On the Dual Nature of Minimal Fractal Manifold and Classical Gravitation

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Abstract

As natural outcome of dimensional regularization, the *minimal fractal manifold* (MFM) describes a space-time continuum equipped with arbitrarily small deviations from four-dimensions $(\varepsilon = 4 - D,\ \varepsilon\ <<\ 1)$. This brief note suggests that the inner connection between MFM and local conformal field theory points to a surprising duality between MFM and classical gravitation.

Key words: Minimal Fractal Manifold, Conformal Field Theory, Dimensional Regularization, Classical Gravitation.

1. Emergence of weak gravitational fields from local conformal transformations

Consider a flat four-dimensional space-time with constant metric having the standard signature $\eta_{\mu\nu} = diag(-1,...,+1)$. A differentiable map $x' = \zeta(x)$ is called a *conformal transformation* if the metric tensor changes as

$$\eta_{\mu\nu} \to \overline{\eta_{\mu\nu}} = \eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Omega^2(x)\eta_{\mu\nu} \tag{1}$$

in which $\Omega^2(x)$ represents the scale factor and Einstein's summation convention is implied. The scale factor is *strictly unitary* on flat space-times ($\Omega^2(x) = 1$), a condition aligning with the group of Lorentz transformations. If the underlying space-time background deviates from flatness and is characterized by a metric $g_{\mu\nu}(x) \neq \eta_{\mu\nu}$, the condition for local conformal transformation (1) reads

$$g_{\mu\nu}(x) \to \overline{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \quad (2)$$

where $\Omega^2(x) \neq 1$ in general . A *nearly conformal transformation* (NCT) is defined by a scale factor departing slightly and continuously from unity, that is,

$$\Omega(x) = 1 \pm \varepsilon(x) \approx e^{\pm \varepsilon(x)} \ , \ \ \varepsilon(x) << 1 \quad (3)$$

It is apparent from (3) that the net effect of the NCT is to induce a slight curvature on the flat space-time background which, when interpreted in classical terms, generates a nearly-vanishing Newtonian-like gravitation. For example, consider the well-known expression of the Newtonian potential created by a point-source of mass m at a radial distance R ,

$$g_{00} = 1 + 2\varphi_N = 1 - 2G\frac{m}{R} \quad (4)$$

or

$$g_{00} = 1 + 2\varphi_N = 1 - 2\frac{m}{M_{Pl}^2 R} \quad (5)$$

where G ~ M_{Pl}^2 in natural units. In light of (3), if $m << M_{Pl}^2 R$, the second term in (5) may be simply regarded as a nearly vanishing deviation from local conformal symmetry responsible for the onset of ultra-weak gravitational fields.

2. Emergence of the MFM from dimensional regularization

We now switch gear and bring into the discussion quantum field theory (QFT) and its regularization procedure. As it is known, the technique of regularization assumes that divergent quantities of perturbative QFT depend on a continuous regulator η. The regulator can be either a large cutoff $\eta = \Lambda_{UV}$ or an infinitesimal deviation of the underlying space-time dimension, that is, $\eta = \varepsilon << 1$, $D \to D - \varepsilon$. A divergent quantity O becomes a function of the regulator, $O = O(\eta)$, asymptotically approaching the original quantity in the limit $\eta^{-1} = \Lambda_{UV}^{-1} \to 0$ or $\eta = \varepsilon \to 0$. As a result, in close proximity to this limit, the quantity of interest is no longer singular ($|O(\eta)| < \infty$). To fix ideas, consider the one-loop momentum integral of the massive φ^4 theory defined on a two-dimensional Euclidean space-time ($D = 2$),

$$\Sigma = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + m^2} \tag{6}$$

The integral is logarithmically divergent at large momenta $\Sigma(p^2) \to \infty$ for $p \to \infty$. One way to regularize (6) is to upper-bound it with a sharp *mass cutoff* $\Lambda_{UV} >> m$ as in

$$\Sigma_c = \int_0^{\Lambda_{UV}^2} \frac{dp^2}{4\pi} \frac{1}{p^2 + m^2} = \frac{1}{4\pi} \ln(\frac{\Lambda_{UV}^2 + m^2}{m^2}) \tag{7}$$

The *Pauli-Villars regularization* method is based on subtracting from (6) the same integral having a larger momentum scale $\Lambda >> m$, that is,

$$\Sigma_{PV} = \int \frac{d^2p}{(2\pi)^2} (\frac{1}{p^2 + m^2} - \frac{1}{p^2 + \Lambda^2}) = \frac{1}{4\pi} \ln(\frac{\Lambda^2}{m^2}) \tag{8}$$

By contrast, *dimensional regularization* posits that the space-time dimension can be analytically continued to $D - \varepsilon$, which turns (6) into

$$\Sigma_{DR} = \mu^{\varepsilon} \int \frac{d^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2 + m^2} \tag{9}$$

where μ is an arbitrary mass scale that preserves the dimensionless nature of Σ_{DR}. It can be shown that (9) amounts to

$$\Sigma_{DR} = \frac{1}{4\pi}[\frac{2}{\varepsilon} - \gamma + \ln(4\pi) - \ln(\frac{m^2}{\mu^2}) + O(\varepsilon)] \tag{10}$$

in which γ stands for the Euler constant. Comparing (8) to (10) and further taking μ to be on the same order of magnitude with m ($\mu = O(m)$, leads to the identification

$$\frac{1}{\varepsilon} \sim \ln(\frac{\Lambda^2}{m^2}) \tag{11}$$

Relation (11) describes the connection between the dimensional parameter ε and the mass scaling $\frac{\Lambda^2}{m^2}$. If ε is assumed to be vanishingly small (ε << 1) and $m = O(\mu)$ << $\Lambda = O(\Lambda_{UV})$, (11) may be reasonably approximated as

$$\varepsilon \sim \frac{m^2}{\Lambda_{UV}^2} \tag{12}$$

<u>3. Duality of MFM and classical gravitation</u>

Consider next the Newtonian potential created by an object of mass m at a distance on the order of its Compton wavelength λ_c, that is, $R \sim \lambda_c \sim m^{-1}$. We obtain

$$g_{00} = 1 - 2(\frac{m}{M_{Pl}})^2 \sim 1 - 2\varepsilon \approx e^{-2\varepsilon} \tag{13}$$

Taking $\Lambda_{UV} = O(M_{Pl})$ in (12), relations (5), (12) and (13) highlight an intriguing duality between the MFM and classical gravitation of Newtonian fields. Loosely speaking, (13) may be associated with the *gravitational coupling of a particle-antiparticle pair separated by a distance scale comparable with the Compton wavelength.*