# Half-Integer Fractional Dirac Magnetic Monopole Charges without Observable Singularities for Tidally-Locked Electron Wavefunctions

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*November 17, 2015; Revised November 27, 2015*

PACS: 11.15.-q; 14.80.Hv

*Abstract:*

*It has long been believed that to avoid unphysical observable string singularities, Dirac monopoles must be quantized in whole integers according to the Dirac Quantization Condition $2eg=n$, where $e$ and $g$ are the electric and magnetic charge strengths respectively, and $n$ is an integer. This is in fact true if the electron wavefunction is not rotated while it traverses a single complete $2\pi$ circuit about the monopole. But it is also well-known that when a spinor undergoes a rotation through $2\pi$, the sign of that spinor is reversed yielding an opposite "version" of that spinor, and that the original sign and version are only restored after a $4\pi$ double rotation. Consequently, it is shown here that when an electron wavefunction is rotated in a tidal lock with the monopole during a single $2\pi$ circuit, and specifically due to the version change that occurs because of this tidally-locked rotation, to avoid unphysical singularities the Dirac condition must change from the usual whole integer condition to a half-integer condition $2eg=n-½$. It is also shown how these half-integer charges would only be detectable by electrically-charged fermions, not bosons.*

## Contents

# 1. Introduction

In 1931 Paul Dirac [1] discovered that if magnetic charges with strength $g$ were to hypothetically exist, this would imply that the electric charge strength $e$ must be quantized. The relationship he found, often written as $2eg = n$ where $n$ is a positive or negative integer or zero, has since come to be known as the Dirac Quantization Condition (DQC). The electric charge strength $e$ in this relationship is the same one which is related to the "running" fine structure coupling via $\alpha = e^2 / 4\pi\hbar c$ which at low probe energy asymptotically approaches the numerical value $\alpha = e^2 / 4\pi\hbar c \cong 1/137.036$, see, e.g., equation [1] in Dirac's [1] (which uses Gaussian units) and Witten's [2], pages 27 and 28.

In the mid-1970s, to remediate the fiction of Dirac's "nodal lines" which subsequently became known as Dirac strings, Wu and Yang [3], [4] developed an approach which achieves completely equivalent results "without strings." The only difference is that this approach is cast in the more-modern language of fiber bundles. In the Wu Yang approach, one uses $U(1)_{em}$ gauge theory to obtain the differential equation $e^{-i\Lambda}de^{i\Lambda} = i2egd\varphi$ (to be derived at (4.2) infra) where $\Lambda$ is the gauge (really, phase) angle and $\varphi$ is the geometric azimuth about the z-axis in the three dimensional physical space of the rotation group SO(3). This equation is easily seen to be solved for constant electric and magnetic charge strengths $de = dg = 0$ by $\exp(i\Lambda) = \exp(i2eg\varphi)$ (as seen at (4.3) infra).

It has long been believed that the only Wu-Yang solution which is free of unphysical observable singularities is $2eg = n$ of the standard DQC (to be derived at (4.7) infra). This is in fact true if the electron wavefunction $\psi$ is not rotated while – to use Dirac's language – it "goes round a closed curve" of $2\pi$ on the SO(3) space about the monopole. However, if the wavefunction is also rotated in a "tidal lock" with the monopole while traversing this $2\pi$ circuit and so itself undergoes a $2\pi$ rotation during this circuit, then its "version" will reverse sign following the completion of this circuit, as taught in section 41.5 of Misner, Thorne and Wheeler's (MTW) definitive work [5]. Consequently, as will be shown in section 5 here, in order to avert observable singularities, these tidally-locked circuits *must* have the half-integer charges $2eg = n - \frac{1}{2}$ derived in (5.14) infra to compensate for this version sign reversal, rather than the usual integer charges $2eg = n$ of the standard DQC.

The only known circumstance in nature under which half-integer charges are observed, is at ultra-low temperatures near 0K in connection with the Fractional Quantum Hall Effect (FQHE). In this environment, fractional fill factors $\nu = n/2$ are in fact experimentally observed, see e.g., [6], [7] for $\nu = 1/2$, [8] for $\nu = 3/2$, [9] for $\nu = 5/2$ and [10] for $\nu = 7/2$. Consequently, the question is raised whether the half-integer fractions found here for wavefunctions tidally-locked to Dirac monopoles might have some connection to these observed half-integer FQHE fractions.

## 2. Local U(1)$_{em}$ Gauge Transformations, in General

We begin by considering a first electron wavefunction $\psi_+(x^\mu)$ which is related to a second electron wavefunction $\psi_-(x^\mu)$ by the local U(1)$_{em}$ gauge transformation (throughout, except in certain particular circumstances, we shall employ natural units $\hbar = c = 1$):

$$\psi_+ \to \psi'_+ = \exp(i\Lambda)\psi_+ \equiv \psi_- , \tag{2.1}$$

where phase angle $\Lambda(x^\mu)$ varies locally as a function of the spacetime coordinates $x^\mu$ as do the wavefunctions $\psi(x^\mu)$. The transformation (2.1) is often written simply as $\psi \to \psi' = \exp(i\Lambda)\psi$, but by placing the label $\psi_+$ on $\psi$ and then $\psi_-$ on $\psi_- \equiv \psi'_+$, we lay the foundation for easily introducing the "north" and "south" gauge patches used to study monopoles starting in section 3.

Next, we define a gauge potential $A_{+\mu}(x^\mu)$ to be an electromagnetic vector potential corresponding with the wavefunction $\psi_+$, and we then use this to define the gauge-covariant derivative $D_{+\mu} \equiv \partial_\mu + ieA_{+\mu}$ where $e$ is the (running) electric charge strength, and where the sign of $ieA_{+\mu}$ is positive in this derivative because we are using a Minkowski metric tensor $\text{diag}(\eta_{\mu\nu}) = (1,-1,-1,-1)$, versus the oppositely-signed convention. Applying this derivative to each side of $\exp(i\Lambda)\psi_+$ in (2.1), we obtain:

$$\begin{aligned} &D_{+\mu}\left(\exp(i\Lambda)\psi_+\right) = \left(\partial_\mu + ieA_{+\mu}\right)\left(\exp(i\Lambda)\psi_+\right) \\ &= i\partial_\mu\Lambda\exp(i\Lambda)\psi_+ + \exp(i\Lambda)\partial_\mu\psi_+ + ieA_{+\mu}\exp(i\Lambda)\psi_+ . \\ &= \exp(i\Lambda)\left(\partial_\mu\psi_+ + \left[ieA_{+\mu} + i\partial_\mu\Lambda\right]\psi_+\right) \end{aligned} \tag{2.2}$$

Based on the inner square-bracketed expression in the bottom line above, we define a second, transformed gauge potential $A_{-\mu} \equiv A'_{+\mu}$ corresponding with the wavefunction $\psi_- \equiv \psi'_+$ by:

$$eA_{-\mu} = eA_{+\mu} + \partial_\mu\Lambda . \tag{2.3}$$

Then, defining a second gauge-covariant derivative $D_{-\mu} \equiv \partial_\mu + ieA_{-\mu}$, (2.2) simplifies to:

$$D_{+\mu}\left(\exp(i\Lambda)\psi_+\right) = \exp(i\Lambda)\left[\partial_\mu + ieA_{-\mu}\right]\psi_+ = \exp(i\Lambda)D_{-\mu}\psi_+ . \tag{2.4}$$

The foregoing represent a fundamental proposition of local gauge theory: the local gauge transformation (2.1) acting on a fermion $\psi$ must be compensated by the introduction of a gauge fields $A_\mu$ transforming according to (2.3) in order to maintain the local gauge invariance of the

electrodynamic Lagrangian and its related field equations. The logical consequence of this proposition is Maxwell's electrodynamics.

The gauge transformation (2.3) may readily be divided through by $e$ and rewritten using the mathematical identity $i\partial_\mu \Lambda = e^{-i\Lambda}\partial_\mu e^{i\Lambda}$ as:

$$A_{-\mu} = A_{+\mu} + e^{-i\Lambda}\partial_\mu e^{i\Lambda} / ie. \tag{2.5}$$

Further, one may generally pack a vector potential into the differential one-form $A = A_\mu dx^\mu$. Therefore (2.5) compacts and rearranges into:

$$A_- - A_+ = e^{-i\Lambda} de^{i\Lambda} / ie. \tag{2.6}$$

This tells us that these two gauge fields $A_-$ and $A_+$ differ from one another by no more than a generalized $U(1)_{em}$ gauge transformation, which is apparent because these are just relabeled names for the one-forms $A$ and $A'$ transforming according to $A' = A + e^{-i\Lambda} de^{i\Lambda} / ie$. Therefore, these two gauge fields are not observably-distinct.

## 3. A Coulomb Magnetic Field which is the Curl of a Vector Potential, i.e., a $U(1)_{em}$ Magnetic Monopole

The electromagnetic field strength two-form $F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$ where $F_{\mu\nu}$ is the field strength tensor / bivector is generally related to the vector potential one-form $A$ by $F = dA$ and so is a locally-exact two-form. The space components $F_{ij} = \partial_i A_j - \partial_j A_i$ are related to the magnetic field vector $B^k = \mathbf{B} = \left(B_x, B_y, B_z\right)$ as represented in Cartesian coordinates, by $F_{ij} = -\varepsilon_{ijk} B^k$, where $\varepsilon_{ijk}$ is the antisymmetric Levi-Civita tensor with $\varepsilon_{123} = +1$. Likewise, using $\text{diag}\left(\eta_{\mu\nu}\right) = \left(1, -1, -1, -1\right)$ to lower indexes in $A^\mu = \left(\phi, \mathbf{A}\right) = \left(\phi, A_x, A_y, A_z\right)$, and with $\partial_i = \boldsymbol{\nabla} = \left(\partial_x, \partial_y, \partial_z\right)$, this means that $F_{ij} = -\varepsilon_{ijk} B^k = \partial_i A_j - \partial_j A_i$, or $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. So whenever we have a field strength $F = dA$ for a given potential $A$, the magnetic field $\mathbf{B}$ will be the curl of the vector potential, $\boldsymbol{\nabla} \times \mathbf{A}$.

Now, to begin a review of magnetic monopole physics, let us define the two four-vector potentials in $A_-$ and $A_+$ of the last section *such that* these are the potentials for a *Coulomb magnetic field* $\mathbf{B}$ which is the curl of the space components these vector potentials, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. That is, let us now define the gauge potentials for a magnetic monopole. We do this by simply postulating the differential forms for these monopole potentials, then showing that these forms do in fact reproduce a Coulomb magnetic field with $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.

We start by positing a (running) magnetic charge strength $g$ for such a monopole, and we then postulate each of the potential one-forms $A_-$ and $A_+$ in a spherical coordinate basis to be:

$$\begin{aligned} A_+ &\equiv g\left(\cos\theta - 1\right)d\varphi \\ A_- &\equiv g\left(\cos\theta + 1\right)d\varphi \end{aligned}. \tag{3.1}$$

Confining our domain to $0 \le \theta \le \pi$, $A_+$ is "northerly" because it is defined everywhere except for $\theta = \pi$, i.e., except due south of the origin, while $A_-$ is a "southerly" potential defined everywhere except for $\theta = 0$, i.e., except due north of the origin. These undefined regions are the Dirac string singularities. But the union of the regions in which $A_\pm$ are well-defined covers the entirety of the SO(3) space of $\mathbb{R}^3$ about the monopole. Often these vector potentials are referred to as the north and south gauge patches, $A_N \equiv A_+$ and $A_S \equiv A_-$. Making this identification, we see via (2.6) that these differ from one another simply by a gauge transformation and so are not observably-distinct. We now show that these will indeed produce a Coulomb magnetic field for which the curl $\mathbf{B} = \nabla \times \mathbf{A}$ for both of the vector potentials $\mathbf{A}_+$, $\mathbf{A}_-$.

First, although we must generally regard *g* as a running magnetic charge strength, for the present analysis let us hold *g* constant, $dg = 0$. That is, we shall not let *g* run over the region of spacetime under consideration, or more precisely, we shall consider a region of spacetime within which any running of *g* may be neglected. Because differential forms geometry teaches that the exterior derivative of an exterior derivative is zero, $dd = 0$ in general, and thus $dd\varphi = 0$ in this specific setting, this all means when we operate on (3.1) with *d* that:

$$F = dA_- = dA_+ = gd\cos\theta d\varphi. \tag{3.2}$$

Therefore, based on what was discussed in the first paragraph of this section, for either potential in (3.1) the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}_+ = \nabla \times \mathbf{A}_-$ is the curl of the gauge potential, as desired.

Of course, $dF = ddA_- = ddA_+ = 0$ via the same identity $dd = 0$, which means that *F* is closed and locally exact. But it is not globally exact. Specifically, if we integrate (3.2) over a closed two-dimensional nonlocal surface with *g* still held constant, and if we also apply Gauss' / Stokes' theorem, then:

$$\iiint dF = \oiint F = \oiint gd\cos\theta d\varphi = g\int_0^\pi d\cos\theta \int_0^{2\pi} d\varphi = g\cos\theta\Big|_0^\pi \varphi\Big|_0^{2\pi} = -4\pi g. \tag{3.3}$$

The fact that the region of spacetime is posited to be one in which any running of *g* may be neglected thus *g* is constant and $dg = 0$ is reflected by our having moved *g* outside the integral after the third equal sign above. Now let us specifically pinpoint the magnetic field.

To do so, we consider the circumstance under which the *electric* fields vanish, that is, under which the electric field vector $F_{0k} = -F_{k0} = \mathbf{E} = 0$. Here, $\oiint F = \oiint \tfrac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \oiint \tfrac{1}{2} F_{ij} dx^i dx^j$. Then, using this in (3.3) also in view of the earlier-noted $F_{ij} = -\varepsilon_{ijk} B^k$, we find that:

$$\oiint F = \oiint \tfrac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \oiint F_{12} dx^1 dx^2 + \oiint F_{23} dx^2 dx^3 + \oiint F_{31} dx^3 dx^1 = -\oiint \mathbf{B} \cdot \mathbf{dS} = -4\pi g \,. \qquad (3.4)$$

So from the final equality above, this means that:

$$\oiint \mathbf{B} \cdot \mathbf{dS} = 4\pi g = \mu \,, \qquad (3.5)$$

where $\mu \equiv 4\pi g$ is defined as the total magnetic field flux across the closed surface. Conversely, the magnetic charge strength $g = \mu / 4\pi$ represents the steradial density of magnetic flux across the closed surface. This, of course, is Gauss' law for magnetism in integral form, but with a non-zero magnetic flux $\mu$ across the closed surface. Consequently, this is the integral formulation of Gauss' law for a non-vanishing magnetic monopole. Because this was arrived at using **E**=0 in (3.4), (3.5), there are no *electric* fields induced by this monopole, and as a result, (3.5) describes this magnetic monopole at rest.

Now, in general, Coulomb's law cannot be derived from Gauss' law alone. However, if the magnetic monopole is stationary – which it is because $\mathbf{E} = 0$ in (3.4) and (3.5) – then the magnetic field **B** in (3.5) will be exactly spherically symmetric. As a result of this spherical symmetry, we can center the coordinates at the monopole so only the radial component $B_r$ of **B** will be non-zero, that is, in spherical coordinates, so that $\mathbf{B} = \left(B_r, B_\varphi, B_\theta\right) = \left(B_r, 0, 0\right)$. Also because of this spherical symmetry, we may remove **B** from the integrand in (3.5). Thus, using a spherical surface $\oiint \mathbf{dS} = 4\pi r^2$ centered about the monopole, we may now write the above as:

$$\oiint \mathbf{B} \cdot \mathbf{dS} = \mathbf{B} \oiint \mathbf{dS} = B_r \cdot 4\pi r^2 = 4\pi g = \mu \,. \qquad (3.6)$$

Finally, (3.6) is easily rearranged to yield:

$$B_r = \frac{g}{r^2} = \frac{\mu}{4\pi r^2} \,. \qquad (3.7)$$

This is indeed a Coulomb magnetic field which has a (constant) magnetic charge strength *g*, and for which the total magnetic flux across any closed surface is $\mu = 4\pi g$. Furthermore, this Coulomb magnetic field is the curl of the vector potentials, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}_+ = \boldsymbol{\nabla} \times \mathbf{A}_-$ as demonstrated at the start of this section. Consequently, we have completed our review of how the potentials postulated in (3.1) do in fact specify a non-vanishing Coulomb magnetic field with $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.

Now, we begin to examine what is required to ensure that this Coulomb magnetic monopole with $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ does not give rise to any observable singularities.

## 4. Conditions under which the U(1)$_{em}$ Magnetic Monopole has No Observable Singularities: The Standard Dirac Quantization Condition

Returning to (3.1), we first find that the difference between the north and south gauge patches:

$$A_- - A_+ = 2gd\varphi . \tag{4.1}$$

Combining the above with the gauge transformation (2.6) then yields the Wu-Yang [3], [4] differential equation:

$$e^{-i\Lambda}de^{i\Lambda} / ie = 2gd\varphi . \tag{4.2}$$

This differential equation is solved for constant *e* and constant *g*, i.e., for $de = 0$ and $dg = 0$ by:

$$\exp(i\Lambda) = \exp(i2eg\varphi), \tag{4.3}$$

as is easily seen by plugging (4.3) back into the left hand side of (4.2) then reducing.

We next employ this solution to operate on $\psi_+$ from the left, and combine this with (2.1), which yields:

$$\psi_+ \to \psi'_+ = \psi_- = \exp(i\Lambda)\psi_+ = \exp(i2eg\varphi)\psi_+ . \tag{4.4}$$

Clearly, for $\varphi = 0$, this yields $\psi_-(0) = \psi_+$, using the notation $\psi(\varphi)$ to denote the wavefunctions at a particular azimuthal disposition. Now, following the course first charted by Dirac, let us move this wavefunction through the Coulomb magnetic field of (3.7) around a closed curve in the azimuthal direction, going from $\varphi = 0$ to $\varphi = 2\pi$. When this single circuit about the monopole is complete, from (4.4) with $\varphi = 2\pi$ we obtain:

$$\psi_+ \to \psi_- = \psi'_+ = \exp(i\Lambda)\psi_+ = \exp(i2eg\cdot 2\pi)\psi_+ = \exp(i4\pi eg)\psi_+ . \tag{4.5}$$

This says that $\psi_-(2\pi) = \exp(i4\pi eg)\psi_+$. Now let's turn to the question of observable singularities.

To avoid observable singularities, it is required that the electron wavefunction at $\varphi = 2\pi$ be the same identical wavefunction as it is at the geometrically identical azimuth $\varphi = 0$ on SO(3). In other words, it is a requirement that the wavefunction be defined so as to have the single value $\psi_+(0) \to \psi_+(2\pi) \equiv \psi_+(0)$ and *not have multiple values at the same azimuthal orientation* on SO(3), see, for example, [11]. This requirement will be satisfied *if and only if*:

$$\psi_+ \to \psi_- = \psi'_+ = \exp(i\Lambda)\psi_+ = \exp(i4\pi eg)\psi_+ \equiv 1\cdot\psi_+ = \exp(i2\pi n)\psi_+ , \tag{4.6}$$

where we make use of the identity $1=\exp\left(i2\pi n\right)$ for $n=0,\pm1,\pm2,\pm3,\pm4...$, i.e., for all positive or negative integers or zero. From $\exp\left(i4\pi eg\right)\psi_{+}=\exp\left(i2\pi n\right)\psi_{+}$, we see that this will occur *if and only if* $4\pi eg=2\pi n$, or more simply:

$$2eg=n. \tag{4.7}$$

This is the Dirac Quantization Condition. In the language of fiber bundles, this all shows how the electromagnetic field is described by a 2-form with integral periods, which is precisely the curvature of a connection on a principal U(1)-bundle, again, see, e.g., [11].

From (4.7), defining the $n=1$ charge units as $e_0\equiv 1/2g$ and $g_0\equiv 1/2e$, we see that the respective electric and magnetic charge strengths are reciprocally quantized by:

$$\begin{aligned} e&=n/2g=ne_0\\ g&=n/2e=ng_0 \end{aligned}, \tag{4.8}$$

Now let's examine the phase behavior. With the Dirac condition $2eg=n$ of (4.7) imposed, (4.4) now becomes:

$$\psi_{+}\rightarrow\psi_{-}=\psi'_{+}=\exp\left(i\Lambda\right)\psi_{+}=\exp\left(in\varphi\right)\psi_{+}, \tag{4.9}$$

which contains the implied quantized relationship:

$$\Lambda=n\varphi \tag{4.10}$$

between the phase angle $\Lambda$ and the azimuth angle $\varphi$. Of course, an *absolute* phase itself is not an observable; all that may be observed is a *change* in phase between two points which we shall denote with a $\Delta$ subscript as $\Lambda_{\Delta}$. So, we may ask, what change in phase occurs after the wavefunction traverses an azimuthal circuit from $\varphi=0$ to $\varphi=2\pi$? For this, we merely insert $\varphi=2\pi$ into (4.10) to find that what Dirac often refers to in [1] as the observable "*change* in phase round" a "closed curve" is:

$$\Lambda_{\Delta}=2\pi n=2\pi,4\pi,6\pi,8\pi.... \tag{4.11}$$

Thus, if we start with a wavefunction at $\varphi=0$ in physical space and assign some unobservable arbitrary angle to the phase $\Lambda$, then after traversing a single circuit to $\varphi=2\pi$ which leads to the same azimuthal orientation in physical space, the observable phase difference will be $\Lambda_{\Delta}=2\pi n$. Thus the phase will likewise have returned to precisely the same angular orientation in the phase space $\exp\left(i\Lambda\right)=\cos\Lambda+i\sin\Delta$ that it had at the start. Likewise, as imposed at (4.6),

there will be no observable singularities, because the wavefunction will maintain the single value $\psi_+(2\pi)=\psi_+(0)$ at both of the $\varphi=0$ and $\varphi=2\pi$ azimuthal orientations.

Using the Dirac quantization condition (4.7) we may finally return to (3.1) to write the monopole potentials as:

$$\begin{array}{l} eA_+ = \frac{1}{2}n(\cos\theta-1)d\varphi \\ eA_- = \frac{1}{2}n(\cos\theta+1)d\varphi \end{array}. \qquad (4.12)$$

All of the foregoing summarizes the present-day understanding of $U(1)_{em}$ magnetic monopoles and the Dirac Quantization Condition $2eg=n$ of (4.7) which is understood to be required if these monopoles are to exist without observable singularities. A very good, parallel review of the above can be studied at [12], which serves the beneficial purpose of clarifying and detailing how the gauge field approach presented above relates to the modern mathematics of fiber bundles. Note that the $A_\pm$ utilized in [12] employ an opposite sign convention from that used here.

Although the Dirac charges $2eg=n$ of (4.7) are presently thought to be the *only* monopole charges that can exist in the natural world without observable singularity, we shall now demonstrate that if the wavefunction in rotated in a tidal lock with the postulated magnetic monopole while it traverses the monopole from $\varphi=0$ to $\varphi=2\pi$, then in order to avoid observable singularities, the Dirac charges must now possess half-integer rather than whole-integer charge quanta.

## 5. Tidally-Locked Electron Wavefunctions and Half-Integer Fractional Monopole Charges

In the derivation of the Dirac Quantization Condition just reviewed, there is an unstated assumption that the electron wavefunction, over the course of traversing its circuit about the monopole from $\varphi=0$ to $\varphi=2\pi$, it not itself undergoing any rotation. But now let us examine what happens if the electron itself rotates in a "tidal lock" with the monopole as it traverses the monopole, so that in the course of traversing from $\varphi=0$ to $\varphi=2\pi$ about the monopole the electron also rotates through $2\pi$ via SU(2) which is the universal cover of the rotation group of SO(3). This is analogous, albeit in the quantum world, to what the moon does when it traverses the earth such that the far side of the moon is never visible from earth. As we shall now see, with such a tidal lock, to avoid observable singularities, the Dirac charge condition must now become a half-integer rather than a whole-integer condition.

We begin with the three 2x2 Pauli matrices $\sigma_i$ of SU(2), posit three associated angles $\theta_i$ in the physical space of spacetime, and form the matrices $U_i=\exp(i\sigma_i\theta_i/2)$ which are unitary, $U^\dagger U=1$, given that $\sigma_i^\dagger=\sigma_i$ are Hermitian. These $U_i$ matrices are used to transform spinors, and when projected via the two-to-one, double-covered, homomorphic, universal covering map $\pi: SU(2)\to SO(3)$ onto physical space, result in rotations through respective angles $\theta_i=\theta_x,\theta_y,\theta_z$

about each of the x, y, z axes in the three-dimensional physical SO(3) space often denoted as $\mathbb{R}^3$. It is well-known how to make use of the series $\exp(ix)=1+ix-\frac{1}{2!}x^2-i\frac{1}{3!}x^3+\frac{1}{4!}x^4...$ together with the fact that $\sigma_i^{2n}=I_i$ and $\sigma_i^{2n+1}=\sigma_i$ to flesh out these unitary matrices, each of which has $\det U_i=1$, into:

$$\begin{aligned}
U_1 &= \exp\left(i\sigma_1\frac{\theta_1}{2}\right)=\begin{pmatrix}\cos(\theta_1/2) & i\sin(\theta_1/2)\\ i\sin(\theta_1/2) & \cos(\theta_1/2)\end{pmatrix}\\
U_2 &= \exp\left(i\sigma_2\frac{\theta_2}{2}\right)=\begin{pmatrix}\cos(\theta_2/2) & \sin(\theta_2/2)\\ -\sin(\theta_2/2) & \cos(\theta_2/2)\end{pmatrix}\\
U_3 &= \exp\left(i\sigma_3\frac{\theta_3}{2}\right)=\begin{pmatrix}\cos(\theta_3/2)+i\sin(\theta_3/2) & 0\\ 0 & \cos(\theta_3/2)-i\sin(\theta_3/2)\end{pmatrix}\\
&=\begin{pmatrix}\exp(i\theta_3/2) & 0\\ 0 & \exp(-i\theta_3/2)\end{pmatrix}
\end{aligned}. \tag{5.1}$$

Continuing with the natural units $\hbar=c=1$ let us next consider an electron traveling with velocity $\beta=v$ along the z axis and thus the Lorentz contraction factor $\gamma=1/\sqrt{1-v^2}$. As is often done, we may then define the boost parameters $\cosh\chi\equiv\gamma$ and $\sinh\chi\equiv\gamma\beta$, and write the Lorentz transformation between the time coordinate *t* and the *z* coordinate using the hyperbolic "rotation":

$$\begin{pmatrix}t\\ z\end{pmatrix}\rightarrow\begin{pmatrix}t'\\ z'\end{pmatrix}=\begin{pmatrix}\cosh\chi & \sinh\chi\\ \sinh\chi & \cosh\chi\end{pmatrix}\begin{pmatrix}t\\ z\end{pmatrix}. \tag{5.2}$$

Several of the points to now be developed are found in Ryder's [13], amidst pages 36 to 42.

The electron wavefunction $\psi$ is a four-component Dirac spinor which we can denote by $\psi^T=(\xi^T,\eta^T)$, where $\xi$ and $\eta$ are each two-component spinors with all components interrelated via Dirac's equation $(i\gamma^\mu\partial_\mu-m)\psi=0$. Under a transformation (5.2) defined by the Lorentz group SO(1,3), which includes a general boost $\boldsymbol{\chi}$ and spatial rotation through $\boldsymbol{\theta}$ on SO(1,3), these spinor components will transform on SL(2,C) according to:

$$\psi=\begin{pmatrix}\xi\\ \eta\end{pmatrix}\rightarrow\psi'=\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}=\begin{pmatrix}\exp\left(i\boldsymbol{\sigma}\cdot(\boldsymbol{\theta}-i\boldsymbol{\chi})/2\right) & 0\\ 0 & \exp\left(i\boldsymbol{\sigma}\cdot(\boldsymbol{\theta}+i\boldsymbol{\chi})/2\right)\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \tag{5.3}$$

where $\boldsymbol{\sigma}=\sigma_i$ are the 2x2 Pauli matrices. So for a non-relativistic electron with $\boldsymbol{\chi}\rightarrow 0$ undergoing simply a rotation without boost, this simplifies to:

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \psi' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} \exp\left(i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2\right) & 0 \\ 0 & \exp\left(i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2\right) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = I_{(2)} \otimes \exp\left( i\boldsymbol{\sigma}\cdot\frac{\boldsymbol{\theta}}{2} \right) \psi \,. \tag{5.4}$$

Here, $I_{(2)}$ is a 2x2 identity matrix and the outer product symbol $\otimes$ is used to compactly represent that $I_{(2)} \otimes \exp\left(i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2\right)$ is a 4x4 matrix formed by placing the 2x2 $\exp\left(i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2\right)$ based on (5.1), on the diagonal twice. For an azimuthal rotation through $\theta_3 = \varphi$ about the z axis only, this becomes $\psi \rightarrow \psi' = I_{(2)} \otimes \exp\left(i\sigma_3\varphi/2\right)\psi = I_{(2)} \otimes U_3\psi$, for which the unitary matrix $U_3$ is explicitly given by the third relation in (5.1) with $\theta_3$ replaced by $\varphi$. Thus, for an azimuthal rotation only, for a non-relativistic electron, the $\exp\left(i\sigma_3\varphi/2\right)$ term in (5.4) will operate identically upon each of the two-spinors $\xi, \eta$. So for the upper spinor $\xi_T = \left(\xi_A, \xi_B\right)$, using $U_3$ from (5.1), the transformation in (5.4) will be:

$$\xi \rightarrow \xi' = U_3\xi = \exp\left( i\sigma_3\frac{\varphi}{2} \right)\xi = \begin{pmatrix} \cos\left(\varphi/2\right) + i\sin\left(\varphi/2\right) & 0 \\ 0 & \cos\left(\varphi/2\right) - i\sin\left(\varphi/2\right) \end{pmatrix} \begin{pmatrix} \xi_A \\ \xi_B \end{pmatrix}. \tag{5.5}$$

For the lower spinor $\eta_T = \left(\eta_A, \eta_B\right)$ the operation is a carbon copy of (5.5) but with the symbol $\xi$ replaced throughout by the symbol $\eta$. The requirement to maintain the two spinors $\xi$ and $\eta$ together within the four-component Dirac wavefunction $\psi^T = \left(\xi^T, \eta^T\right)$ arises because these are interchanged $\xi \leftrightarrow \eta$ under parity. But when the boost is removed the overall $\psi$ as well as each of $\xi$ and $\eta$ will transform in identical fashion and so may be separately considered.

We finally consolidate the transformation (5.5) on both $\xi, \eta$ into one expression by representing the third 2x2 matrix (5.1) compactly as $U_3 \Rightarrow \cos\left(\varphi/2\right) \pm i\sin\left(\varphi/2\right) = \exp\left(\pm i\varphi/2\right)$ while also using the $\psi_+$ labelling of (2.1) to recast $\psi \Rightarrow \psi_+$, and thus also recast $\xi \Rightarrow \xi_+$ and $\eta \Rightarrow \eta_+$, yielding:

$$\psi_+ = \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} \rightarrow \psi'_+ = \begin{pmatrix} \xi'_+ \\ \eta'_+ \end{pmatrix} = I_{(2)} \otimes \exp\left( i\sigma_3\frac{\varphi}{2} \right)\psi_+ = I_{(2)} \otimes U_3\psi_+$$
$$= \begin{pmatrix} \cos\left(\varphi/2\right) \pm i\sin\left(\varphi/2\right) & 0 \\ 0 & \cos\left(\varphi/2\right) \pm i\sin\left(\varphi/2\right) \end{pmatrix} \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} = \begin{pmatrix} \exp\left(\pm i\varphi/2\right) & 0 \\ 0 & \exp\left(\pm i\varphi/2\right) \end{pmatrix} \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} \tag{5.6}$$

In this compact notation, the $\pm$ signs denote the respective operations on each component of $\xi_{+T} = \left(\xi_{+A}, \xi_{+B}\right)$ and $\eta_{+T} = \left(\eta_{+A}, \eta_{+B}\right)$. This is a more explicit form of (5.4) for an azimuthal rotation with $\boldsymbol{\sigma}\cdot\boldsymbol{\theta} = \sigma_3\theta_3 = \sigma_3\varphi$, also adopting the labelling of (2.1).

Now, let us return to the gauge transformation $\psi_+ \rightarrow \psi'_+ = \exp\left(i\Lambda\right)\psi_+$ of (2.1) and contrast this against (5.6). As already noted, now quoting Dirac from page 63 of [1], "the value of [the phase] at a particular point has no physical meaning and only the difference between the values of [the phase] at two different points is of any importance." So, if we are comparing phases between two different azimuthal points (for the non-relativistic electron presently under examination), then we should also inquire whether the electron has been rotated at all when moving from one such point to the next. If the electron has not rotated but the phase has changed, then the transformation will be $\psi_+ \rightarrow \psi'_+ = \exp\left(i\Lambda\right)\psi_+$ from (2.1). Conversely, if the electron has rotated but the phase has not changed, then the transformation will be $\psi_+ \rightarrow \psi'_+ = I_{(2)} \otimes \exp\left(i\sigma_3\varphi/2\right)\psi_+$ from (5.6). But, if both the phase has changed *and* the electron has rotated, then the complete transformation will be a combination of both operations (2.1) and (5.6), namely:

$$\psi_+ \rightarrow \psi'_+ = I_{(2)} \otimes U_3 \exp\left(i\Lambda\right)\psi_+ = I_{(2)} \otimes \exp\left(i\sigma_3 \frac{\varphi}{2}\right)\exp\left(i\Lambda\right)\psi_+ = I_{(2)} \otimes \exp i\left(\sigma_3 \frac{\varphi}{2} + \Lambda\right)\psi_+ . \quad (5.7)$$

With (5.7) we are now equipped to ask what happens if the electron makes a complete circuit "round a closed curve" about the monopole through a $2\pi$ azimuth *and* simultaneously does so in a tidal lock with the monopole thus also rotating through $2\pi$, all on SO(3).

To avoid observable singularities, as in section 4, we must still have a single-valued wavefunction after the full $2\pi$ circuit is complete, that is, we must still impose $\psi'_+ = \psi_+(2\pi) \equiv \psi_+(0)$. But now, the condition required to avoid a singularity will be imposed by defining $\psi'_+ \equiv \psi_+$ when $\varphi = 2\pi$ using (5.7). So to impose the condition that the tidally-locked wavefunction be single-valued after completing a $2\pi$ circuit, we simultaneously set $\varphi = 2\pi$ in (5.7) and require $\psi'_+ \equiv \psi_+ = I_{(4)} \exp\left(i2\pi n\right)\psi_+$, which uses $1 = \exp\left(i2\pi n\right)$ as before. By setting $\varphi = 2\pi$, we are also now implicitly examining an observable phase difference as between $\varphi = 0$ and $\varphi = 2\pi$, which we again denote by replacing the absolute phase $\Lambda$ with the phase difference $\Lambda_\Delta$. Consequently, from (5.7), with these conditions, we obtain:

$$\psi_+ \rightarrow \psi'_+ = I_{(2)} \otimes \exp\left(i\sigma_3\pi\right)\exp\left(i\Lambda_\Delta\right)\psi_+ = I_{(2)} \otimes \exp i\left(\sigma_3\pi + \Lambda_\Delta\right)\psi_+ \equiv \psi_+ = I_{(4)} \exp\left(i2\pi n\right)\psi_+ . \quad (5.8)$$

Now turning the "≡" used to designate the imposing of a single-valued wavefunction into an equal sign, this will be recognized as, and may be restructured into, an eigenvalue equation:

$$\left(I_{(2)} \otimes \exp i\left(\sigma_3\pi + \Lambda_\Delta\right) - I_{(4)} \exp\left(i2\pi n\right)\right)\psi_+ = 0 \quad (5.9)$$

for the phase difference $\Lambda_\Delta$ that is introduced when going from $\varphi = 0$ to $\varphi = 2\pi$. Now, we merely need to solve this eigenvalue equation.

To simplify solving (5.9), we may use (5.1) to deduce that when $\theta_3 = \varphi = 2\pi$ as it is in (5.9), the 2x2 matrix $U_3 = \exp\left(i\sigma_3\pi\right) = -I_{(2)}$, which produces a sign reversal. So rather than solve (5.9) using explicit matrices, we may use this observation together with $I_{(2)} \otimes I_{(2)} = I_{(4)}$ to directly simplify then reduce (5.8) to:

$$\psi_+ \rightarrow \psi'_+ = -\exp\left(i\Lambda_\Delta\right)\psi_+ \equiv \psi_+ . \tag{5.10}$$

Except for the sign reversal and the fact that we are now using the notation $\Lambda_\Delta$ introduced at (4.11) to represent that this is an observable phase *difference*, this is the same as (4.6) from which we obtained the standard DQC of (4.7). This sign reversal, which is a consequence of the rotation from the tidal lock, is, however, not a trivial matter, because it changes the Dirac condition needed to avert observable singularities. Let us see how:

Now, in lieu of $1 = \exp\left(i2\pi n\right)$ used in (4.6), we use the mathematical identity $-1 = \exp\left(i\pi\left(2n-1\right)\right)$, i.e., we use the fact that the Euler formula $\exp\left(i\vartheta\right) = -1$ at angles $\vartheta = \pi, 3\pi, 5\pi... = \left(2n-1\right)\pi$ for which the coefficient of $\pi$ is an odd-integer $\left(2n-1\right) = 1,3,5....$ So now, flipping the signs in (5.10) and using this identity for -1, we have:

$$\exp\left(i\Lambda_\Delta\right)\psi_+ = -\psi_+ = \exp\left(i\pi\left(2n-1\right)\right)\psi_+ . \tag{5.11}$$

As a result, for the tidally-locked electron, we may extract from (5.11) that after a single tidally-locked $2\pi$ circuit "round a closed curve," the change in phase will be:

$$\Lambda_\Delta = \left(2n-1\right)\pi = \pi, 3\pi, 5\pi, 7\pi..., \tag{5.12}$$

which is likewise an odd-integer multiple of $\pi$, contrast (4.11) which is an even-integer multiple of $\pi$.

Most importantly, if now combine (5.11) with the Wu-Yang equation (4.4) also obtained from a single $2\pi$ circuit about the monopole, that is, if we combine (5.11) with (4.4) (with the notation $\Lambda_\Delta$ for the phase difference) for the same $\varphi = 2\pi$ azimuthal circuit, we now obtain:

$$\exp\left(i\Lambda_\Delta\right)\psi_+ = \exp\left(i\pi\left(2n-1\right)\right)\psi_+ = \exp\left(i2eg\varphi\right)\psi_+ = \exp\left(i4\pi eg\right)\psi_+ . \tag{5.13}$$

From $\exp\left(i\pi\left(2n-1\right)\right)\psi_+ = \exp\left(i4\pi eg\right)\psi_+$ above, we may finally extract $\pi\left(2n-1\right) = 4\pi eg$ which reduces to:

$$2eg = \left(2n-1\right)/2 = n - \tfrac{1}{2} = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2}..., \tag{5.14}$$

for the positive integers $n = 1, 2, 3, 4, 5....$ *This is the charge condition required to avoid observable singularities when the wavefunction traverses the monopole in a tidal lock.* Contrasting the usual DQC $2eg = n$ of (4.7), we see that to avoid observable singularities for a tidally-locked electron which rotates in synchronization with its circuit about the monopole, we must now have a Dirac quantization condition for which the charges are *half-integer charge fractions that skip over the whole integer charges.*

Under this condition, using the same charge quanta $e_0 \equiv 1/2g$ and $g_0 \equiv 1/2e$ defined after (4.8), the electric and magnetic charge strengths are now reciprocally quantized according to:

$$\begin{array}{l} e = \left(n - \frac{1}{2}\right)/2g = \left(n - \frac{1}{2}\right)e_0 \\ g = \left(n - \frac{1}{2}\right)/2e = \left(n - \frac{1}{2}\right)g_0 \end{array}, \tag{5.15}$$

Likewise, using (5.14) in (3.1), in contrast to the earlier (4.12), the vector potential one-forms are now quantized according to:

$$\begin{array}{l} eA_+ \equiv \frac{1}{2}\left(n - \frac{1}{2}\right)\left(\cos\theta - 1\right)d\varphi \\ eA_- \equiv \frac{1}{2}\left(n - \frac{1}{2}\right)\left(\cos\theta + 1\right)d\varphi \end{array}. \tag{5.16}$$

Now, let's step back to gain some perspective on what has happened here to reveal these half-integer Dirac charges.

## 6. Why the Half-Integer Dirac Charges are Simply a Consequence of Wavefunctions Changing their Version when Undergoing Rotations

In their classic exposition at section 41.5 of [5], Misner, Thorne and Wheeler (MTW) teach that a spinor will reverse sign after any $2\pi = 360°$ rotation, and will only regain its original sign after a $4\pi = 720°$ rotation. A four-component Dirac wavefunction $\psi$ houses two spinors $\xi$, $\eta$, and as reviewed in the last section, these two spinors and thus the overall wavefunction will transform identically under rotations absent boost. Thus, the entire non-relativistic Dirac wavefunction will exhibit this sign reversal after a $2\pi$ rotation. On close inspection, it will be seen that the rotation reviewed in [41.48] through [41.50] of [5] when taken about the z-axis is the same as that used in (5.5) here. We discuss this for an electron by saying that the electron changes to an oppositely-signed "version" after a $2\pi$ rotation and only recovers its original version after a $4\pi$ rotation. So if the electron is traversed through a $2\pi$ azimuthal circuit about the hypothesized magnetic monopole reviewed in section 3, and if it is tidally-locked to the monopole and thus has a rotation synchronized to this traversal, then the electron will return to its original azimuth, but with its version sign reversed. And this means that the wavefunction at this azimuth is *not single-valued but is double-valued* with a leading $\pm$ sign. *This would give rise to an unphysical observable Dirac string singularity if not compensated in some way.*

MTW analogize this version change to the macroscopic and entirely classical "orientation-entanglement" phenomenon wherein an object connected to its environment by a set of threads

will only regain its original state of entanglement after it is rotated twice over $4\pi$, but will have an opposite entanglement following only a $2\pi$ rotation. But an electron is a quantum object not a classical one, and it is not necessary here to use this macroscopic analogy. The angles $\theta_i$ in (5.1) are rotation angles in physical space which are mapped onto SO(3) through the homomorphic double-covering projection $\pi : SU(2) \to SO(3)$, and when the complete Dirac theory is taken into account, through the mapping $\pi : SL(2,C) \to SO(1,3)$. Numerically, this is encoded in the denominator of 2 first appearing in (5.3). Thus, the spinor transformation (5.5) makes very clear that the sign of the electron wavefunction will invert following a $2\pi$ rotation and only be restored after $4\pi$. Specifically, when the rotation azimuth $\varphi = 2\pi$ we have $\xi \to \xi' = U_3(2\pi)\xi = -\xi$ which will carry through to the entire non-relativistic wavefunction $\psi^T = \left(\xi^T, \eta^T\right)$, but when $\varphi = 4\pi$ we have $\xi \to \xi' = U_3(4\pi)\xi = \xi$ which restores the sign to its original value. All of this is well-known and well-settled physics. Indeed, this two-valued $\pm$ version sign is directly related to the double covering of SO(3) by its *simple* universal cover SU(2). *But the question of what happens in Dirac monopole theory when electron wavefunctions are tidally-locked to a postulated magnetic monopole and so undergo this well-known version change after executing a $\varphi = 2\pi$ circuit does not appear to have been previously considered in the literature.*

Because a version charge is simply a sign change, this may be encoded in the identity $-1 = \exp\left(i\pi\left(2n-1\right)\right)$, which represents the primitive square root of unity using Euler's formula $\exp\left(i\vartheta\right) = -1$ at angles $\vartheta = \pi, 3\pi, 5\pi... = \left(2n-1\right)\pi$ which are oriented at the Euler angle $\vartheta = \pi = 180°$ and at angles differing from this simply by integer multiples of $2\pi$. Indeed, the two signs of the wavefunction versions taught by MTW may be illustratively represented in the simplest and most transparent form by writing the square roots of unity using Euler angles as:

$$\sqrt[2]{1} = \pm 1 = \exp\left(i\vartheta\right) = \begin{cases} \exp\left(i\pi\left(2n\right)\right) & \text{for } \vartheta = 2\pi\text{n} \\ \exp\left(i\pi\left(2n-1\right)\right) & \text{for } \vartheta = \pi\left(2n-1\right) \end{cases}. \tag{6.1}$$

So to maintain the single-valued wavefunction $\psi'_+ = \psi_+(2\pi) = \psi_+(0)$ required to avoid observable string singularities, we need to compensate for this sign change that occurs when there is a version change. When calculated through, this compensation is reflected and absorbed into the phase difference $\Lambda_\Delta = \left(2n-1\right)\pi$ of (5.13) for a tidally-locked electron. This is in contrast to the phase difference $\Lambda_\Delta = 2\pi n$ of (4.11) required when there is no tidal lock and thus no version change. And this, in turn, finally cascades through to the requirement that the Dirac condition for tidally-locked wavefunctions must be the half-integer $2eg = \left(2n-1\right)/2 = n - \frac{1}{2}$ found in (5.14), rather than the customary whole-integer $2eg = n$ of (4.7).

There is a related way to look at all of this which focuses on the phase difference rather than the charge fraction. The result in (5.12) teaches that the phase difference for a tidally-locked wavefunction after traversing a $\varphi = 2\pi$ azimuth must be $\Lambda_\Delta = \left(2n-1\right)\pi = \pi, 3\pi, 5\pi...$, which means that the wavefunction orientation becomes $180°$ out of phase after this single azimuthal

circuit in the complex phase space of $\exp(i\Lambda)=\cos\Lambda+i\sin\Lambda$ used to perform the gauge transformation $\psi_+\rightarrow\psi'_+=\exp(i\Lambda)\psi_+\equiv\psi_-$ of (2.1) on the electron wavefunction. So, what sort of azimuthal traversal is required to restore the original phase orientation of the wavefunction? If we traverse a first $2\pi$ azimuth and then a second $2\pi$ azimuth for a total $4\pi$ circuit using $n_1$ and $n_2$ to denote the characteristic integers $n$ from the first and second traversals, then the phase differences will add together in the form of $\Lambda_\Delta=(2n_1-1)\pi+(2n_2-1)\pi=2\pi(n_1+n_2-1)=2\pi n'$, where in the final step we simply rename $n_1+n_2-1=n'$ to another integer. So after a $\varphi=4\pi$ azimuthal traversal – but not after only a $2\pi$ traversal – the phase difference becomes $\Lambda_\Delta=2\pi n'$ like that in (4.11), and so the phase returns to its original orientation. This means that in general, the *phase* for a tidally-locked electron will return to its original orientation only after circuits of $\varphi=4\pi n$, and *not* after only $\varphi=2\pi n$ circuits – *just like the wavefunction version itself.*

So, stepping back from the mathematical detail, we may summarize all of this by saying that to avoid singularities for an electron wavefunction traversing a magnetic monopole, *the wavefunction phase orientation in phase space must be synchronized to the wavefunction version orientation in physical space.* If the wavefunction does not rotate in a tidal lock during a $\varphi=2\pi$ circuit, then both the version and the phase will be restored to their original orientations in their respective spaces after the $2\pi$ azimuthal circuit is complete. However, if the wavefunction *does* rotate in a tidal lock during this circuit, then once the circuit is complete, the version will have an opposite sign, hence opposite orientation from what it had at the outset, and synchronized to this, the phase will also have an opposite orientation in the phase space. Here, both the phase and the version – synchronized to one another – will only revert to their original orientations in their respective spaces after traversing $\varphi=4\pi n$ circuits, which is an extension of the teachings of Misner, Thorne and Wheeler in section 41.5 of [5] to the Wu-Yang analysis [3], [4] of Dirac monopoles. Whether tidally-locked or not, the wavefunctions themselves remain single-valued $\psi'_+=\psi_+(2\pi)=\psi_+(0)$ after each $\varphi=2\pi n$ circuit which is required to avoid observable string singularities, and the resulting Dirac monopole charges are the half-integer $2eg=n-\frac{1}{2}$ when there is a tidal lock, and the standard whole-integer $2eg=n$ when the is no tidal lock.

## 7. Why Half-Integer Dirac Charges would only be Detectable by Fermions, not Bosons

While there are many important differences between fermions and bosons including the fact that fermions must adhere to the Dirac statistics of the Exclusion Principle whereas bosons do not, for purposes of the present discussion the most important difference is that fermion wavefunctions invert their version sign under a $2\pi$ rotation and only regain their original signs under a $4\pi$ rotation, whereas boson wavefunctions do not behave in this way. So for example, if we postulate a negatively charged "scalar electron" $\phi$ with spin zero which we denote at a particular azimuthal state of rotation by $\phi(\varphi)$, then in contrast to the non-relativistic electron $\psi(\varphi)$ reviewed in section 5 for which $\psi(2\pi)=-\psi(0)$, the scalar electron will have $\phi(2\pi)=\phi(0)$ following a $2\pi$ rotation, with no version sign reversal. At the same time, the physics must still be

symmetric under the local $U(1)_{em}$ gauge transformations reviewed in section 2, which transformations will now be $\phi \to \phi' = \exp(i\Lambda)\phi$ for our posited scalar, contrast (2.1). This will lead after the same development reviewed in section 2 to the same gauge field $A$ for which the one-form transforms as $A' = A + e^{-i\Lambda} de^{i\Lambda} / ie$, contrast (2.6) and note the discussion right afterwards. This means that once we make the definitional associations $A_N \equiv A_+ \equiv A$ and $A_S \equiv A_- \equiv A'$, all the results of section 3 will carry through intact.

So now to the question: what would be detectable by our posited scalar electron, were it to traverse the monopole in a tidal lock? Would it be the whole-integer Dirac monopole condition $2eg = n$ of (4.7), or the half-integer condition $2eg = n - \frac{1}{2}$ of (5.14)? We shall show in this section that such a scalar electron could only detect the whole-integer, not the half-integer, condition.

For a scalar electron traversing a $2\pi$ azimuth about a magnetic monopole in a tidal lock and so at the same time rotating through $2\pi$ in harmony with this circuit, to ensure there are no observable string singularities we must require $\phi(2\pi) = \phi(0)$, similarly to what we did for the spin-half electron at (4.6) and (5.8). But we can cut to the chase by recognizing that the minus sign in (5.10) resulted directly from the oppositely-signed version $\psi(2\pi) = -\psi(0)$ following a $2\pi$ rotation. Thus, changing the fermion $\psi_+$ in (5.10) into a spinless scalar $\phi$ and removing the minus sign which is a direct consequence of how *fermions but not bosons* behave under $2\pi$ rotations, then to avoid observable singularities, the applicable equation in place of (5.10) will be:

$$\phi \to \phi' = \exp(i\Lambda_\Delta)\phi \equiv \phi . \tag{7.1}$$

This, it will be seen, leads to a variant of (4.6), and *not* of (5.11).

Specifically, if we now use (4.3) to operate from the left on $\phi$, we may write:

$$\exp(i\Lambda)\phi = \exp(i2eg\varphi)\phi . \tag{7.2}$$

Then, taking the *change in phase* going from $\varphi = 0$ to $\varphi = 2\pi$ while combining (7.1) and (7.2) and again using the identity $1 = \exp(i2\pi n)$, we next obtain (contrast (4.6)):

$$\phi \to \phi' = \exp(i\Lambda)\phi = \exp(i2eg\varphi)\phi = \exp(i4\pi eg)\phi \equiv 1\cdot\phi = \exp(i2\pi n)\phi , \tag{7.3}$$

where the definition $\equiv 1\cdot\phi$ is what imposes a single-valued wavefunction and so avoids observable singularities. Once again, as in (4.6), we extract $\exp(i4\pi eg)\phi = \exp(i2\pi n)\phi$, and this contains the ordinary, whole-integer Dirac condition

$$2eg = n . \tag{7.4}$$

Therefore, a hypothetical spin-zero electron would *not* detect the half-integer magnetic monopoles, even in a tidal lock. Rather, because bosons do not flip their versions under $2\pi$ rotations, this spinless electron would only detect the whole-integer monopoles of the customary Dirac monopole analyses, whether or not it was tidally-locked to the monopole. Consequently, the half-integer monopole charges $2eg = n - \frac{1}{2}$ – were they to physically exist – would be a phenomenon associated uniquely detectable by fermions and not bosons.

## 8. Conclusion

Just as Dirac's finding at page 68 of [1] of the quantization condition $2eg = n$ of (4.7) raised the question whether this might provide the theoretical explanation for why electric charge is quantized, the finding at (5.14) here that a wavefunction tidally locked to a magnetic monopole obeys the half-integer condition $2eg = n - \frac{1}{2}$ of (5.14) raises the question whether this might be the theoretical reason why half-integer FQHE charge fractions are observed in conductive materials at ultra-low temperatures near absolute 0K when very strong perpendicular magnetic fields are applied to two-dimensional *systems of electrons*, which electrons are fermions not bosons.

It is left for future study to examine whether such a physical connection can in fact be established between the half-integer charge fractions found here and the half-integer charge fractions observed in the FQHE. But it is important in and of itself to recognize, using the Wu-Yang analysis for maintaining a single-valued wavefunction to avoid observable string singularities, that *when a spin-half electron wavefunction is rotated in a tidal lock as it traverses a circuit about a hypothesized magnetic monopole, the Dirac condition not merely admits, but indeed requires, the existence of half-integer charge fractions*.

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