A variant of Dirac Equation with Super-symmetric partner

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Abstract - A variant of Dirac equation has been proposed wherein in addition to particles and anti-particles (fermions) with positive and negative energies respectively, one also obtains a bosonic solution with spin 0 (a scalar particle) with the same mass. The boson also has positive and negative energies. In the context of supersymmetry the boson can be seen as an unbroken super partner of the corresponding fermion. Quaternion coordinates have been used to obtain this result.

We know that the Dirac equation (in natural units) is given by :

$$(i\gamma^{\mu}\partial\mu - m)\psi(x) = 0 \quad \text{...(1)}$$

wherein .

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{...(2)}$$

and I is a 2*2 identity matrix and

$$\gamma^i = \begin{pmatrix} 0 & \sigma i \\ -\sigma i & 0 \end{pmatrix} \quad \text{...(3)}$$

where $\sigma i\ are\ 2 * 2\ Pauli\ matrices$ given by :

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;;\quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;;\quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{...(4)}$$

The γ matrices satisfy the relations :

$$(\gamma^0)^2 = 1,\ (\gamma^i)^2 = -1 \quad \text{...(5)}$$

and $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 0$ for $\mu \neq \nu$ ....(6)

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With $i = 1, 2, 3$, $\mu, \nu = 0, 1, 2, 3$

Now , putting the solution :

$$\psi(x) = u(p)e^{-ip\cdot x} \qquad \text{.......................................(7)}$$

in the Dirac equation (1) we obtain :

$$(\gamma^{\mu}p_{\mu} - m)u(p) = 0 \qquad \text{.......................................(8)}$$

Where u(p) is a 4*1 column matrix.

We shall now extend the set of γ matrices using quaternion , a number system that extends the complex numbers. A feature of quaternion is that multiplication of two quaternion is non-commutative and their square is -1.

Thus if $\hat{j}, \hat{k}$ are quaternion then

$\hat{j}\hat{k} = -\hat{k}\hat{j}$; and

$\hat{j}^2 = \hat{k}^2 = -1$

Using these, we define a new set of matrices I_b as follows :

$$I_b^{\,0} = \begin{pmatrix} i\hat{k} & 0 \\ 0 & -i\hat{k} \end{pmatrix} \qquad \text{................................(9)}$$

$$I_b^{\,1} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \;;\quad I_b^{\,2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;;\quad I_b^{\,3} = \begin{pmatrix} \hat{j} & 0 \\ 0 & -\hat{j} \end{pmatrix} \qquad \text{...............................(10)}$$

It may be noted that *i* is a complex number whereas $\hat{j}, \hat{k}$ are quaternion.

Using these we define a new set of γ matrices called θ matrices as follows :

$$\theta^0 = \begin{pmatrix} I & 0 & 0 \\ 0 & {I_b}^0 & 0 \\ 0 & 0 & -I \end{pmatrix} \qquad \text{.......................................(11)}$$

$$\theta^i = \begin{pmatrix} 0 & 0 & \sigma^i \\ 0 & {I_b}^i & 0 \\ -\sigma^i & 0 & 0 \end{pmatrix} \qquad \text{.......................................(12)}$$

where *i = 1, 2, 3*

These θ $m$atrices also satisfy the relations (5) and (6) as satisfied by γ matrices of Dirac equation.

Substituting the θ matrices in place of γ matrices in equation (8) we obtain :

$$(\theta^{\mu} p_{\mu} - m)u(p) = 0 \qquad \text{.......................................(13)}$$

With θ matrices as defined above, u(p) are now 6*1 column matrices.

To solve equation (13) let us define u(p) as consisting of three matrices Ua, Ub and Uc i.e

$$u(p) = \begin{pmatrix} \mathrm{Ua} \\ \mathrm{Ub} \\ \mathrm{Uc} \end{pmatrix}$$

where each of Ua, Ub, Uc is a 2*1 column matrix

Substituting this in equation (13) and using (11) and (12) we obtain :

$$\begin{pmatrix} E - m & 0 & -\sigma.p \\ 0 & E{I_b}^0 - m - I_b.p & 0 \\ \sigma.p & 0 & -E - m \end{pmatrix} \begin{pmatrix} Ua \\ Ub \\ Uc \end{pmatrix} = 0$$

This gives,

(E-m)Ua - $\sigma.p\ Uc$ = 0 ……………………………………………(14)

(E$I_b{}^0$-m-$I_b.p)Ub = 0$ ……………………………………………(15)

σ.p Ua-(E+m)Uc=0 ……………………………………………(16)

from (14) and (16) we obtain :

Ua = $[\sigma.p/(E-m)]\ Uc$ ……………………………………………(17)

Uc = $[\sigma.p/(E+m)]\ Ua$ ……………………………………………(18)

It may be noted that these equations are identical to the solutions obtained in the original Dirac equation with γ matrices and thus with substitution successively of $\binom{1}{0}$ and $\binom{0}{1}$ for Ua and Uc one obtains Dirac spinors for fermions as the solutions. Thus usual solutions of Dirac equation emerge as solutions of the modified equation (13).

Now, putting p =0 (for particle at rest) in equation (15) we obtain :

(E$I_b{}^0$-m) Ub =0 ………………………………(19)

Wherein Ub is a 2*1 column matrix. On substituting for $I_b{}^0$ in equation (15) we get :

$$\begin{pmatrix} i\hat{k}\,E-m & 0 \\ 0 & -i\hat{k}\,E-m \end{pmatrix} Ub = 0 \qquad \text{………………………………(20)}$$

This gives E = $\pm m$ corresponding to particles and antiparticles respectively.

For the general case when particle has momentum p let us analyze separately the term containing p in (15) i.e. $I_b.p$, we obtain :

$I_b.p$ = √[$I_b.p$]^2 = SQRT [$\begin{pmatrix} \hat{\jmath}\,\mathrm{Pz} & i\mathrm{Px}-\mathrm{Py} \\ i\mathrm{Px}+\mathrm{Py} & -\hat{\jmath}\mathrm{Pz} \end{pmatrix} \begin{pmatrix} \hat{\jmath}\,\mathrm{Pz} & i\mathrm{Px}-\mathrm{Py} \\ i\mathrm{Px}+\mathrm{Py} & -\hat{\jmath}\mathrm{Pz} \end{pmatrix}$]

On further solving,

$I_b.p$ = SQRT [$\begin{pmatrix} -(\mathrm{Pz}^2+\mathrm{Py}^2+\mathrm{Px}^2) & 0 \\ 0 & -(\mathrm{Pz}^2+\mathrm{Py}^2+\mathrm{Px}^2) \end{pmatrix}$]

$I_b.p$ = SQRT [$\begin{pmatrix} -p^2 & 0 \\ 0 & -p^2 \end{pmatrix}$]

$I_b.p = \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}$

Substituting this in equation (15) along with the value of I_b^0 we obtain :

$$\begin{pmatrix} i\hat{k}\,E - m & p \\ -p & -i\hat{k}\,E - m \end{pmatrix} Ub = 0 \qquad \text{..............................(21)}$$

This gives the familiar relationship E = $\pm$ $[m^2+p^2]^{1/2}$ corresponding to positive and negative energy solutions for particles and anti-particles respectively .

Writing Ub = $\begin{pmatrix} x \\ y \end{pmatrix}$ in (21), we obtain :

$(i\hat{k}\,E - m)$ x + py = 0

-px – $(i\widehat{k + E}m)$ y = 0

or

x = -py/ $(i\hat{k}\,E - m)$

y = -px /($i\hat{k}\,E + m$)

For non-trivial case , successively putting y =1 and x =1 in these equations we obtain the solutions :

Ub = $\begin{pmatrix} -\mathrm{p}/\ (i\hat{k}\,E - m) \\ 1 \end{pmatrix}$ for negative energy, representing anti-particles

Ub = $\begin{pmatrix} 1 \\ -\mathrm{p}\ /(\ i\hat{k}\,E + m) \end{pmatrix}$ for positive energy, representing particles.

It is observed that with the new definition of γ matrices – the θ matrices as defined by equations (11) and (12), in addition to usual Dirac spinors as solutions, we also obtain a solution Ub satisfying equation (15). Ub admits only one state respectively for positive and negative energies – i.e. it has only one component each for positive and negative energy. It is a scalar with the same mass m as the Dirac spinor.

It is conjectured that Ub is the unbroken super-symmetric boson corresponding to the super-symmetric partner of the fermion obeying the usual Dirac equation.

Thus modification of γ matrices as above can generate an additional scalar solution of Dirac equation and this solution may correspond to the unbroken super-symmetric partner of the corresponding fermion. It is however noted that this requires addition of quaternion coordinates $\hat{\jmath}$ and $\hat{k}$.

References :

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