Reconciling the Dirac Quantization Condition (DQC) with the Apparent Non-Observation of $U(1)_{em}$ Magnetic Monopoles

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Abstract: The Dirac Quantization Condition (DQC) for magnetic charges and its elegant Dirac-Wu-Yang (DWY) derivation based on $U(1)_{em}$ gauge theory predicts an electric / magnetic duality which to the best of our knowledge simply has never been observed in nature, as well as a charge quantization which is observed. The fact that this predicted duality has never been observed to our knowledge means as a matter of elementary logic that this DWY derivation (and the DQC itself) is either elegant but physically wrong, or elegant and correct but physically incomplete. This paper pinpoints a flawed assumption deeply-hidden in the DWY derivation that the south gauge field patch of the posited monopole charge differs from the north patch merely by an unobservable gauge-transformation. By correcting this assumption by defining an observable difference between the north and south patches, the DQC is made fully compatible with the non-observation of magnetic charges and its correct prediction of electric charge quantization is maintained, while the incomplete DWY derivation is made complete. Some concurrences among the corrected DWY derivation and the FQHE and the electronic structure of electrons in atoms are reported without present claim, and several experiments designed to empirically arbitrate these concurrences are proposed.

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Contents

1. Introduction: Wu and Yang and the Dirac Monopole without Strings

In 1931 Dirac [1] discovered that the existence of magnetic monopoles would imply that the electric charge must be quantized. While charge quantization had been known for several decades based on the experimental work of Thompson [2] and Millikan [3], Dirac was apparently the first to lay out a possible theoretical imperative for this quantization. Using a hypothesized solenoid of singularly-thin width known as the Dirac string to shunt magnetic field lines out to mathematical infinity, Dirac established that a magnetic charge strength μ would be related to the electric charge strength e according to $e\mu = 2\pi n$, where n is an integer. This became known as the Dirac Quantization Condition (DQC). This electric charge strength is the same one which, at low probe energies, is related to the running "fine structure" coupling via $4\pi\alpha = e^2 / \hbar c \cong 1/137.036$, see, e.g., Witten's [4], pages 27 and 28. Subsequently, Wu and Yang used gauge potentials, which are locally- but not globally-exact, to obtain the exact same DQC without strings [5], [6]. Their approach is concisely summarized by Zee on pages 220-221 of [7] and will be briefly reviewed here. Throughout we shall use the natural units of $\hbar = c = 1$.

Using the differential one form $A = A_\mu dx^\mu$ for the electromagnetic gauge field a.k.a. vector potential and the differential two-form $F = \frac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu$, a hypothetical magnetic charge μ may be *defined* as the total net magnetic flux $\mu \equiv \oiint F$ passing through a closed two-dimensional surface S^2 which for convenience and symmetry we may take to be a sphere. Differential exterior calculus in spacetime geometry teaches that the exterior derivative of an exterior derivative is zero, dd=0, which means that the three-form equation $dF = ddA = 0$. Thus, via Gauss / Stokes, $\iiint 0 = \iiint dF = \oiint F = \mu$. In classical electrodynamics prior to Dirac, this was taken to mean that the magnetic charge μ=0. But a close consideration of gauge symmetry, which is locally but not globally exact, tells a different story:

When a spin ½ fermion wavefunction (which we shall generally regard as that of the electron) undergoes a local gauge (really, phase) transformation $\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)$, the gauge field one-form transforms under $U(1)_{em}$ as

$$A \to A' = A + e^{-i\Lambda} de^{i\Lambda} / ie \,. \tag{1.1}$$

More generally for larger non-abelian gauge groups with gauge potential G and charge g, this transformation is $G \to G' = U^\dagger (G + d) U / ig$ where U is a unitary matrix $U^\dagger U = 1$. If we represent F in polar coordinates (r, φ, θ) in the three-dimensional space of physical spacetime as $F = (\mu / 4\pi) d\cos\theta d\varphi$, then because $F = dA$ and dd=0, we can deduce that $A = (\mu / 4\pi)\cos\theta d\varphi$. However, $d\varphi$ is indeterminate on the north and south poles, which is an inherent feature of three-dimensional space. To remove this indeterminacy and create a smooth geometric interface, we may define north and south gauge field patches $A_N \equiv (\mu / 4\pi)(\cos\theta - 1) d\varphi$ and $A_S \equiv (\mu / 4\pi)(\cos\theta + 1) d\varphi$, respectively. But at places where these patches overlap, these gauge potentials are not the same, and specifically, their difference is $A_S - A_N = (\mu / 2\pi) d\varphi$, or written slightly differently:

$$A_N \to A'_N \equiv A_S = A_N + (\mu / 2\pi) d\varphi . \tag{1.2}$$

To unite the two patches, using (1.1) written for the north patch as $A'_N = A_N + e^{-i\Lambda} de^{i\Lambda} / ie$, we regard A_S as a gauge-transformed state $A_S \equiv A'_N$ of A_N. Combined with (1.2) this means that:

$$\frac{1}{ie} e^{-i\Lambda} de^{i\Lambda} = \frac{\mu}{2\pi} d\varphi . \tag{1.3}$$

We simply note for the moment that $A'_N = A_S$ which yields (1.3) from (1.1) for the north patch combined with (1.2) is actually a commonly-made *assumption* that the north and south gauge field patches differ from one another by no more than a gauge transformation and so are not observably distinct, in order to yield a smooth unbroken geometric relationship between the north and south patches. *Whether the physics we observe in the natural world agrees with this assumption is a separate question* which we shall deeply explore starting in the next section.

Defining a "reduced azimuth" $\cancel{\varphi} \equiv \varphi / 2\pi = 0,1,2,3...$ which represents the quantized number of rotations or "windings" over a complete 2π circumference about the z axis, this differential equation (1.3) for Λ and φ in relation to e and μ is solved by:

$$\exp(i\Lambda) = \exp\left(ie\mu \frac{\varphi}{2\pi} \right) = \exp(ie\mu\cancel{\varphi}) . \tag{1.4}$$

This can be seen simply by plugging $e^{i\Lambda}$ from (1.4) into the left hand side of (1.3) and reducing. This relates the azimuth angle φ which is one of the three physical space coordinates in spherical coordinates $x^\mu = (t, r, \varphi, \theta)$, to the local gauge (phase) angle Λ, and thereby connects rotations about the z axis through φ in physical space to rotations through Λ in the gauge space in a manner that we shall now explore in detail.

Using the simplest states $\varphi = 0$ and $\varphi = 2\pi$ a.k.a. $\cancel{\varphi} = 0$ and $\cancel{\varphi} = 1$ in (1.4), we have:

$$\exp(i\Lambda) = \exp(ie\mu\cancel{\varphi}) = \exp(ie\mu \cdot 0) = 1 = \exp(ie\mu \cdot 1) . \tag{1.5}$$

Specifically, this means that $\exp(ie\mu) = 1$. Mathematically, the general solution for an equation of this form is $\exp(i2\pi n) = 1$ for any integer $n = 0, \pm 1, \pm 2, \pm 3...$, which is infinitely degenerate but quantized. As a result, the solution to (1.5) based on the $\varphi = 0$ and $\varphi = 2\pi$ states only, is:

$$\Lambda = e\mu = 2\pi n . \tag{1.6}$$

Defining a reduced gauge angle $\not{\Lambda} \equiv \Lambda / 2\pi$ this solution may be recast as

$$\not{\Lambda} = n = e\mu / 2\pi \,, \tag{1.7}$$

where $\not{\Lambda} = n$ is the number of gauge "windings," and is a *topological quantum number* naturally arising from $\mathrm{U(1)_{em}}$ gauge theory.*

This $\varphi = 2\pi$ a.k.a. $\not{\varphi} = 1$ result, which solves $\exp\left(ie\mu\not{\varphi}\right) = 1$ in (1.4), (1.5) for $\not{\varphi} = 1$, is of course the Dirac Quantization Condition (DQC). It will be immediately apparent that this equation has an electric / magnetic duality symmetry under $e \leftrightarrow \mu$ interchange. Further, (1.6) with simple rearrangement tells us that the electric charge is quantized according to:

$$e = n\frac{2\pi}{\mu} = ne_{\mathrm{u}} = \not{\Lambda} e_{\mathrm{u}} \,, \tag{1.8}$$

where the "unit" (u) of electric charge $e_{\mathrm{u}} \equiv 2\pi / \mu$ is *defined* as 2π times the inverse of the magnetic charge. As already noted from [4], this $\not{\varphi} = 1$ charge solution to $\exp\left(ie\mu\not{\varphi}\right) = 1$ is the precise same running electric charge strength which appears in $4\pi\alpha = e^2 / \hbar c$ and so is the running electric charge strength of the electron. So when $\not{\Lambda} = 1$ in addition to $\not{\varphi} = 1$, (1.8) becomes $e = e_{\mathrm{u}}$ which describes the unit charge strength of a single electron. Consequently, we may think of this unit DWY electron as the $\not{\Lambda} = \not{\varphi} = 1$ topological solution to (1.4), and $e = ne_{\mathrm{u}}$ generally as the $\not{\varphi} = 1$ solution for all $\not{\Lambda} = n$. In turn, (1.4) is the general solution to (1.3), which in turn assumes that $A_S \equiv A'_N$ differ by nothing more than a $\mathrm{U(1)_{em}}$ gauge transformation.

Finally we may go back to the original definition $\mu \equiv \oiint F$ and isolate μ in (1.6), thus:

$$\oiint F = \mu = n\frac{2\pi}{e} = n\mu_{\mathrm{u}} = \not{\Lambda}\mu_{\mathrm{u}} \,, \tag{1.9}$$

where we also define a $\not{\Lambda} = n = 1$ unit of magnetic charge $\mu_{\mathrm{u}} \equiv 2\pi / e$, similarly quantized. By appropriate local gauge transformation, and specifically by choosing *n*=0 which is the same as choosing the phase angle $\Lambda = 0$, this nonzero surface integral can be made to vanish, $\oiint F = 0$. But this does not invalidate (1.8) and (1.9) nor does it prevent us from seeking to draw physical conclusions from these. It simply means that $\not{\Lambda} = n = 0$ with no monopoles and no electric charges is one of an infinite number of quantized solutions to (1.3).

* It should be noted that when we implicitly used the local angles $\varphi(x) = \varphi_0 = 0$ and $\varphi(x) = \varphi_0 + 2\pi$ in (1.4), the choice of $\varphi_0 = 0$ had no special physical significance. We could have used any other $0 < \varphi_0 < 2\pi$ or indeed any φ_0 whatsoever and still ended up with the exact same DQC in (1.6); $\varphi_0 = 0$ was merely the easiest mathematical choice. This means the DQC (1.6) is invariant under local gauge symmetry, as it must be to have possible physical meaning.

The customary interpretation of $\Lambda = e\mu = 2\pi n$ in (1.6) and $e = n(2\pi/\mu)$ in (1.8), ever since Dirac first found this relationship, is the conditional *logical* statement that *if* this magnetic charge "exists," *then* there is a duality symmetry between electricity and magnetism, and electric charge is quantized in units of $e_{\mathrm{u}} \equiv 2\pi/\mu$. We also see that the quantum number of electric charges $\cancel{\Lambda} = n$ is a *topological quantum number* naturally arising from the $\mathrm{U(1)_{em}}$ gauge theory corresponding to the number of 2π circumferential windings $\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)$ of the electron wavefunction through the complex gauge space defined by $e^{i\Lambda} = \cos\Lambda + i\sin\Lambda = a + bi$. While an *absolute* phase angle is not observable, $\cancel{\Lambda} = n$ is observable because it represents a topologically-quantized *difference* between (reduced) electron phase angles which all have the same orientation (but not entanglement) in the gauge space.

This is how Wu and Yang obtain Dirac monopoles and the DQC without strings.

It is extremely elegant theoretically that the Dirac monopole [1] and its associated charge quantization and electric / magnetic duality can be derived entirely from $\mathrm{U(1)_{em}}$ gauge theory as taught by Wo and Yang [5], [6] as reviewed above. It is also very theoretically attractive that the charge quantum number $n = \cancel{\Lambda}$ has a *topological* meaning as a gauge space winding number, and that the unit electron charge $e = e_{\mathrm{u}}$ may be represented as the $\cancel{\Lambda} = \cancel{\varphi} = 1$ topological winding state of a DWY monopole. And it is well-established that electric charge is indeed quantized, albeit on the basis of the charge generators $Q = Y/2 + I^3$ which emerge in Yang-Mills gauge theory following the electroweak symmetry breaking of $SU(2)_W \times U(1)_Y$ down to $U(1)_{em}$ and not on the basis of DWY monopoles.

There is only one problem however, and that problem is empirical: a century and a half of experimental study since the time of James Clerk Maxwell informs us that these magnetic charges do *not* exist in nature, or, that if they do, they exist only under some very specialized set of physical conditions which have yet to be understood. By *contrapositive logic*, *if* there is *not* a duality symmetry between electricity and magnetism, *then* $\Lambda = e\mu = 2\pi n$ in (1.6) and the consequent $e = n(2\pi/\mu)$ in (1.8) are *not* true, and consequently there are no DWY monopoles. More precisely: *for all natural circumstances* under which there is no observed electric / magnetic duality, there are also no observed DWY monopoles, and to the best of present knowledge, there are no natural circumstances under which electric / magnetic duality is observed. So to the best of our present knowledge and even though they represent a deep theoretical elegance, DWY monopoles do not exist in the natural world.

But the very fact that magnetic monopoles and electric-magnetic duality are not generally physically observed in nature tells us that there *must* be one or more deeply-hidden *physical* omissions or unrecognized assumptions in this DWY derivation. This DWY derivation (and the DQC itself) is either elegant but physically wrong, or elegant and correct but physically incomplete. So we need to carefully diagnose this DWY derivation to pinpoint what is being routinely overlooked. To do this, we now examine the contrapositive logic of the DWY monopoles more closely.

2. The Magnetic Monopole Residue: How to Make the Dirac-Wu-Yang (DWY) Analysis and the Dirac Quantization Condition (DQC) Logically Consistent with the Empirical non-Observation of Magnetic Monopoles

In the last section we made a linguistic logical statement $A \rightarrow B$ (A implies B) and its contrapositive logical statement $\sim B \rightarrow \sim A$ (not-B implies not-A) about the existence of DWY monopoles which led us to conclude that because magnetic monopoles are not generally observed, there must be some deep omission or unrecognized assumption in the DWY derivation precisely because that derivation leads a result – electric/ magnetic duality – which is empirically proven to be unobserved in general. Now let us formalize this logic so we know where to look for whatever is being overlooked in the DWY derivation.

We start with equation (1.3), $e^{-i\Lambda} de^{i\Lambda} / ie = (\mu / 2\pi) d\varphi$. *If* equation (1.3) is true, *then* its solution (1.4) is true, and thus the solution $\Lambda = e\mu = 2\pi n$ in (1.6) – namely the DQC based on the simplest like-oriented states $\varphi = 0$ and $\varphi = 2\pi$ – is also true. (We are at present continuing to neglect all other φ which differ from these by integer multiples of 2π; we shall consider these in the next section.) Putting this into a formal logical statement using (1.3) and (1.6) we may write $\left[e^{-i\Lambda} de^{i\Lambda} / ie = (\mu / 2\pi) d\varphi \right] \rightarrow \left[\Lambda = e\mu = 2\pi n \right]$ (from the azimuth winding states $\varphi = 0$ and $\varphi = 2\pi$). But in the physical world, we do not observe $\Lambda = e\mu = 2\pi n$, because this expression is invariant under $e \leftrightarrow \mu$ electric / magnetic interchange, and we do not observe electric / magnetic duality. Rather, what we generally observe is $\sim \left[\Lambda = e\mu = 2\pi n \right]$. So the formal contrapositive logic statement must be $\sim \left[\Lambda = e\mu = 2\pi n \right] \rightarrow \sim \left[e^{-i\Lambda} de^{i\Lambda} / ie = (\mu / 2\pi) d\varphi \right]$. We *must* therefore conclude that because we do not generally observe electric / magnetic duality, $e^{-i\Lambda} de^{i\Lambda} / ie = (\mu / 2\pi) d\varphi$ in (1.3) is *not generally true*. So given that (1.3) is *disproven by empirical observations* showing no duality at least in general, whatever is routinely being overlooked in the DWY monopole derivation is already being overlooked before we even get to (1.3). Thus, we need to scour everything that gets us to (1.3) to find out what is being missed.

We know that $F = dA$ is a generally true relationship, because its consequences are observed throughout electrodynamics. We know that *dd*=0 is a mathematical identity of differential forms geometry which states that the exterior derivative of an exterior derivative is zero, and is also true in general. We know that $A \rightarrow A' = A + e^{-i\Lambda} de^{i\Lambda} / ie$ in (1.1) is a correct and generally-true statement of how a $U(1)_{em}$ gauge field transforms, and we know that this gauge symmetry is manifest throughout electrodynamics and that its non-abelian extensions $G \rightarrow G' = U^\dagger (G + d) U / ig$ appear throughout nature generally such as in the weak and strong interactions. We know that *if* a magnetic charge μ exists, *then* it will be defined by $\mu \equiv \oiint F$. We know that using $F = (\mu / 4\pi) d\cos\theta d\varphi$ in this surface integral properly reproduces $\mu = \oiint F$ mathematically, because $\oiint (\mu / 4\pi) d\cos\theta d\varphi = (\mu / 4\pi) \int_0^\pi d\cos\theta \int_0^{2\pi} d\varphi$ evaluates upon definite integration to $(\mu / 4\pi) \cos\theta \big|_0^\pi \varphi \big|_0^{2\pi} = \mu$. Further, because $F = dA$ and *dd*=0 we know that $F = dA = (\mu/4\pi)\, d\cos\theta d\varphi = (\mu/4\pi)\, d(\cos\theta - K)\, d\varphi$ will be correctly reproduced for any constant K in $A = (\mu/4\pi)(\cos\theta - K)\, d\varphi$. Of course, these relationships containing μ presuppose a magnetic charge μ. But the existence of a hypothesized magnetic charge is the *hypothetical proposition being tested for its implications*, not an oversight in logic. Finally, because general coordinate invariance allows us any choice of coordinates, we can choose $A_N = (\mu/4\pi)(\cos\theta - 1)\, d\varphi$ and $A_S = (\mu/4\pi)(\cos\theta + 1)\, d\varphi$ (with $K = \pm 1$ respectively) to avoid any indeterminacy at the north and south poles. And we know that none of the foregoing is limited to any special physical circumstances. Therefore, we find that $A_S - A_N = (\mu/2\pi)\, d\varphi$ obtained precedent to (1.2) is a proper and perfectly general relationship between these two gauge field patches of the $U(1)_{em}$ magnetic monopole were it to exist, in a generally valid and fully determinate system of coordinates. So with all of these ingredients being correct and generally true, what are we missing?

Starting with $A_S - A_N = (\mu/2\pi)\, d\varphi$, we can easily rewrite this as $A_S = A_N + (\mu/2\pi)\, d\varphi$ as in (1.2), and we are still on *terra firma*. But now, when we take the next step and regard $A_S \equiv A'_N$ as simply a gauge-transformed state A'_N of A_N, and proceed to write $A_S = A'_N = A_N + (\mu/2\pi)\, d\varphi$ as in (1.2), the problem begins. For as soon as we write $A_S = A_N + (\mu/2\pi)\, d\varphi$ in the form of the gauge transformation $A'_N = A_N + (\mu/2\pi)\, d\varphi$, then the combination with the generally-true gauge transformation $A \to A' = A + e^{-i\Lambda} de^{i\Lambda}/ie$ in (1.1) leads us to (1.3), and (1.3) in turn inexorably leads us to the electric / magnetic duality of (1.6) that we do not generally observe. So what is wrong here?

When we regard A_S as a gauge-transformed A_N, i.e., when we assume that $A'_N = A_S$, *at least in general*, we are assuming that the north and south gauge field patches differ from one another by nothing more than a $U(1)_{em}$ gauge transformation. Because a gauge transformation is not observable, this assumption that $A'_N = A_S$ is an assumption that the north and south gauge field patches about a magnetic monopole – were one to exist – would not be observably distinct. This is understandable in terms of wishing for there to be a smooth transformation between the two hemispheres, but that does not mean that nature will necessarily cooperate with us to make our wishes so. In fact, the DWY derivation tells us that if nature were to cooperate such that $A'_N = A_S$ so that there were no observable distinctness between the hemispheres, then nature would also cooperate such that $\Lambda = e\mu = 2\pi n$ (for the states $\varphi = 0$ and $\varphi = 2\pi$), and we would therefore observe electric / magnetic duality. That is, refining the logic, this would mean that $[A'_N = A_S] \to [\Lambda = e\mu = 2\pi n]$. But we do *not* observe $\Lambda = e\mu = 2\pi n$, at least in general. Rather, $\sim[\Lambda = e\mu = 2\pi n]$ is the correct logical statement of what is empirically observed in general. Therefore, the correct contrapositive logic statement is $\sim[\Lambda = e\mu = 2\pi n] \to \sim[A'_N = A_S]$, which means that $A'_N = A_S$, in general, is *disproven by nature*, and particularly, by the very-well-established generalized absence of $U(1)_{em}$ magnetic monopoles.

So if $A'_N = A_S$ as a general proposition is disproven by the non-observation of duality, this means that in general, A'_N must differ from A_S by *something more than an ordinary gauge transformation, such that there is a* *physically-observable* *distinctness between the north and south patches*. That is, there must be some *physically-observable* difference ε formally defined by $A'_N \equiv A_S + \varepsilon$ between the south patch and the gauge-transformed north patch. Because each of these gauge patches $A'_N = A'_{N\mu} dx^\mu$ and $A_S = A_{S\mu} dx^\mu$ is a differential one-form, this difference ε must also be a differential one-form $\varepsilon \equiv \varepsilon_\mu dx^\mu$. Because the gauge potential four-vectors $A'_{N\mu}$ and $A_{S\mu}$ are energy/momentum-dimensioned spacetime four-vectors, so too ε_μ must be an energy/momentum-dimensioned four-vector.

But in contrast to the unobservable phase gradient $\partial_\mu \Lambda$ contained in $e^{-i\Lambda} de^{i\Lambda} / ie = d\Lambda / e = \partial_\mu \Lambda dx^\mu / e$ in (1.1), this ε_μ *must be observable*. Why? Because if ε_μ was not observable, then it could always be gauged away so that $A'_N = A_S + \varepsilon$ could be turned back into $A'_N = A_S$ which would once again imply the existence of an electric / magnetic duality that is not observed in nature in general. *It is the general non-observation of U(1)$_{em}$ magnetic monopoles which requires ε_μ to be observable.* Consequently, this sets us on the path of needing to study all that we can about ε_μ and its related differential one-form $\varepsilon = \varepsilon_\mu dx^\mu$, because it is only via this observable ε_μ that we can understand why U(1)$_{\mathrm{em}}$ DWY magnetic monopoles and electric magnetic duality – as theoretically elegant as they are – are not generally observed. So let us commence this study.

First, let us rewrite the ε definition $A'_N \equiv A_S + \varepsilon$ as $A_S = A'_N - \varepsilon$ and combine this with the generally-valid expression $A_S = A_N + (\mu / 2\pi) d\varphi$ in (1.2) to obtain:

$$A_S = A'_N - \varepsilon = A_N + (\mu / 2\pi) d\varphi . \tag{2.1}$$

This now replaces (1.2) and is synonymous with (1.2) when $\varepsilon = 0$. Then, we write the generally-valid U(1)$_{\mathrm{em}}$ gauge transformation (1.1) for the north patch as $A'_N = A_N + e^{-i\Lambda} de^{i\Lambda} / ie$, and combine this with (2.1) to obtain $A_S = A_N + e^{-i\Lambda} de^{i\Lambda} / ie - \varepsilon = A_N + (\mu / 2\pi) d\varphi$. After subtracting A_N throughout this becomes $A_S - A_N = e^{-i\Lambda} de^{i\Lambda} / ie - \varepsilon = (\mu / 2\pi) d\varphi$, or, in two forms that will be useful for development:

$$\frac{1}{ie} e^{-i\Lambda} de^{i\Lambda} = \frac{\mu}{2\pi} d\varphi + \varepsilon \tag{2.2}$$

and

$$\varepsilon = A_N - A_S + \frac{1}{ie} e^{-i\Lambda} de^{i\Lambda} = A_N - A_S + d\Lambda / e . \tag{2.3}$$

The above (2.2) is the generalization of $e^{-i\Lambda}de^{i\Lambda}/ie=(\mu/2\pi)d\varphi$ in (1.3) to the circumstance where there is an observable distinctness between the north and south gauge patches which cannot be gauged away by a gauge transformation. Meanwhile, we can apply the gauge transformation $\varepsilon \to \varepsilon' = \varepsilon - d\Lambda/e$ to (2.3) together with the generally-valid expression $A_N - A_S = -(\mu/2\pi)d\varphi$ based on (1.2), to obtain the result that:

$$\varepsilon' = A_N - A_S = -(\mu/2\pi)d\varphi . \tag{2.4}$$

If we then rename ε' back to ε, we find that ε can always be placed into a gauge *such that* it specifies the observable difference $\varepsilon = A_N - A_S$ between the north and south gauge field patches about the hypothesized magnetic charge $\oiint F = \mu$. In this gauge, extracting vectors from the differential forms, the covariant (lower-indexed) $\varepsilon_\mu = A_{N\mu} - A_{S\mu}$.

Because any gauge potential $A_\mu(x^\mu)$ four-vector is a function of the spacetime coordinates x^μ and so is a field in spacetime, this means that $\varepsilon_\mu(x^\mu) = A_{N\mu}(x^\mu) - A_{S\mu}(x^\mu)$ is likewise a four-vector field in spacetime. But we know that a gauge potential $A_\mu(x^\mu)$ by itself is not observable. What is observable is a *difference* between two potentials. Specifically, the time component of $A^\mu \equiv (\phi, \mathbf{A})$ represents a scalar potential ϕ, and the difference $V = \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)$ in this scalar potential as between *two different points in space* $\mathbf{x}_1, \mathbf{x}_2$ at a given time *t* in the observer's frame of reference represents an observable *voltage drop*. Often, one of these points is *arbitrarily chosen* as an electrical ground, for example, $\phi(\mathbf{x}_2) \equiv 0$. So because a gauge potential $A_\mu(x^\mu)$ is not observable, what must make $\varepsilon_\mu = A_{N\mu} - A_{S\mu}$ observable is the fact that it represents a *difference of potential* between the north and south gauge field patches of the hypothesized magnetic charge μ. Specifically, stated in the gauge (2.4), the time component $V \equiv \varepsilon^0 = \phi_N - \phi_S$ of $\varepsilon^\mu \equiv (V, \boldsymbol{\varepsilon})$ would have to be the *observable energy* of a *voltage drop between the north potential and the south potential of a magnetic monopole, were such a monopole to exist*.

Now, what makes this potential energy $V = \phi_N - \phi_S$ unusual aside from the fact that this would only be observed *if* one had a magnetic monopole which as far as is known has never been observed in the material world, is the fact that because $A_{N\mu}(x^\mu)$ and $A_{S\mu}(x^\mu)$ and $\varepsilon_\mu(x^\mu) = A_{N\mu}(x^\mu) - A_{S\mu}(x^\mu)$ are all *fields*, this difference in potential, dimensioned as an observable energy, is defined for the monopole *at each and every point in spacetime.* That is, $V(\mathbf{x}_1) = \phi_N(\mathbf{x}_1) - \phi_S(\mathbf{x}_1)$ is an observably-defined energy at any single selected spatial coordinate $\mathbf{x}_1$ at a given time *t* for an observer, and likewise at any and all other space coordinates at the same time *t* for the observer, without having to take a voltage difference between two *separate* points in space. Whereas the potential energy $A_\mu(x^\mu)$ is not absolutely defined at each spacetime event but is only defined relative to an arbitrary ground at a separate location, the potential energy $\varepsilon_\mu(x^\mu)$ does have an absolute, non-arbitrary, energy-dimensioned definition at each and every event in spacetime *without reference to any other event*. However, this naturally-defined $\varepsilon_\mu(x^\mu)$ potential uniquely arises as one of the gauge-theory consequences of positing a magnetic monopole which so far as is known has never been observed. Now we return to (2.2).

To solve (2.2), let us posit a zero-form dimensionless scalar field $\tau(x^\mu)$ related in some to-be-determined way to the one-form $\varepsilon(x^\mu)$. In a spherical coordinate system $x^\mu = (t, r, \varphi, \theta)$, the azimuth φ is one of the coordinates of which this is a function. Using this τ in a posited test expression we write:

$$\exp(i\Lambda) = \exp\left(ie\mu\frac{\varphi}{2\pi} + ie\tau\right) = \exp\left(ie\mu\frac{\varphi}{2\pi}\right)\exp(ie\tau) = \exp(ie\mu\cancel{\varphi})\exp(ie\tau). \tag{2.5}$$

If we insert this in (2.2) and reduce, we find that this does indeed solve (2.2), *if and only if*

$$\varepsilon = d\tau = \varepsilon_\mu dx^\mu = \partial_\mu \tau dx^\mu, \tag{2.6}$$

that is, extracting the vectors, *iff*

$$\varepsilon_\mu = \partial_\mu \tau. \tag{2.7}$$

It is clear from (2.7) that the vector $\varepsilon_\mu(x^\mu)$ is the spacetime gradient of $\tau(x^\mu)$. So we will also wish to study this dimensionless scalar τ along with the energy-dimensioned vector ε_μ in the one-form $\varepsilon = \varepsilon_\mu dx^\mu$.

Because *dd*=0 when applied to any differential form, one of the immediate things we know via (2.6) is that:

$$d\varepsilon = dd\tau = 0. \tag{2.8}$$

Therefore, if we apply applies the Gauss / Stokes theorem $\int_M dH = \oint_{\partial M} H$ where H is a generalized *p*-form and ∂M is the closed exterior boundary of a *p*+1-dimensional manifold, the integral form of the above is:

$$\iint d\varepsilon = \iint dd\tau = \oint \varepsilon = \oint d\tau = 0 \left(= \iint 0\right). \tag{2.9}$$

Next, as we did at (1.5), let us examine (2.5) using the simplest states $\varphi = 0$ and $\varphi = 2\pi$ a.k.a. $\cancel{\varphi} = 0$ and $\cancel{\varphi} = 1$, still ignoring until the next section, the other states differing from these by integer multiples of 2π. This azimuth $\varphi \subset t, r, \varphi, \theta$ is one of the four spherical coordinates i.e., a subset of these coordinates, which means that $\tau(x^\mu)$ is some to-be-determined function of the reduced $\cancel{\varphi} = \varphi / 2\pi$ plus the three other coordinates. Because much of our interest will be focused on how $\tau(x^\mu)$ behaves as a function of $\cancel{\varphi}$ given that $\cancel{\varphi} = 0,1,2,3,4...$ is a topological azimuth quantum winding number in physical space, let us generally suppress showing the other three x^μ, and simply highlight the fact that that $\tau(x^\mu) = \tau(t, r, \cancel{\varphi}, \theta) \supset \tau(\cancel{\varphi}) \equiv \tau_{\cancel{\varphi}}$. Then, in contrast to (1.5), with this $\tau \to \tau_{\cancel{\varphi}}$ now defined, (2.5) yields:

$$\exp(i\Lambda) = \exp(ie\mu\cancel{\varphi})\exp(ie\tau_{\cancel{\varphi}}) = \exp(ie\mu \cdot 0)\exp(ie\tau_0) = 1 \cdot \exp(ie\tau_0) = \exp(ie\mu \cdot 1)\exp(ie\tau_1) . \tag{2.10}$$

Now we multiply through by $\exp(-ie\tau_0)$ and also use the general expression $1 = \exp(i2\pi n)$. This leads to:

$$\exp(i\Lambda - ie\tau_0) = \exp(ie\mu + ie\tau_1 - ie\tau_0) = 1 = \exp(i2\pi n) . \tag{2.11}$$

This is then solved by all states for which:

$$\Lambda - e\tau_0 = e[\mu + \tau_1 - \tau_0] = 2\pi n . \tag{2.12}$$

We see from $\Lambda - e\tau_0 = 2\pi n$ that the gauge angles Λ which solve the above are still separated from one another by multiples of $2\pi n$, but now we have an absolute offset phase $e\tau_0$. As noted in the last section, an absolute phase angle is not observable; only phase *differences* are observable. Therefore, we may gauge this offset to $e\tau_0 \equiv 0$ without changing the observed physics in any way, in effect establishing $e\tau_0 = e\tau(\varphi = 0) \equiv 0$ as a "ground." Doing so, (2.12) simplifies to:

$$\Lambda = e[\mu + \tau_1] = 2\pi n . \tag{2.13}$$

The above, which is again based on the $\varphi = 0$ and $\varphi = 2\pi$ states only, should be contrasted to the usual DQC of (1.6) and (1.7) to which it reduces when $\tau_1 = 0$. The reduced gauge angle $\cancel{\Lambda} = \Lambda / 2\pi = n$ is a topological quantum number as before, but what is new is $e\tau_1$. Let us see what now changes.

Similarly to (1.8), we may write (2.13) above in terms of the electric charge strength, as:

$$e = n\frac{2\pi}{\mu + \tau_1} = ne_{\mathrm{u}} \tag{2.14}$$

where the unit of electric charge, previously $e_{\mathrm{u}} \equiv 2\pi / \mu$, is now defined as:

$$e_{\mathrm{u}} \equiv \frac{2\pi}{\mu+\tau_1} . \tag{2.15}$$

Likewise, if we isolate μ in (2.13), then as in (1.9) we may write:

$$\oiint F = \mu = n\frac{2\pi}{e} - \tau_1 = n\mu_{\mathrm{u}} - \tau_1 , \tag{2.16}$$

which continues to use the same unit $\mu_{\mathrm{u}} \equiv 2\pi / e$ of magnetic charge as before. Finally, the scalar potential $\tau_1 = \tau(\varphi = 1)$ in (2.13), which did not appear at all in (1.6), is isolated as such:

$$\tau_1 = n\frac{2\pi}{e} - \mu = n\mu_{\mathrm{u}} - \mu = \frac{\Lambda}{e} - \mu . \tag{2.17}$$

Let us now consider two reductions of (2.13). First, in the circumstance where $\tau_1 = 0$, this reduces to $\Lambda = e\mu = 2\pi n$, which is synonymous with the usual DQC (1.6). All of (2.14) through (2.17) then reduce to the section 1 results of the usual DWY formulation. This is to be expected, because the only new objects we have introduced are ε and τ related by $\varepsilon = d\tau$.

Second, because our main purpose is to reconcile the prediction of DWY monopoles with $\Lambda = e\mu = 2\pi n$ as in (1.6) with the apparent absence of these monopoles in nature, let us now do exactly that. For any natural circumstances under which there are no magnetic charges – and to the best of our knowledge this describes all observed natural circumstances – the net magnetic flux $\oiint F = \mu = 0$. So we can examine this widely-observed circumstance by setting $\mu = 0$ in either (2.13) or (2.16), with the result that:

$$e\tau_1 = 2\pi n = \Lambda . \tag{2.18}$$

Now, in contrast to (1.8), with $\mu = 0$, (2.14) for the electric charge strength reduces to:

$$e = n\frac{2\pi}{\tau_1} = ne_{\mathrm{u}} = \cancel{\Lambda}e_{\mathrm{u}} , \tag{2.19}$$

while the unit electric charge (2.15) reduces to:

$$e_{\mathrm{u}} = \frac{2\pi}{\tau_1} . \tag{2.20}$$

Then (2.17) becomes:

$$\tau_1 = n\frac{2\pi}{e} = n\mu_u = \frac{\Lambda}{e} . \tag{2.21}$$

Now – even with $\mu = 0$ – the electric charge in (2.19) is quantized – as is observed – but this quantization no longer depends upon the existence of DWY monopoles which are not generally observed. That is, even when the DWY monopoles are set to $\mu = 0$ in accordance with what is observed, Dirac's original prediction of electric charge quantization remains intact. But the electric / magnetic duality is broken – as is also observed – and we see from contrasting (2.19) with (1.8) that the monopole charge μ is replaced by $\tau_1 = \tau(\not{\varphi} = 1)$, that is, $\mu \Rightarrow \tau_1$. Because τ_1 replaces the magnetic charge μ when that charge is zeroed out i.e., because $\mu \Rightarrow \tau$, and because τ acts as a dimensionless scalar potential τ in equations (2.8) and (2.9), we shall refer to τ as the "magnetic monopole residue potential." The introduction of this residue potential and its one-form exact differential $\varepsilon = d\tau = A'_N - A_S$ is what logically reconciles the DWY analysis with the general non-observation of magnetic monopoles in nature. Moreover, this retains the Dirac prediction of electric charge quantization even in the absence of monopoles as is generally observed, now with a unit electric charge $e_{\mathrm{u}} = 2\pi / \tau_1$ in lieu of $e_{\mathrm{u}} = 2\pi / \mu$.

3. Charge Fractionalization in the Extended DWY Analysis

In the first two sections, we developed (1.4) and (2.5) using only the simplest like-oriented states $\varphi = 0$ and $\varphi = 2\pi$ a.k.a. $\not{\varphi} = 0$ and $\not{\varphi} = 1$, and ignored all other states in the complete topologically-quantized set $\varphi = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi ...$ a.k.a. $\not{\varphi} = 0, \pm 1, \pm 2, \pm 3, \pm 4...$ of azimuth windings. Let us now remove this restriction and consider all these other states.

First, mindful that the scalar potential $\tau = \tau(x^\mu) \supset \tau(\not{\varphi}) \equiv \tau_{\not{\varphi}}$ is some to-be-determined function of the reduced azimuth $\not{\varphi} = 0, \pm 1, \pm 2, \pm 3, \pm 4...$, we rewrite (2.5) as shown in (2.11):

$$\exp(i\Lambda) = \exp(ie\mu\not{\varphi})\exp(ie\tau_{\not{\varphi}}). \tag{3.1}$$

Then, we expand the above for the first several states $\not{\varphi} = m = 0,1,2,3,4,5...$, where $m \equiv \not{\varphi}$ is a topological quantum number specifying the number of *azimuth* windings, which is a number different from the topological quantum number $n = \not{\Lambda}$ specifying the number of *gauge-space* windings. Now, with the top line effectively the same as (2.10), (3.1) expands to:

$$\begin{aligned} &\exp(i\Lambda) = \exp(ie\mu\not{\varphi})\exp(ie\tau_{\not{\varphi}}) = \exp(ie\mu \cdot 0)\exp(ie\tau_0) = \exp(ie\mu \cdot 1)\exp(ie\tau_1) \\ &= \exp(ie\mu \cdot 2)\exp(ie\tau_2) = \exp(ie\mu \cdot 3)\exp(ie\tau_3) = \exp(ie4 \cdot \not{\varphi})\exp(ie\tau_4) = \exp(ie\mu \cdot 5)\exp(ie\tau_5)... \end{aligned} . \tag{3.2}$$

Multiplying through by $\exp(-ie\tau_0)$, and with $\exp(ie\mu \cdot 0) = 1 = \exp(i2\pi n)$, this becomes:

$$\begin{aligned} &\exp(i\Lambda - ie\tau_0) = \exp(ie\mu + ie\tau_1 - ie\tau_0) = \exp(2ie\mu + ie\tau_2 - ie\tau_0) = \exp(3ie\mu + ie\tau_3 - ie\tau_0) \\ &= \exp(4ie\mu + ie\tau_4 - ie\tau_0) = \exp(5ie\mu + ie\tau_5 - ie\tau_0)... = 1 = \exp(i2\pi n) \end{aligned} . \tag{3.3}$$

Then, consolidating using $\not{\varphi} = m = 0,1,2,3,4,5...$, we obtain a general expression for all m:

$$\exp\left(i\Lambda - ie\tau_0\right) = \exp\left(mie\mu + ie\tau_m - ie\tau_0\right) = \exp\left(i2\pi n\right). \tag{3.4}$$

Similarly to (2.12), this is solved by all states for which:

$$\Lambda - e\tau_0 = e\left[m\mu + \tau_m - \tau_0\right] = 2\pi n. \tag{3.5}$$

As before, see (2.12) and (2.13), $\Lambda - e\tau_0 = 2\pi n$ contains an unobservable phase angle offset $e\tau_0$ which may be gauged away by setting $e\tau_0 \equiv 0$ as a ground state phase without impacting observable physics. Doing so, (3.5) then simplifies to:

$$\Lambda = e\left[m\mu + \tau_m\right] = 2\pi n. \tag{3.6}$$

So the earlier (2.12) and (2.13) are simply the $m=1$ reductions of (3.5) and (3.6) above.

As before, contrast (1.8) and (2.14), we write this in terms of the electric charge strength:

$$e = n\frac{2\pi}{m\mu + \tau_m} = ne_{\mathrm{u}} \tag{3.7}$$

which continues to be quantized. Here, however, contrast $e_{\mathrm{u}} \equiv 2\pi/\mu$ and (2.15), the unit electric charge is now:

$$e_{\mathrm{u}} \equiv \frac{2\pi}{m\mu + \tau_m}. \tag{3.8}$$

Likewise, if we isolate μ in (3.6), then as in (1.9) and (2.16), for the monopole we may write:

$$\oint\!\!\!\oint F = \mu = \frac{n}{m}\frac{2\pi}{e} - \frac{\tau_m}{m} = \frac{n}{m}\mu_{\mathrm{u}} - \frac{\tau_m}{m} = \frac{\cancel{\Lambda}}{\cancel{\varphi}}\frac{2\pi}{e} - \frac{\tau_{\cancel{\varphi}}}{\cancel{\varphi}} = \frac{\cancel{\Lambda}}{\cancel{\varphi}}\mu_{\mathrm{u}} - \frac{\tau_{\cancel{\varphi}}}{\cancel{\varphi}}, \tag{3.9}$$

in which the unit magnetic charge $\mu_{\mathrm{u}} \equiv 2\pi/e$ is unaltered. Finally, in contrast to (2.17), we may isolate the monopole residue scalar potential in (3.6) by writing:

$$\tau_m = \tau_{\cancel{\varphi}} = n\frac{2\pi}{e} - m\mu = nu_{\mathrm{u}} - m\mu = \cancel{\Lambda}\frac{2\pi}{e} - \cancel{\varphi}\mu = \cancel{\Lambda}u_{\mathrm{u}} - \cancel{\varphi}\mu. \tag{3.10}$$

Now let's consider the same two reductions which we considered in the last section. First, when we set the dimensionless scalar potential $\tau_m = 0$, (3.7) will become:

$$e=\frac{n}{m}\frac{2\pi}{\mu}=\frac{n}{m}e_{\mathrm{u}}=\frac{\cancel{A}}{\cancel{\varphi}}\frac{2\pi}{\mu}=\frac{\cancel{A}}{\cancel{\varphi}}e_{\mathrm{u}}\,, \tag{3.11}$$

and the unit electric charge will reduce to the customary $e_{\mathrm{u}}=2\pi/\mu$. Also, (3.9) becomes:

$$\oiint F=\mu=\frac{n}{m}\frac{2\pi}{e}=\frac{n}{m}\mu_{\mathrm{u}}=\frac{\cancel{A}}{\cancel{\varphi}}\frac{2\pi}{e}=\frac{\cancel{A}}{\cancel{\varphi}}\mu_{\mathrm{u}}\,. \tag{3.12}$$

Finally, as can also be seen by restructuring (3.11) and (3.12), (3.10) leads to:

$$\frac{e\mu}{2\pi}=\frac{e}{e_{\mathrm{u}}}=\frac{\mu}{u_{\mathrm{u}}}=\frac{n}{m}=\frac{\cancel{A}}{\cancel{\varphi}}\,. \tag{3.13}$$

We see from the ratio $n/m=\cancel{A}/\cancel{\phi}$ appearing in (3.11) through (3.13) that for $\tau_m=0$, the electric and magnetic charges are now *topologically quantized and fractionalized*, where $n=\cancel{A}$ is the quantization numerator denoting an integer number of topological rotations in the complex gauge space, and $m=\cancel{\varphi}$ is the fractionalization denominator denoting an integer number of topological rotations about the z-axis through the azimuth φ in the real three-dimensional physical space of spacetime.

Second, as we did starting at (2.18), let's consider what happens when the net magnetic flux $\oiint F=\mu=0$, i.e., under the widely-observed empirical conditions where there are no magnetic monopoles observed. Now, (3.7) reduces to:

$$e=n\frac{2\pi}{\tau_m}=ne_{\mathrm{u}} \tag{3.14}$$

with the unit electric charge (3.8) reducing to:

$$e_{\mathrm{u}}=\frac{2\pi}{\tau_m}\,. \tag{3.15}$$

The magnetic charge $\mu=0$ in (3.12) is zero by definition in this specialization, and from (3.10) with $\mu=0$ we obtain:

$$\tau_m=\tau_{\cancel{\varphi}}=n\frac{2\pi}{e}=nu_{\mathrm{u}}=\cancel{A}\frac{2\pi}{e}=\cancel{A}u_{\mathrm{u}}\,. \tag{3.16}$$

Comparing (3.11) with (3.14), when $\mu\neq 0$ and $\tau_m=0$ the electric charge $e=(n/m)e_{\mathrm{u}}$ is both quantized and fractionalized with $e_{\mathrm{u}}=2\pi/\mu$, while when $\tau_m\neq 0$ and $\mu=0$ the electric charge $e=ne_{\mathrm{u}}$ is quantized only, with no fractionalization, and $e_{\mathrm{u}}=2\pi/\tau_m$. Further comparing the $\tau_m = 0$ specialization to the $\mu = 0$ specialization, in the former case we find from (3.11) that $2\pi n / m = e\mu$ and in the latter we find from (3.14) that $2\pi n / m = e\tau_m / m$. This means that when going from $\mu \neq 0$, $\tau_m = 0$ to $\mu = 0$, $\tau_m \neq 0$, the topologically-quantized fraction $2\pi n / m = 2\pi\cancel{\Lambda} / \cancel{\varphi}$ goes from $e\mu$ to $e\tau_m / m = e\tau_{\cancel{\varphi}} / \cancel{\varphi}$. Thus, the magnetic charge goes from:

$$\mu \Rightarrow \tau_{\cancel{\varphi}} / \cancel{\varphi} . \tag{3.17}$$

In the last section, when limited to $\cancel{\varphi} = 0, 1$, we found the magnetic monopole residue potential $\tau(x^\mu)$ to be $\tau_1 = \tau(t, r, \cancel{\varphi} = 1, \theta)$, which again, is a dimensionless scalar. Now we see that this residue potential generalizes to $\tau_{\cancel{\varphi}} / \cancel{\varphi} = \tau(t, r, \cancel{\varphi}, \theta) / \cancel{\varphi}$ for all integer $\cancel{\varphi}$, and that the residue obtained in section 2 was merely the $\cancel{\varphi} = 1$ specialization of this residue.

It is also of interest to examine the role of the topological quantum number $n = \cancel{\Lambda}$ as between the $\tau_m = 0$ and $\mu = 0$ specializations. For $\tau_m = 0$, and with $\cancel{\varphi} = 1$, we have $e = ne_{\mathrm{u}}$ with $e_{\mathrm{u}} = 2\pi / \mu$, so that *n* specifies charge quantization as was first found by Dirac. But for $\mu = 0$, we can apply $\varepsilon = d\tau$ found in (2.6) to (3.16) along with $4\pi\alpha = e^2$, see also (2.3), to find:

$$\varepsilon_m = d\tau_m = n\left(-\frac{2\pi}{e^2} de\right) = n\left(-\frac{1}{2\alpha} de\right) = A_{Nm} - A_{Sm} + d\Lambda / e . \tag{3.18}$$

Extracting the four-vector from the differential forms, and restoring $\hbar$ and *c*, we have:

$$\varepsilon_{m\mu} = \partial_\mu \tau_m = n\left(-\frac{(\hbar c)^{1.5}}{2\alpha} \partial_\mu e\right) = A_{Nm\mu} - A_{Sm\mu} + \partial_\mu \Lambda / e . \tag{3.19}$$

The expression $-\left((\hbar c)^{1.5} / 2\alpha\right)\partial_\mu e$ has dimensions of energy. The fact that $\varepsilon_{m\mu}$ is an energy vector equal to this expression times the topological quantum number $\cancel{\Lambda} = n$ means that after we set $\mu = 0$ and so only have the monopole residue τ_m, this quantum number goes from representing quantization of charge, to representing *quantization of energy*. Specifically, going back to natural units, the vector potential which is $\varepsilon_{m\mu} = A_{Nm\mu} - A_{Sm\mu}$ in the gauge $\partial_\mu \Lambda = 0$, at a given spacetime event coordinate x^μ, has an energy four-vector which is a topologically quantized integer multiple $\cancel{\Lambda} = n$ of $-\partial_\mu e / 2\alpha$ at that same event, where $\cancel{\Lambda} = n$ continues to be a winding number through the complex gauge space $e^{i\Lambda} = \cos\Lambda + i \sin\Lambda = a + bi$.

If we finally consider orientation and entanglement as reviewed by as Misner, Thorne and Wheeler (MTW) in one of the most widely-regarded discussions of this topic in [8] at section 41.5, and if we start with the electron state $m = \cancel{\varphi} = 1$ for which $e = ne_{\mathrm{u}}$, see the discussion following (1.8), then the only states *which can be disentangled* back to the original $m = \cancel{\varphi} = 1$

electron state are those states with $m=\cancel{\varphi}=1,3,5,7...$ which are all *odd integers*. We may write this set of states as $m=\cancel{\varphi}=2l+1$ with $l=0,1,2,3...$. So, starting from (3.11) for $\tau_m=0$, if we now include only those fractionalized, quantized electron charges which can be disentangled back to an unfractionalized $\cancel{\varphi}=1$ electron with $e=ne_{\text{u}}$, then this restricted set of charges is:

$$\begin{aligned}&e=\frac{n}{2l+1}\frac{2\pi}{\mu}=\frac{n}{2l+1}e_{\text{u}}=\frac{n}{m}e_{\text{u}}=\nu e_{\text{u}};\quad \nu\equiv\frac{n}{2l+1}=\frac{n}{2\left(l+\frac{1}{2}\right)}=\frac{n}{2\left(l+s\right)}=\frac{n}{2j}\\&n=\cancel{\Lambda}=0,\pm1,\pm2,\pm3,\pm4...;\quad m=\cancel{\varphi}=2l+1=1,3,5,7...;\quad l=0,1,2,3,4...;\quad s=\tfrac{1}{2};\quad j=l+s\end{aligned}\tag{3.20}$$

Except for the even denominator $m=2$, this fill factor $\nu=n/\left(2l+1\right)$ precisely describes the set of charge states observed in the FQHE, which we simply report without claim. And by naturally eliminating $m=0$, this also avoids the solution with an infinite $e=\infty$. Note too, that it was this same $m=0$, $e=\infty$ state which we earlier gauged away, first in (2.15), then in (3.6).

We have used the quantum numbers $l=0,1,2,3,4...$ and $s=\frac{1}{2}$ and $j=l+s$ in (3.20) to be suggestive of the quantum numbers in the Casimir operations $\mathbf{L}^2\left|\xi\right\rangle=l\left(l+1\right)\left|\xi\right\rangle$, $\mathbf{S}^2\left|\xi\right\rangle=s\left(s+1\right)\left|\xi\right\rangle$ and $\mathbf{J}^2\left|\xi\right\rangle=j\left(j+1\right)\left|\xi\right\rangle$ as applied to spinor eigenstates $\left|\xi\right\rangle$ whereby the total angular momentum **J** is observable because it commutes with the Dirac Hamiltonian, $\left[\mathbf{J},H\right]=0$ and thereby *j* which sits in the (3.20) denominator is observable, without at this moment claiming a physical linkage. We simply report the fact that the use of these Casimir quantum numbers from atomic theory does correctly describe the DWY charge quantization and fractionalization that emerges when $\tau_m=0$ and when one discards the azimuths which cannot be topologically disentangled back to $\cancel{\varphi}=1$, and the fact that the fractions are all odd integers. And we also note without claim that the FQHE likewise has only odd integers with the sole exception of the even integer 2 which is not described in (3.20). To describe this $2j=2\left(l+s\right)=2$ denominator, one would need an $l=\frac{1}{2}$ state in addition to the $s=\frac{1}{2}$ state, that is, one would need a *pair* of electron states each with a half unit of orbital / spin angular momentum. *If* (3.20) was to describe an actual physical connection between (3.20) and FQHE, then because (3.20) is observed only as $\tau_m\to0$, and because the FQHE is observed only as the temperature $T\to0\text{K}$, this means that τ_m would have to approach zero as the temperature approaches absolute zero.

Finally, ever since (2.10), we have observed that $\tau_{\cancel{\varphi}}=\tau\left(\cancel{\varphi}\right)$ is a function of the topologically-quantized number of azimuth windings in three-dimensional physical space (as well as the remaining spherical coordinates t,r,θ). But we have not yet discerned exactly what that function of $\cancel{\varphi}$ might be. *If* the accurately-descriptive linkages with the Casimir quantum numbers reported in (3.20) do represent a genuine physical connection and not mere coincidence, and specifically if $m=\cancel{\varphi}=2\left(l+s\right)=2j$ is real, then because (3.19) tells us that $n=\cancel{\Lambda}$ when the magnetic charge $\mu=0$ is a topological gauge space winding number for *energy quantization*, and because *j, l* and *s* are all of the Casimir numbers used to describe orbital angular momentum states in electron shells, and because all we have not yet discussed is the principal quantum number $n > l$ which like $n = \cancel{\Lambda}$ also specifies energy quantization, we are motivated to observe that $n = \cancel{\Lambda}$ could be made to correspond to the principal quantum number, if, when the magnetic charge $\mu = 0$ so all we have is the residue $\tau_{\cancel{\varphi}} = \tau_{2j}$, we connect $n > l$ to one another using the more explicit $n = n_r + l + 1$, where n_r is the radial quantum number equal to the number of nodes in the radial wavefunction for the electron. Writing this via $l = j - s = j - \frac{1}{2}$ as $n = n_r + j + s$, we may replace *n* in (3.19) to explicitly show the required *j*-dependency, together with the *s* and n_r parameterization, namely:

$$\varepsilon_\mu(n_r, j, s) = \partial_\mu \tau_m(n_r, j, s) = -(n_r + j + s)\frac{\partial_\mu e}{2\alpha} = A_{N\mu}(n_r, j, s) - A_{S\mu}(n_r, j, s) + \partial_\mu \Lambda / e, \qquad (3.21)$$

where $\cancel{\varphi}/2 = j = l + s$ and $\cancel{\Lambda} = n = n_r + l + 1$ serve to establish a topological quantization for all of the Casimir numbers plus the principal quantum numbers needed to completely characterize the exclusionary electronic structure of atoms and thus the Periodic Table of the Elements.

4. Proposed Experimental Validation

Although these possible concurrences are reported without claim, there is an apparent route for direct experimental confirmation or contradiction of these results. The odd FQHE denominators *are*, *objectively*, equal to twice the total angular momentum, $2j = 1, 3, 5, 7...$, which electrons in atomic shells are empirically permitted to have. The question is whether these FQHE denominators are a direct physical consequence of this, or whether this is just a coincidence in which two disconnected physical effects happen to each have an odd-integer spectrum. Now, for any given electronic *j*, the total number of observed spin states is equal to $4j$. In other words, for $j = \frac{1}{2}$ correlating to denominator 1, there are two (2) spins states of the *s* ("sharp") orbital shell, $s = \pm\frac{1}{2}$. For $j = \frac{3}{2}$ hence denominator 3 there are six (6) spin states, namely the three states $m = 0, \pm 1$ times the two states $s = \pm\frac{1}{2}$ of the *p* ("principal") orbital shell. For $j = \frac{5}{2}$ hence denominator 5 there are ten (10) spin states, namely the five states $m = 0, \pm 1, \pm 2$ times the two states $s = \pm\frac{1}{2}$ of the *d* ("diffuse") orbital shell. And so on for the *f*, *g*, *h*, etc. states. This would means that a close inspection of spin correlation in FQHE – if these reported but unclaimed concurrences in (3.20) do represent a genuine physical connection – should reveal 2, 6, 10, 14… distinct spin states associated with each of the FQHE denominators 1, 3, 5, 7… respectively.

Further, non-prime multiples of prime number denominators should display particularly robust spins characteristics. The unit state 1=3/3=5/5=7/7… should be highly robust, exhibiting all of s, p, d, f… and other shell characteristics. And a state such as 1/3=3/9=9/27… should display 6 spin states of a *p* shell plus 18 spin states of a *g* shell plus 54 spins states =2x27, etc.

Additionally, if the only even denominator $2j = 2(l + s) = 2$ in the FQHE does result from a *pairing* of electrons each with a half unit of angular momentum, this composite spin state should be observed as the four (4) spin states of a composite $2\otimes 2=3\oplus 1$ boson representation of SU(2). The absence of higher-integer even denominators beyond 2 can be understood based the need for individual fermions to occupy exclusionary states with $2j=1,3,5,7...$ whereas there is no such need for bosons to do the same.

Finally, when the host metal used to observe FQHE has a high atomic number *Z*, such that there are many accessible outer-shell electrons grounded in *d* ($2j=5$) orbitals with $4j=10$ spin states (transition metals), or in *f* ($2j=7$) orbitals with $4j=14$ spin states (lanthanides or actinides), it should be possible to correlatively observe the larger FQHE denominators 5, 7, 9, 11… with the application of smaller perpendicular magnetic fields, because there are already electrons naturally subsisting in *d* or *f* states simply by Exclusion, before any field is applied whatsoever.

5. Conclusion

The DWY analysis can be made fully consistent with the apparent non-observation of magnetic monopoles in nature, if we replace the usual assumption that the south gauge field patch of the posited monopole charge $\mu\equiv\oiint F$ differs from the north patch merely by a gauge-transformation such that $A'_N=A_S$, with the relationship $A'_N\equiv A_S+\varepsilon$ where ε defines an *observable difference* between these north and south patches. Indeed, if the DWY analysis is to have any applicability to physics rather than being simply an elegant but physically-wrong line of development, then the widespread non-observation of electric / magnetic duality in nature *disproves the assumption* that $A'_N=A_S$ and *requires* that there be an *observable* difference ε. Then, the DWY solution requires that $\varepsilon=d\tau$ where τ is a monopole residue $\mu\to\tau_\varphi/\cancel{\varphi}$ which remains behind after the magnetic charge is set to $\mu=0$, consistent with the non-observation of monopoles. The electric charge remains quantized as predicted by DWY in the form $e=ne_{\mathrm{u}}$, but now the unit charge is $e_{\mathrm{u}}=2\pi/\tau_m$, rather than the $e_{\mathrm{u}}\equiv 2\pi/\mu$ obtained when $\tau=0$ and $\mu\neq 0$ in the usual DWY analysis.

What is of particular interest for further study, is that when $\mu\neq 0$ and $\tau=0$, the electric charge states are quantized and fractionalized according to $e=\left(\cancel{A}/\cancel{\varphi}\right)e_{\mathrm{u}}$ as found in (3.11), and when we restrict consideration to only those states which can be disentangled into the $\cancel{\varphi}=1$ azimuth winding of an $e=ne_{\mathrm{u}}$ electron, this becomes $e=\left(n/2\left(l+\tfrac{1}{2}\right)\right)e_{\mathrm{u}}$ which happens to correctly reproduce the odd-denominator FQHE charge states. And what is also of interest is that $2j=2l+2s=1,3,5,7...$ with $s=\tfrac{1}{2}$ happens to also describe the observable Casimir quantum number for the total angular momentum *j* of electrons in atomic shells. We leave as a question for further study and review without present claim, whether this fractionalization might provide a microscopic, per-electron basis for understanding the Fractional Quantum Hall Effect, with the conditions $\mu\neq 0$ and $\tau=0$ prevailing for the two-dimensional FQHE electron configurations near 0K, and the opposite conditions $\mu=0$ and $\tau\neq 0$ with $\mu\to\tau_\varphi/\cancel{\varphi}$ prevailing otherwise where the temperature is higher and electrons have more ample freedom in all three space dimensions. And we also leave open for further study and review without present claim, whether $2j = 2l + 2s = 1, 3, 5, 7...$ represents a real physical linkage between odd-integer fractionalization and the electronic structure of atoms, and whether $n = \not{\Lambda}$ which is the topological quantum number of gauge space windings and does become an energy quantum number when the monopole charge $\mu = 0$ bears a real physical link to the principal quantum number $n = n_r + j + s$. The experiments proposed in section 4, if conducted, could perhaps shed further light on these questions by objectively arbitrating their empirical validity.

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