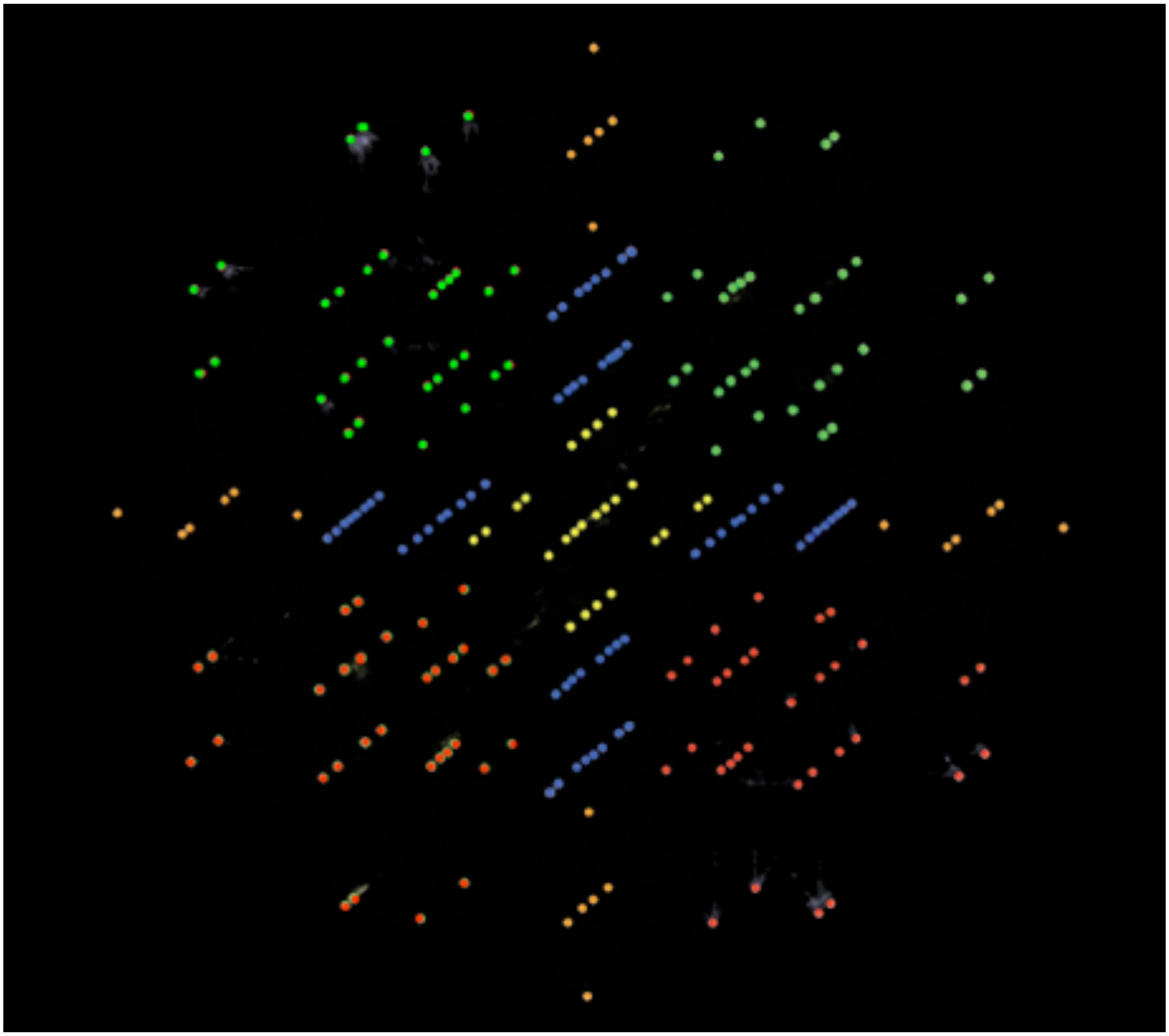
so
Extension of Local Lagrangian E8 Root Vector Physics over large Spacetime regions
can be described in terms of Flat 2D and 3D projections of E8 Root Vectors
by tiling 3D Space using Regular Cubic Tiling



can be extended to fully and exactly tile 3D space
with
the two axes representing the two parts of (4+4)D M4xCP2 Kaluza-Klein Spacetime
and the third axis representing the momentum components of 8D Spacetime
and
Gauge Bosons of Gravity and the Standard Model living on the two K-K position axes
and
Fermion Particles and AntiParticles living in 2^3 = 8 off-axis octants at each vertex.

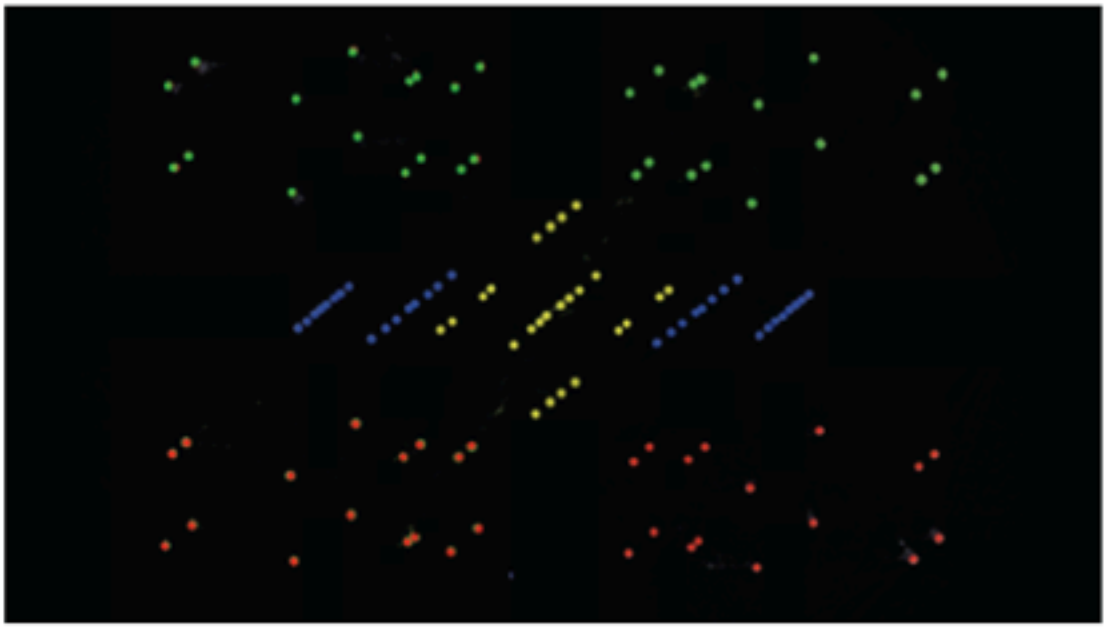
The physical interpretations of the 240 E8 Root Vectors are clear:
green / cyan and red / magenta for fermion particles and antiparticles (E8 / D8)
blue for M4 x CP2 Kaluza-Klein SpaceTime (D8 / D4xD4)
yellow for Gravity + Dark Energy (one of the D4)
orange for Standard Model SU(3) (the other D4)

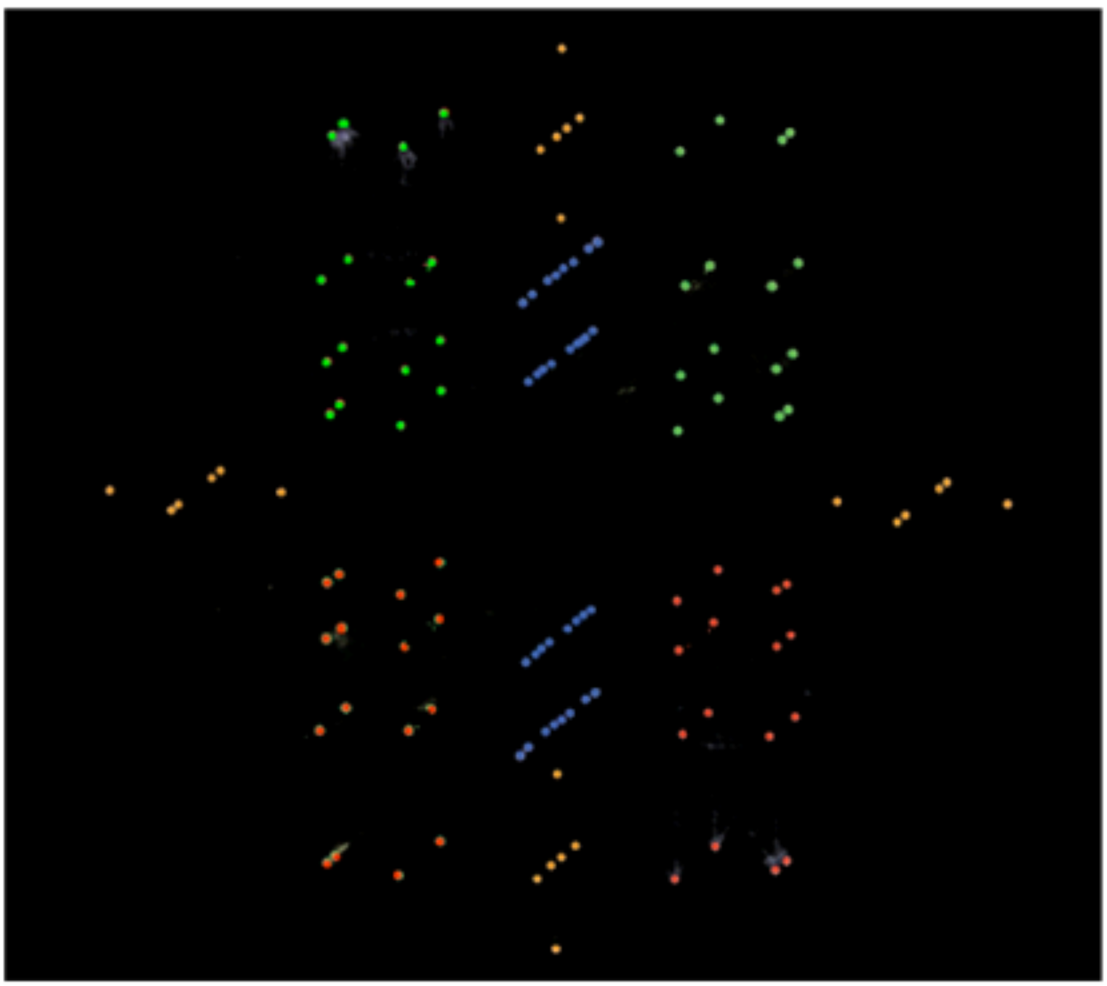
In this Cube-type 3D projection the 240 Root Vectors of E8



can be seen as

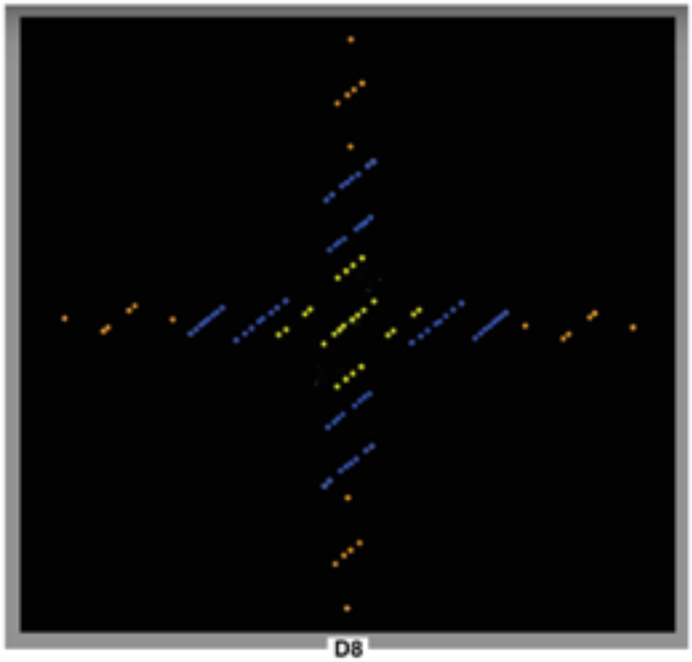
120 vertices of the 600-cell containing the D4 of Conformal Gravity and M4
and
120 vertices of the 600-cell containing the D4 of the Standard Model and CP2



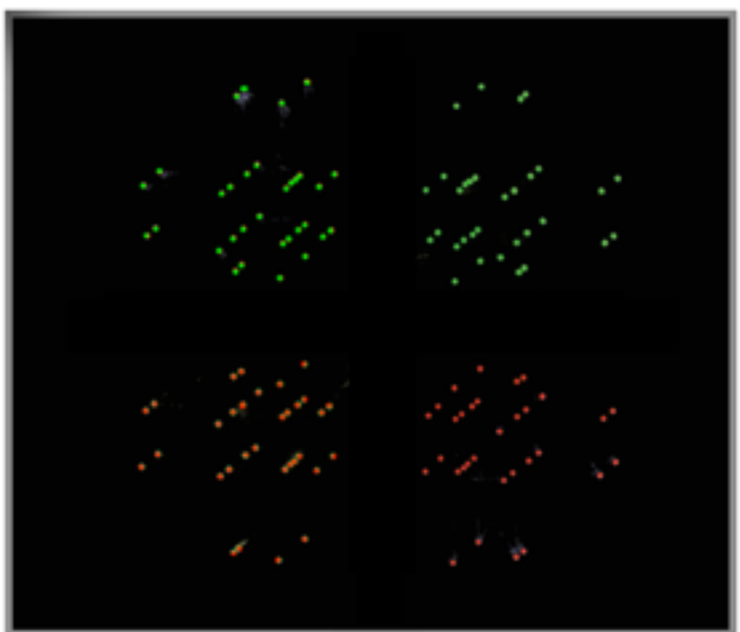


or, since D8 = Cl(16) bivectors,

240 E8 Root Vectors = 112 D8 Root Vectors + 128 Cl(16) half-spinors
E8 Lattice = D8 Lattice + ([1] + D8 Lattice)
where the lattice shifting glue vector [1] = (1/2, ... , 1/2)



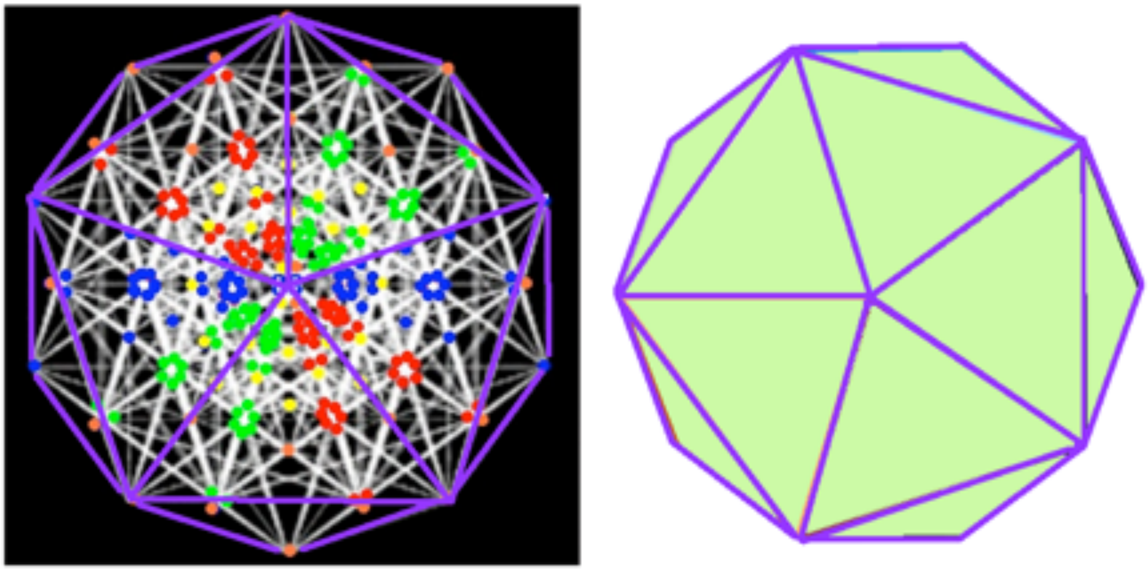
D8



E8 / D8

What about the QuasiCrystal / approximant in flat 3-dim space ?

Icosahedral 3D Projection

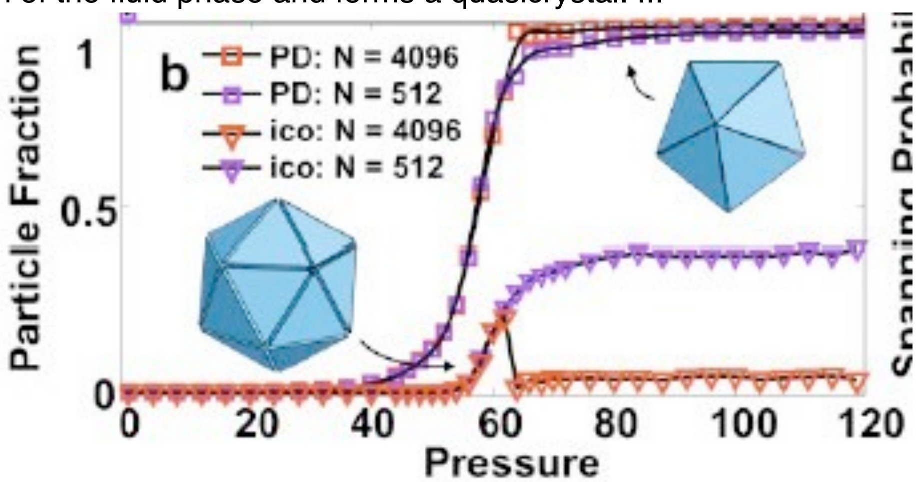


does NOT produce a Regular Tiling
but
gives an Icosahedron (purple) that can produce 3D Quasi-Crystals
in which
the root vector correspondence to E8 Physics
is not as easy to see as with the Cube-type 3D projection.

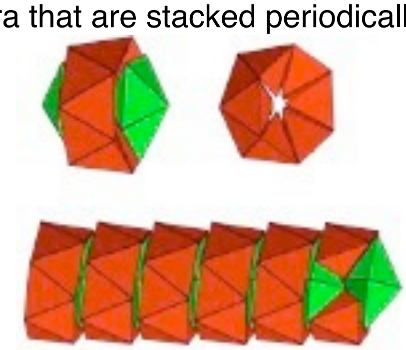
Also, if you make an Icosahedral Quasi-Crystal,
some of the Quasi-Crystal vertices will not be at the center of a full Icosahedron
so some E8 Physics information will be lost from the Quasi-Crystal.

Haji-Akbari1, Engel, Keys, Zheng, Petschek, Palffy-Muhoray, and Glotzer in arXiv 1012.5138 say: “... a fluid of hard tetrahedra undergoes a first-order phase transition to a dodecagonal quasicrystal,
which can be compressed to a packing fraction of $\phi = 0.8324$. By compressing a crystalline approximant of the quasicrystal, the highest packing fraction we obtain is $\phi = 0.8503$.
...

To obtain dense packings of hard regular tetrahedra, we carry out Monte-Carlo (MC) simulations ... of a small system with 512 tetrahedra and a large system with 4096 tetrahedra. ... The large system undergoes a first order transition on compression of the fluid phase and forms a quasicrystal. ...

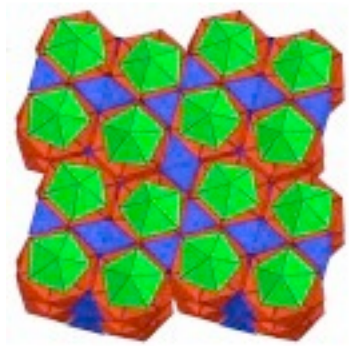


... the quasicrystal consists of a periodic stack of corrugated layers ... Recurring motifs are rings of twelve tetrahedra that are stacked periodically to form “logs”...

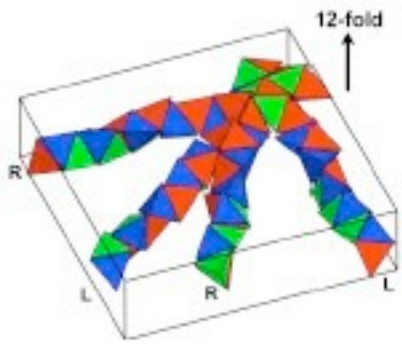


... Perfect quasicrystals are aperiodic while extending to infinity; they therefore cannot be realized in experiments or simulations, which are, by necessity, finite. ...
Quasicrystal approximants are periodic crystals with local tiling structure identical to that in the quasicrystal. Since they are closely related, and they are often observed in experiments, we consider them as candidates for dense packings.

The dodecagonal approximant with the smallest unit cell (space group) has 82 tetrahedra ...



... At each vertex we see the logs of twelve-member rings (shown in red) capped by single PDs (green). The logs pack well into squares and triangles with additional, intermediary tetrahedra (blue). The vertex configuration of the tiling is ...



...".

The QuasiCrystal approximant is not as dense as the 4N Tetra Cluster packing, so I do not regard it as being as useful for fundamental physics as the 4N Tetra packing.

The true QuasiCrystal is less dense than the QuasiCrystal approximant, so I regard it as being less useful for fundamental physics. However,
as Sadoc and Mosseri say in their book "Geometrical Frustration" (Cambridge 2005)
"... quasiperiodic structures [can be] derived from the eight-dimensional lattice E8. ...
... using the cut and project method, it is possible to generate
a four-dimensional quasicrystal having the symmetry of the [600-cell] polytope {3,3,5} ... a shell-by-shell analysis ...

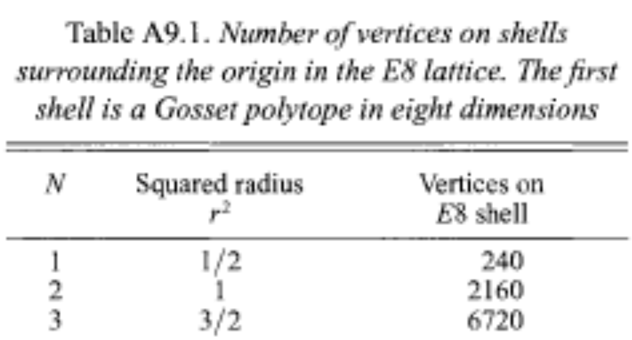


Table A9.1. *Number of vertices on shells surrounding the origin in the E8 lattice. The first shell is a Gosset polytope in eight dimensions*

N	Squared radius r^2	Vertices on $E8$ shell
1	1/2	240
2	1	2160
3	3/2	6720

... recalls in some respects ... the Fibonacci chain ...

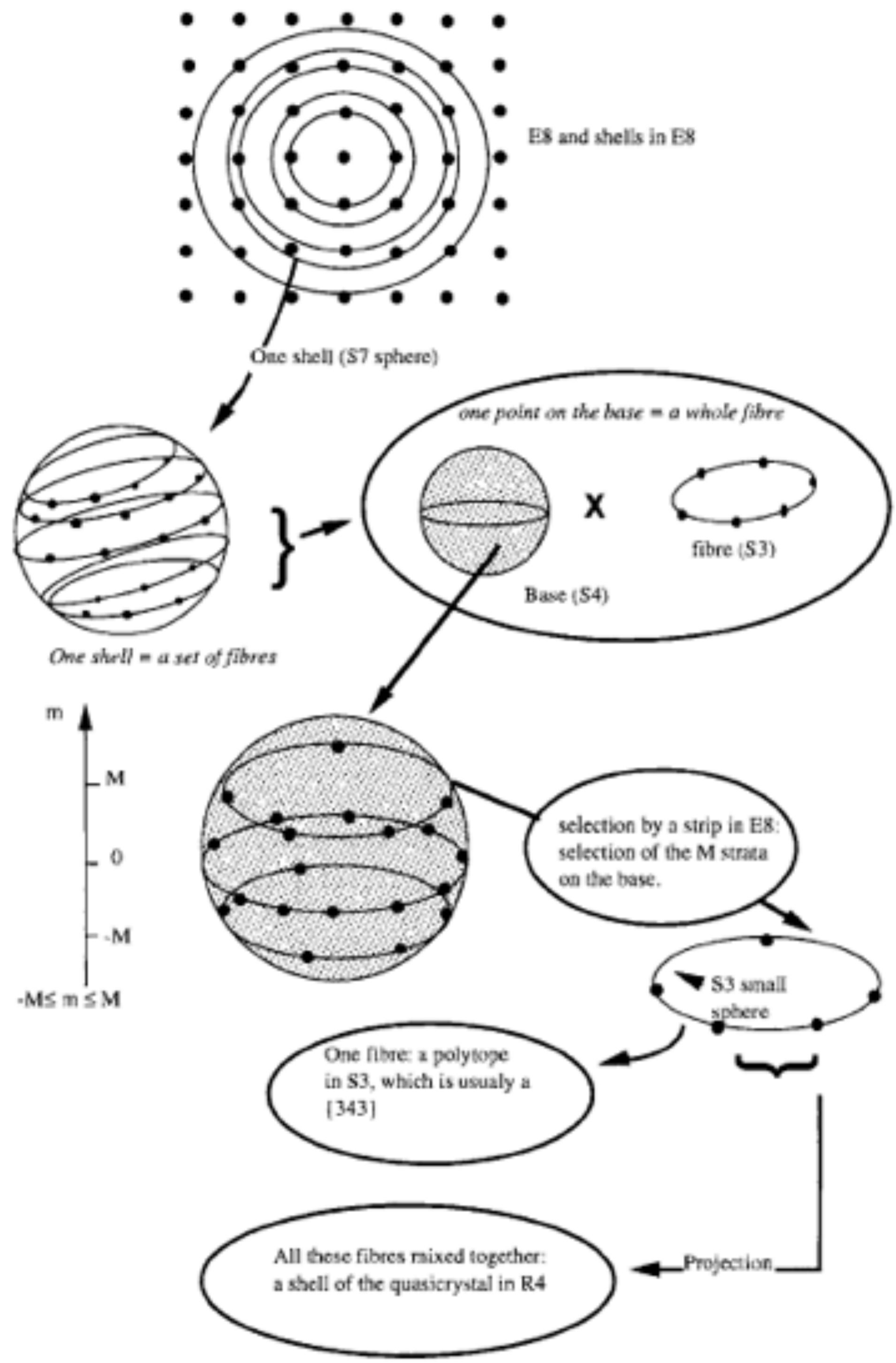


Fig. A9.1. Scheme summarizing the four-dimensional construction method: take an $E8$ shell, considered as a discrete fibration of S^7, select the fibres which map (H-map) onto a stratum M of the base of the fibration, and finally orthogonally map (O-map) the selected sites onto R^4.

The relationship between QuasiCrystals and QuasiCrystal approximants is discussed by An Pang Tsai in an IOP review “Icosahedral clusters, icosahedral order and stability of quasicrystals - a view of metallurgy”:

“... we overview the stability of quasicrystals ... in relation to phason disorder ...
the phonon variable leads to long wavelength and low energy distortion of crystals, the phason variable in quasicrystals leads to a ... type of distortion ...
Let a two-dimensional lattice points sit at the corners of squares in a grid.
... a strip with a slope of an irrational number ... golden mean ... is ... a Fibonacci sequence and is exactly a one-dimensional quasicrystal ...
... [if] the slope of the strip is ... a rational number ...[it]... is a periodic sequence ... [and]... is called an approximant ...
in the approximant where the sequence changes by a flip ... This flip is called phason flip ... a flipping of tiles in two-dimensions or three-dimensions ...

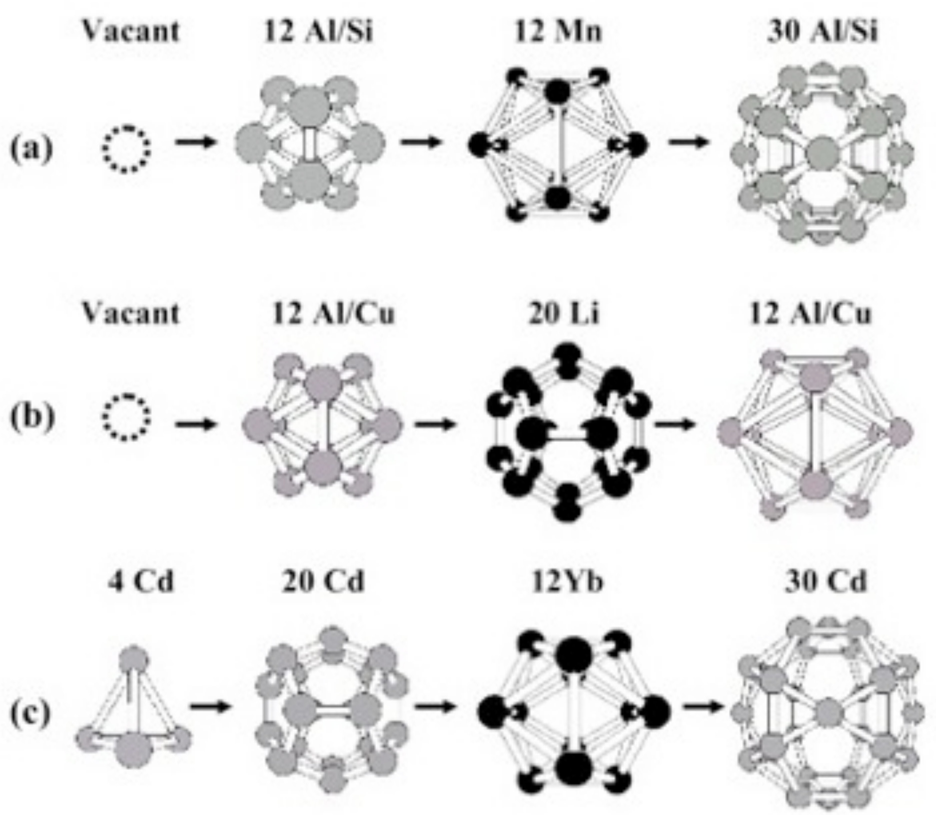


Figure 3. Concentric structures of three types of icosahedral clusters derived from three 1/1 approximants of quasicrystals. (a) The Al–Mn–Si class or Mackay icosahedral cluster: the center is vacant, the 1st shell is an Al/Si icosahedron, the 2nd shell is a Mn icosahedron, and the 3rd shell is an Al/Si icosidodecahedron. (b) The Zn–Mg–Al class or Bergman cluster: an example is R-AlLiCu: the center is vacant, the 1st shell is an Al/Cu icosahedron, the 2nd shell is a Li dodecahedron, the 3rd shell is a larger Al/Cu icosahedron. (c) The Cd–Yb class: the center is a Cd tetrahedron, the 1st shell is a Cd dodecahedron, the 2nd shell is a Yb icosahedron, and the 3rd shell is a Cd icosidodecahedron.

... ‘phason strain’ ... is the characteristic disorder for quasicrystals
but does not exist in crystals ...
a fully annealed stable iQc [icosahedral quasicrystal]...
is almost free of phason disorder ...”.