Approximate formula for periodic windows of the logistic map

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Abstract

In the present work approximate formula for periodic windows of the logistic map is derived.

1 Introduction

We consider the logistic map:

$$x_{n+1} = 4bx_n\left(1 - x_n\right), \tag{1}$$

where b is a parameter. This simple nonlinear system has very complicated dynamics [1, 2]. Various dynamical modes, universal for some classes of one-dimensional mappings of form $x_{n+1} = f\left(x_n\right)$ [3, 4], were observed in several experiments [1, 5].

Recently, equation for 3-cycles, stable and unstable, was solved exactly, condition for the onset of the 3-cycles was found, and the map of dynamics for $b \in [0.5,\ 1]$ onto dynamics for $b \in [-0.5,\ 0]$ was constructed [6]; see also [7] for a survey of rigorous results for the logistic map. In this note we attempt to obtain approximate localization of periodic windows in the parameter space for the map (1). The present work is based on results obtained in Refs. [8, 9, 6].

2 Map on the sphere

Let us consider the following map on the $SU\left(2\right)$ group [10]:

$$R_{n+1} = R_n S R_n^{-1}, \tag{2a}$$

$$R_n = \exp\left(i\tfrac{\beta}{2}\sigma\cdot\mathbf{r}_n\right),\ S = \exp\left(i\tfrac{\beta}{2}\sigma\cdot\mathbf{s}\right), \tag{2b}$$

where σ_1, σ_2, σ_3, are Pauli matrices and $\mathbf{r}_n$, $\mathbf{s}$ are unit vectors. Components of $\mathbf{r}_n$ evolve on a unit sphere, see Eqs. (7) in [10]. Due to symmetry $R_n \longrightarrow SR_nS^{-1}$

dynamics of variables $z_n = \mathbf{s} \cdot \mathbf{r}_n$ decouples from other equations. If we choose $\mathbf{s} = [0,\ 0,\ 1]$ then we get one-dimensional map:

$$z_{n+1} = f\left(z_n\right) = \cos\theta + \left(1 - \cos\theta\right) z_n^2, \tag{3}$$

and $z_n = \cos\left(\theta_n\right)$ where θ_n is the polar angle on the sphere. In the case of a more general map, $R_{n+1} = QR_nSR_n^{-1}Q^{-1}$, $Q = \exp\left(i\frac{\gamma}{2}\sigma \cdot \mathbf{q}\right)$, $QS \neq SQ$, where the symmetry $R_n \longrightarrow SR_nS^{-1}$ is destroyed, we obtain two-dimensional dynamics on the sphere [11]. Geometry of the map (2) is shown below

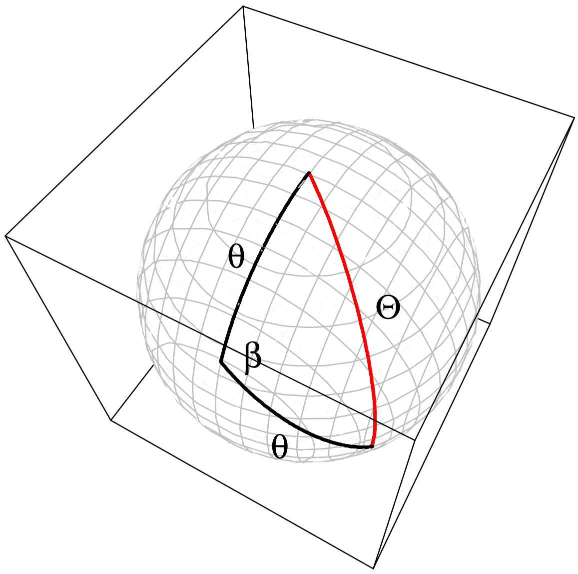

Figure 1: Geometry of the map (2).

where $\theta = \theta_n$, $\Theta = \theta_{n+1}$, see also Fig. 1a in [10].

There is a simple linear transformation converting (1) into (3), namely:

$$x_n = -\frac{1}{2}z_n + \frac{1}{2}, \tag{4a}$$

$$b = \frac{1}{2} - \frac{1}{2}\cos\beta. \tag{4b}$$

For $\cos\left(\beta\right) = -1$ equation (3) reads:

$$z_{n+1} = -1 + 2z_n^2, \tag{5}$$

and due to identity $\cos\left(2\theta_n\right) = 2\cos^2\left(\theta_n\right) - 1$ we obtain:

$$z_n = \cos\left(\theta_n\right), \qquad \theta_n = 2^n\theta_0. \tag{6}$$

3 Periodic windows for $b \lessapprox 1$ and $b \gtrapprox -0.5$

Condition for a supercycle is that the first derivative of the map vanishes in a fixed point z_*:

$$f'(z_*) = 0. \tag{7}$$

Application of this condition to $f(z) = \cos(\beta) + (1 - \cos(\beta))\, z^2$ yields:

$$\cos(\theta_*) = 0, \tag{8}$$

and from (3) the next value ot the cycle is computed as $\cos(\theta_{*1}) = \cos(\beta)$. We shall look for stable k-cycles in the case $\cos(\beta) \cong -1$. We can expect that in this case the solution (6) will be a reasonable approximation:

$$\cos(\theta_{*1}) = \cos\left(2^k \theta_{*1}\right), \quad \theta_{*1} = \beta, \quad \cos(\beta) \cong -1. \tag{9}$$

Solving (9) for β we get:

$$\beta = \pm 2^k \beta + 2m\pi, \tag{10}$$

and $\beta \cong \pi$. Finally, we obtain:

$$\beta_{\pm} = \tfrac{2m}{2^k \pm 1}\pi, \qquad \beta \cong \pi, \tag{11}$$

where for a given k-cycle (k is the number of parallels on the sphere) we have to choose such integer m that the condition $\beta_{\pm} \cong \pi$ is indeed fulfilled. Since $b = \frac{1}{2} - \frac{1}{2}\cos(\beta)$ we obtain approximate expression for k-cycles in terms of parameter b.

Periodic windows, present in the interval of control parameter $b \in \left[\frac{1}{2},\, 1\right]$, can be mapped onto periodic windows in $b \in [-0.5,\, 0]$. Indeed, if we put $x_n = \frac{2-4b}{4b}\hat{x}_n + \frac{4b-1}{4b}$ into (1) then we get

$$\hat{x}_{n+1} = 4\hat{b}\hat{x}_n\left(1 - \hat{x}_n\right), \qquad \hat{b} = \tfrac{2-4b}{4}, \tag{12}$$

see the map defined in [6] with $r = 4b$.

4 Computations

To localize a periodic window we choose value of integer parameter k (which corresponds to number of parallels on the sphere) in (11) and then we select such integer m that $\frac{2m}{2^k \pm 1} \simeq 1$. It follows that $m = 2^{k-1}$. We thus obtain the following set of parametrs k and m fulfilling this condition:

TABLE I

k	$m = 2^{k-1}$	$\beta_+^{(k)}$	$b_+^{(k)}$	$\beta_-^{(k)}$	$b_-^{(k)}$
3	4	$\frac{8}{9}\pi$	0.969 846 310 4	$\frac{8}{7}\pi$	0.950 484 434 0
4	8	$\frac{16}{17}\pi$	0.991 486 549 8	$\frac{16}{15}\pi$	0.989 073 800 4
5	16	$\frac{32}{33}\pi$	0.997 735 961 3	$\frac{32}{31}\pi$	0.997 434 661 7
6	32	$\frac{64}{65}\pi$	0.999 416 113 4	$\frac{64}{63}\pi$	0.999 378 460 6

In Table I values of $\beta_{\pm}^{(k)}$, as well as $b_{\pm}^{(k)} = \frac{1}{2} - \frac{1}{2}\cos\left(\beta_{\pm}^{(k)}\right)$, have been also listed.

In all cases shown in Table I the stable k-cycle is in the interval $b \in \left[b_{-}^{(k)},\ b_{+}^{(k)}\right]$. Moreover, the mean value, $\bar{b}^{(k)} = \frac{1}{2}\left(b_{+}^{(k)} + b_{-}^{(k)}\right)$, is a good approximation of the corresponding supercycle. For the sake of just one example we show results of computations for $k = 5$.

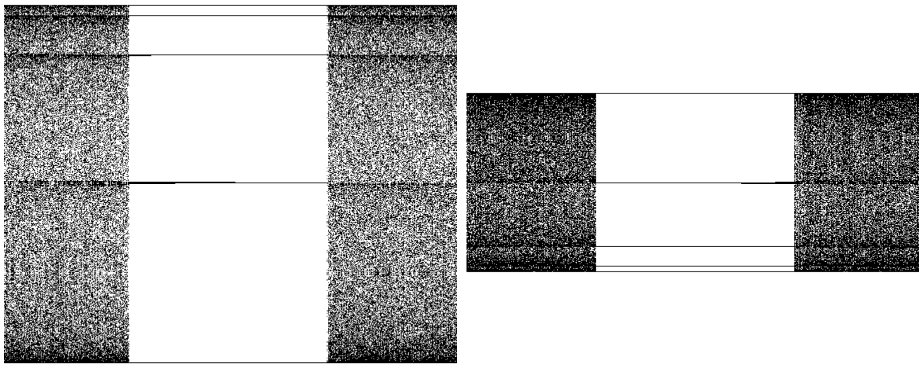

Figure 2: The logistic map (1): $b \in [-0.4976,\ -0.4975]$ – left figure, $b \in [0.99755,\ 0.99760]$ – right figure.

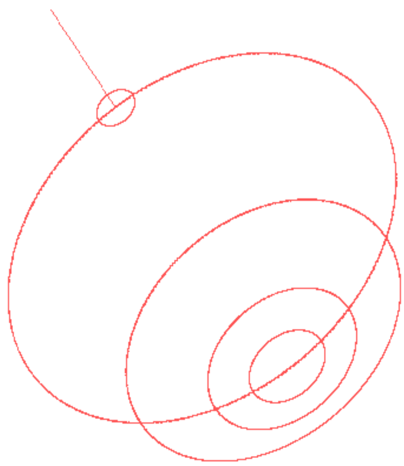

Figure 3: Periodic window for the map (2), $b = \frac{1}{2} - \frac{1}{2}\cos\beta = 0.99758$.

We have shown in Fig. (2) two periodic windows: the window in the interval $b \in [0.997\,55,\ 0.997\,60]$, which was localized with help of $b_{\pm}^{(5)}$, cf. Table I, and another window, in the interval $b \in [-0.497\,6,\ -0.497\,5]$, where the latter interval was computed from the former via formula $b \to \frac{2-4b}{4}$, cf. Eq. (12).

5 Summary

We have shown that periodic windows of the logistic map can be localized for $b \lessapprox 1$ and $b \gtrapprox -0.5$, where geometric interpretation of the logistic map, as well as transformation $b \to \frac{2-4b}{4}$, were used.

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