Gauge Invariance of Sedeonic Klein-Gordon Equation

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Abstract

We discuss the sedeonic Klein-Gordon wave equation based on sedeonic space-time operators and wave function. The generalization of the gauge invariance for a wider class of scalar-vector substitutions is demonstrated.

Keywords: Sedeonic Klein-Gordon wave equation; Proca-Maxwell equation; Gauge invariance.

1 Introduction

The operator Klein-Gordon (KG) wave equation corresponds to the fundamental Einstein relation between energy and momentum [1]. Substantially it is the basis for the description of relativistic quantum particles and fields. However the scalar KG equation describes only scalar bosons but does not specify the spin properties of particles. Therefore the generalization of KG equation to a wider class of operators and multicomponent wave functions is the subject of intensive investigations.

First, to describe the quantum particles with spin 1/2 P.A.M.Dirac proposed the matrix equation for the spinor wave function [2], which leads to non-scalar KG equation in the case of electromagnetic interaction. This approach was further generalized for the integer spin 0 and 1 in the frames of Duffin-Kemmer-Petiau (DKP) formalism [3-5].

On the other hand, in recent years many attempts have been made to generalize KG equation using different algebras of hypercomplex numbers, such as four-component quaternions and eight-component octonion [6-8]. The authors discuss the reformulation of the Proca equation [9] as the system of first-order equations similar to the equations of electromagnetic field but with a massive "photon". However, in comparison with equations of electrodynamics the resulting Proca-Maxwell (PM) equations for massive field are not gauge invariant [10,11].

Recently we proposed an alternative approach to the generalization of KG operator based on sixteen-component sedeons generating noncommutative associative space-time algebra [12]. The sedeons take into account the space-time properties of physical values and realize the scalar-vector representation of Poincare group. In particular, we proposed the symmetric sedeonic second-order and first-order wave equations describing massive and massless fields [13]. In present paper we focus main attention on the gauge invariance of sedeonic KG wave equation based on sedeonic space-time operators and sedeonic wave function.

2 Sedeonic space-time algebra

To begin with we briefly review the basic properties of sedeons. The sedeonic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

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Table 1:

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	1	$i\mathbf{a_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	1

Table 2:

	$\mathbf{e_t}$	$\mathbf{e_r}$	$\mathbf{e_{tr}}$
$\mathbf{e_t}$	1	$i\mathbf{e_{tr}}$	$-i\mathbf{e_r}$
$\mathbf{e_r}$	$-i\mathbf{e_{tr}}$	1	$i\mathbf{e_t}$
$\mathbf{e_{tr}}$	$i\mathbf{e_r}$	$-i\mathbf{e_t}$	1

1. Absolute scalars (A) and absolute vectors ($\vec{A}$) are not transformed under spatial and time inversion.

2. Time scalars ($B_\mathbf{t}$) and time vectors ($\vec{B}_\mathbf{t}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.

3. Space scalars ($C_\mathbf{r}$) and space vectors ($\vec{C}_\mathbf{r}$) are changed under spatial inversion and are not transformed under time inversion.

4. Space-time scalars ($D_\mathbf{tr}$) and space-time vectors ($\vec{D}_\mathbf{tr}$) are changed under spatial and time inversion.

The indexes **t** and **r** indicate the transformations (**t** for time inversion and **r** for spatial inversion), which change the corresponding values. The space-time sedeon $\tilde{\mathbf{S}}$ is defined by the following expression:

$$\tilde{\mathbf{S}} = A + \vec{A} + B_\mathbf{t} + \vec{B}_\mathbf{t} + C_\mathbf{r} + \vec{C}_\mathbf{r} + D_\mathbf{tr} + \vec{D}_\mathbf{tr}. \tag{1}$$

Here and further we indicate the sedeon by bold symbol with wave. The components of sedeon (1) can be written in the sedeonic space-time basis as

$$\begin{aligned}
&A = \mathbf{e_0} A \mathbf{a_0}, \\
&\vec{A} = \mathbf{e_0}\left(A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}\right), \\
&B_\mathbf{t} = \mathbf{e_t} B \mathbf{a_0}, \\
&\vec{B}_\mathbf{t} = \mathbf{e_t}\left(B_1\mathbf{a_1} + B_2\mathbf{a_2} + B_3\mathbf{a_3}\right), \\
&C_\mathbf{r} = \mathbf{e_r} C \mathbf{a_0}, \\
&\vec{C}_\mathbf{r} = \mathbf{e_r}\left(C_1\mathbf{a_1} + C_2\mathbf{a_2} + C_3\mathbf{a_3}\right), \\
&D_\mathbf{tr} = \mathbf{e_{tr}} D \mathbf{a_0}, \\
&\vec{D}_\mathbf{tr} = \mathbf{e_{tr}}\left(D_1\mathbf{a_1} + D_2\mathbf{a_2} + D_3\mathbf{a_3}\right),
\end{aligned} \tag{2}$$

where values $\mathbf{a_0}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are scalar-vector basis ($\mathbf{a_0} \equiv 1$ is absolute scalar unit and the values $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are absolute unit vectors generating the right Cartesian basis) and values $\mathbf{e_0}$, $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ are space-time basis ($\mathbf{e_0} \equiv 1$ is a absolute scalar unit; $\mathbf{e_t}$ is a time unit; $\mathbf{e_r}$ is a space unit; $\mathbf{e_{tr}}$ is a space-time unit). Further we will omit the units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity.

The multiplication and commutation rules for the sedeonic absolute unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and space-time units $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ are presented in the tables 1 and 2 respectively (in the tables and further the value i is the imaginary unit ($i^2 = -1$)). Note that sedeonic units $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ commute with $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$.

Thus the sedeon $\tilde{\mathbf{S}}$ is the complicated space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

In sedeonic algebra we assume the Clifford multiplication of vectors. For example, the sedeonic product of two absolute vectors $\vec{A}$ and $\vec{B}$ can be presented in the following form:

$$\vec{A}\vec{B} = \left(\vec{A} \cdot \vec{B}\right) + \left[\vec{A} \times \vec{B}\right]. \tag{3}$$

Here we denote the sedeonic scalar multiplication of two vectors (internal product) by symbol "·" and round brackets

$$\left(\vec{A} \cdot \vec{B}\right) = A_1 B_1 + A_2 B_2 + A_3 B_3, \tag{4}$$

and sedeonic vector multiplication (external product) by symbol "×" and square brackets

$$\left[\vec{A} \times \vec{B}\right] = i\left(A_2 B_3 - A_3 B_2\right)\mathbf{a_1} + i\left(A_3 B_1 - A_1 B_3\right)\mathbf{a_2} + i\left(A_1 B_2 - A_2 B_1\right)\mathbf{a_3}. \tag{5}$$

Note that in sedeonic algebra the definition of the vector product differs from analogous expression in Gibbs-Heaviside vector algebra.

3 Sedeonic Klein-Gordon equation

The sedeonic KG equation [13] can be written in the following compact form:

$$\overset{\frown}{\nabla}\overset{\frown}{\nabla}\ \tilde{\mathbf{W}}\ = 0. \tag{6}$$

Here the wave function $\tilde{\mathbf{W}}$ is the sixteen-component sedeon and we use the complex operator

$$\overset{\frown}{\nabla} = \left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right), \tag{7}$$

with

$$\begin{aligned} \partial &= \frac{1}{c}\frac{\partial}{\partial t}, \\ \vec{\nabla} &= \frac{\partial}{\partial x}\,\mathbf{a_1} + \frac{\partial}{\partial y}\,\mathbf{a_2} + \frac{\partial}{\partial z}\,\mathbf{a_3}, \\ m &= \frac{m_0 c}{\hbar}, \end{aligned} \tag{8}$$

where $\mathbf{e_t}, \mathbf{e_r}, \mathbf{e_{tr}}$ are the sedeonic space-time units; $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ are the sedeonic unit vectors; c is the speed of light; m_0 is the mass of quantum; $\hbar$ is the Planck constant. On the other hand, if we introduce the filed strength

$$\tilde{\mathbf{E}} = \overset{\frown}{\nabla}\ \tilde{\mathbf{W}}, \tag{9}$$

then the wave equation (6) takes the following form:

$$\overset{\frown}{\nabla}\ \tilde{\mathbf{E}}\ = 0. \tag{10}$$

The formalism of sedeonic $\overset{\frown}{\nabla}$ operator enables the direct conclusion that the equation (6) and field strength definition (9) are invariant with respect to the following replacement of wave function

$$\tilde{\mathbf{W}} \Rightarrow \tilde{\mathbf{W}} + \tilde{\mathbf{F}} + \overset{\frown}{\nabla}\tilde{\mathbf{G}}, \tag{11}$$

where $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are arbitrary sedeons satisfy the following conditions:

$$\overset{\frown}{\nabla}\ \tilde{\mathbf{F}} = 0, \tag{12}$$

$$\overset{\frown}{\nabla}\overset{\frown}{\nabla}\ \tilde{\mathbf{G}} = 0. \tag{13}$$

Besides, the equation (10) is invariant with respect to the replacement

$$\tilde{\mathbf{E}} \Rightarrow \tilde{\mathbf{E}} + \tilde{\mathbf{F}} + \overset{\frown}{\nabla}\tilde{\mathbf{G}}. \tag{14}$$

The gauge relations (11) - (14) are the direct consequence of the operator $\overset{\frown}{\nabla}$ formalism.

Let us consider the field equations and gauge conditions in detail. We choose the wave function as

$$\begin{aligned} \tilde{\mathbf{W}} = \;& ia_1\mathbf{e_t} - ia_2\mathbf{e_r} + a_3 - ia_4\mathbf{e_{tr}} \\ & +\vec{A}_1\mathbf{e_r} + \vec{A}_2\mathbf{e_t} - \vec{A}_3\mathbf{e_{tr}} \; + i\vec{A}_4, \end{aligned} \tag{15}$$

where components a_{s} and $\vec{A}_{\mathrm{s}}$ are real functions of coordinates and time. Here and further the index s = 1, 2, 3, 4. Multiplying the operators in the left part of equation (6) and separating the values with different space-time properties, we obtain the system of KG equations for the components of wave function:

$$\begin{aligned} \left(\partial^2 - \Delta + m^2\right) a_{\mathrm{s}} = 0, \\ \left(\partial^2 - \Delta + m^2\right) \vec{A}_{\mathrm{s}} = 0, \end{aligned} \tag{16}$$

where $\triangle$ is the Laplace operator. On the other hand, if we take the sedeon $\tilde{\mathbf{E}}$ as

$$\begin{aligned} \tilde{\mathbf{E}} = -\varepsilon_1 + i\varepsilon_2\mathbf{e_{tr}} + i\varepsilon_3\mathbf{e_t} - i\varepsilon_4\mathbf{e_r} \\ +\vec{E}_1\mathbf{e_{tr}} - i\vec{E}_2 + \vec{E}_3\mathbf{e_r} + \vec{E}_4\mathbf{e_t}, \end{aligned} \tag{17}$$

then we have the following definitions for scalar ε_{s} and vector $\vec{E}_{\mathrm{s}}$ components:

$$\begin{aligned} \varepsilon_1 &= \partial a_1 + \left(\vec{\nabla}\cdot\vec{A}_1\right) + ma_4, \\ \varepsilon_2 &= \partial a_2 + \left(\vec{\nabla}\cdot\vec{A}_2\right) - ma_3, \\ \varepsilon_3 &= \partial a_3 + \left(\vec{\nabla}\cdot\vec{A}_3\right) + ma_2, \\ \varepsilon_4 &= \partial a_4 + \left(\vec{\nabla}\cdot\vec{A}_4\right) - ma_1, \\ \vec{E}_1 &= -\partial\vec{A}_1 - \vec{\nabla}a_1 + i\left[\vec{\nabla}\times\vec{A}_2\right] + m\vec{A}_4, \\ \vec{E}_2 &= -\partial\vec{A}_2 - \vec{\nabla}a_2 - i\left[\vec{\nabla}\times\vec{A}_1\right] - m\vec{A}_3, \\ \vec{E}_3 &= -\partial\vec{A}_3 - \vec{\nabla}a_3 - i\left[\vec{\nabla}\times\vec{A}_4\right] + m\vec{A}_2, \\ \vec{E}_4 &= -\partial\vec{A}_4 - \vec{\nabla}a_4 + i\left[\vec{\nabla}\times\vec{A}_3\right] - m\vec{A}_1, \end{aligned} \tag{18}$$

and the sedeonic wave equation (10) is equivalent to the following system of equations for the field strengths:

$$\begin{aligned} &\partial\varepsilon_1 + \left(\vec{\nabla}\cdot\vec{E}_1\right) - m\varepsilon_4 = 0, \\ &\partial\varepsilon_2 + \left(\vec{\nabla}\cdot\vec{E}_2\right) + m\varepsilon_3 = 0, \\ &\partial\varepsilon_3 + \left(\vec{\nabla}\cdot\vec{E}_3\right) - m\varepsilon_2 = 0, \\ &\partial\varepsilon_4 + \left(\vec{\nabla}\cdot\vec{E}_4\right) + m\varepsilon_1 \; = 0, \\ &\partial\vec{E}_1 + \vec{\nabla}\varepsilon_1 + i\left[\vec{\nabla}\times\vec{E}_2\right] + m\vec{E}_4 = 0, \\ &\partial\vec{E}_2 + \vec{\nabla}\varepsilon_2 - i\left[\vec{\nabla}\times\vec{E}_1\right] - m\vec{E}_3 = 0, \\ &\partial\vec{E}_3 + \vec{\nabla}\varepsilon_3 - i\left[\vec{\nabla}\times\vec{E}_4\right] + m\vec{E}_2 = 0, \\ &\partial\vec{E}_4 + \vec{\nabla}\varepsilon_4 + i\left[\vec{\nabla}\times\vec{E}_3\right] - m\vec{E}_1 = 0. \end{aligned} \tag{19}$$

Now we consider the gauge invariance for the components of wave function and field strength. Let us take the arbitrary sedeons $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ as

$$\begin{aligned}\tilde{\mathbf{F}} = \ & if_1\mathbf{e_t} - if_2\mathbf{e_r} + f_3 - if_4\mathbf{e_{tr}} \\ & + \vec{F}_1\mathbf{e_r} + \vec{F}_2\mathbf{e_t} - \vec{F}_3\mathbf{e_{tr}} \ + i\vec{F}_4,\end{aligned} \tag{20}$$

and

$$\begin{aligned}\tilde{\mathbf{G}} = \ & -g_1 + ig_2\mathbf{e_{tr}} + ig_3\mathbf{e_t} - ig_4\mathbf{e_r} \\ & + \vec{G}_1\mathbf{e_{tr}} - i\vec{G}_2 + \vec{G}_3\mathbf{e_r} + \vec{G}_4\mathbf{e_t}.\end{aligned} \tag{21}$$

Then the replacement (11) leads us to the following substitutions:

$$\begin{aligned}
& a_1 \Rightarrow a_1 + f_1 - \partial g_1 - \left(\vec{\nabla}\cdot\vec{G}_1\right) + mg_4, \\
& a_2 \Rightarrow a_2 + f_2 - \partial g_2 - \left(\vec{\nabla}\cdot\vec{G}_2\right) - mg_3, \\
& a_3 \Rightarrow a_3 + f_3 - \partial g_3 - \left(\vec{\nabla}\cdot\vec{G}_3\right) + mg_2, \\
& a_4 \Rightarrow a_4 + f_4 - \partial g_4 - \left(\vec{\nabla}\cdot\vec{G}_4\right) - mg_1, \\
& \vec{A}_1 \Rightarrow \vec{A}_1 + \vec{F}_1 + \partial\vec{G}_1 + \vec{\nabla}g_1 + i\left[\vec{\nabla}\times\vec{G}_2\right] + m\vec{G}_4, \\
& \vec{A}_2 \Rightarrow \vec{A}_2 + \vec{F}_2 + \partial\vec{G}_2 + \vec{\nabla}g_2 - i\left[\vec{\nabla}\times\vec{G}_1\right] - m\vec{G}_3, \\
& \vec{A}_3 \Rightarrow \vec{A}_3 + \vec{F}_3 + \partial\vec{G}_3 + \vec{\nabla}g_3 - i\left[\vec{\nabla}\times\vec{G}_4\right] + m\vec{G}_2, \\
& \vec{A}_4 \Rightarrow \vec{A}_4 + \vec{F}_4 + \partial\vec{G}_4 + \vec{\nabla}g_4 + i\left[\vec{\nabla}\times\vec{G}_3\right] - m\vec{G}_1.
\end{aligned} \tag{22}$$

which do not change the field strengths definitions (18) and equations (19). The generalized substitutions (22) include the gradient invariance of electromagnetic potentials [14] as a particular case. Indeed, if we take $m = 0$, $f_s = 0$, $\vec{F}_s = 0$, $\vec{G}_s = 0$ then the substitutions (22) are rewritten as

$$\begin{aligned} & a_s \Rightarrow a_s + \partial g_s, \\ & \vec{A}_s \Rightarrow \vec{A}_s - \vec{\nabla}g_s. \end{aligned} \tag{23}$$

Similarly, the replacement (14) is equivalent to the following substitutions for the field strengths

$$\begin{aligned}
& \varepsilon_1 \Rightarrow \varepsilon_1 - f_3 + \partial g_3 + \left(\vec{\nabla}\cdot\vec{G}_3\right) - mg_2, \\
& \varepsilon_2 \Rightarrow \varepsilon_2 - f_4 + \partial g_4 + \left(\vec{\nabla}\cdot\vec{G}_4\right) + mg_1, \\
& \varepsilon_3 \Rightarrow \varepsilon_3 + f_1 - \partial g_1 - \left(\vec{\nabla}\cdot\vec{G}_1\right) + mg_4, \\
& \varepsilon_4 \Rightarrow \varepsilon_4 + f_2 - \partial g_2 - \left(\vec{\nabla}\cdot\vec{G}_2\right) - mg_3, \\
& \vec{E}_1 \Rightarrow \vec{E}_1 - \vec{F}_3 - \partial\vec{G}_3 - \vec{\nabla}g_3 + i\left[\vec{\nabla}\times\vec{G}_4\right] - m\vec{G}_2, \\
& \vec{E}_2 \Rightarrow \vec{E}_2 - \vec{F}_4 - \partial\vec{G}_4 - \vec{\nabla}g_4 - i\left[\vec{\nabla}\times\vec{G}_3\right] + m\vec{G}_1, \\
& \vec{E}_3 \Rightarrow \vec{E}_3 + \vec{F}_1 + \partial\vec{G}_1 + \vec{\nabla}g_1 + i\left[\vec{\nabla}\times\vec{G}_2\right] + m\vec{G}_4, \\
& \vec{E}_4 \Rightarrow \vec{E}_4 + \vec{F}_2 + \partial\vec{G}_2 + \vec{\nabla}g_2 - i\left[\vec{\nabla}\times\vec{G}_1\right] - m\vec{G}_3.
\end{aligned} \tag{24}$$

which do not change the equations (19).

4 Sedeonic Proca-Maxwell equation

The equations (19) include the PM equations as the partial case. Indeed, if we suppose that the field is described only by a_1 and $\vec{A}_1$ components:

$$\tilde{\mathbf{W}} = ia_1\mathbf{e_t} + \vec{A}_1\mathbf{e_r}, \tag{25}$$

with Lorentz gauge

$$\partial a_1 + \left(\vec{\nabla} \cdot \vec{A}_1\right) = 0, \tag{26}$$

then we have only the following nonzero field's strengths (see the definitions (18)):

$$\begin{aligned}
&\varepsilon_4 = -ma_1, \\
&\vec{E}_1 = -\partial\vec{A}_1 - \vec{\nabla}a_1, \\
&\vec{E}_2 = -i\left[\vec{\nabla} \times \vec{A}_1\right], \\
&\vec{E}_4 = -m\vec{A}_1,
\end{aligned} \tag{27}$$

and the system (19) is rewritten as

$$\begin{aligned}
&\left(\vec{\nabla} \cdot \vec{E}_1\right) - m\varepsilon_4 = 0, \\
&\left(\vec{\nabla} \cdot \vec{E}_2\right) = 0, \\
&\partial\varepsilon_4 + \left(\vec{\nabla} \cdot \vec{E}_4\right) = 0, \\
&\partial\vec{E}_1 + i\left[\vec{\nabla} \times \vec{E}_2\right] + m\vec{E}_4 = 0, \\
&\partial\vec{E}_2 - i\left[\vec{\nabla} \times \vec{E}_1\right] = 0, \\
&i\left[\vec{\nabla} \times \vec{E}_4\right] - m\vec{E}_2 = 0, \\
&\partial\vec{E}_4 + \vec{\nabla}\varepsilon_4 - m\vec{E}_1 = 0.
\end{aligned} \tag{28}$$

The system (28) is the sedeonic analog of PM equations with considerably reduced symmetry and broken gauge invariance.

Note that all obtained results were derived in the sedeonic algebra. For the transition to the common used Gibbs-Heaviside vector algebra the change of vector product

$$i[\vec{\nabla} \times \vec{V}] \Rightarrow -[\vec{\nabla} \times \vec{V}] \tag{29}$$

should be made in all vector expressions.

5 Summary

Thus we have demonstrated that the gauge invariance of the sedeonic KG equation can be generalized for a wider class of scalar-vector substitutions. On the example of PM equations we have shown that the decrease in the number of components of wave function leads to the symmetry breaking and gauge invariance violation. The proposed approach can be applied for the analysis of second-order wave equations based on Dirac and DKP matrix operators.

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References

[1] V.B. Berestetskii, E.M. Lifshits and L.P. Pitaevskii – Quantum electrodynamics (2nd ed.). London: Pergamon. Vol. 4 of the Course of Theoretical Physics. (1982).

[2] P.A.M. Dirac, The quantum theory of the electron, Proceedings of Royal Society of London, A., **117**, 610 (1928).

[3] G. Petiau, University of Paris thesis, Published in: Academie Royale de Belgique, Classes des Sciences, Memoires, Collection 8, **16**(2), 1114 (1936).

[4] N. Kemmer, Quantum theory of Einstein-Bose particles and nuclear interaction, Proceedings of Royal Society of London, A **166**, 127 (1938).

[5] R. J. Duffin, On the characteristic matrices of covariant systems, Physical Review, **54**, 1114 (1938).

[6] S. Demir and M. Tanisli, A compact biquaternionic formulation of massive field equations in gravi-electromagnetism, European Physical Journal - Plus, **126**(11), 115 (2011).

[7] N. Candemir, M. Tanisli, K. Ozdas, and S. Demir, Hyperbolic octonionic Proca-Maxwell equations, Zeitschrift für Naturforschung A, **63a**, 15 (2008).

[8] T. Tolan, M. Tanisli and S. Demir, Octonic form of Proca-Maxwell equations and relativistic derivation of electromagnetism, International Journal of Theoretical Physics, **52**, 4488 (2013).

[9] A. Proca, Sur la theorie ondulatoire des electrons positifs et negatifs, Journal de Physique et Le Radium, **7**, 347 (1936).

[10] J. D. Jackson and L. B. Okun, Historical roots of gauge invariance, Reviews of Modern Physics, **73**(3), 663 (2001).

[11] L.-Ch. Tu, J. Luo, and G. T. Gillies, The mass of the photon, Reports on Progress in Physisc, **68**, 77 (2005).

[12] V. L. Mironov and S. V. Mironov, Reformulation of relativistic quantum mechanics equations with non-commutative sedeons, Applied Mathematics, **4**(10C), 53 (2013).

[13] S. V. Mironov and V. L. Mironov, Sedeonic equations of massive fields, International Journal of Theoretical Physics, **54**(1), 153 (2015).

[14] L.D. Landau and E.M. Lifshits - Classical Theory of Fields (New York: Pergamon Press, 1975).