Tensor Fields in Relativistic Quantum Mechanics

Valeriy V. Dvoeglazov
UAF, Universidad de Zacatecas
Apartado Postal 636, Suc. 3, Zacatecas 98061 Zac., México
Web: http://fisica.uaz.edu.mx/˜valeri/
E-mail: valeri@fisica.uaz.edu.mx

Abstract

We re-examine the theory of antisymmetric tensor fields and 4-vector potentials. We discuss corresponding massless limits. We analize the quantum field theory taking into account the mass dimensions of the notoph and the photon. Next, we deduced the gravitational field equations from relativistic quantum mechanics.

In this presentation we re-examine the theory of the 4-potential field, the antisymmetric tensor fields of the second ranks and the spin-2 fields, coming from the modified Bargmann-Wigner formalism. In the series of the papers [1, 2, 3, 4] we tried to find connection between the theory of the quantized antisymmetric tensor (AST) field of the second rank (and that of the corresponding 4-vector field) with the $2(2s+1)$ Weinberg-Tucker-Hammer formalism [5, 6]. Several previously published works [7, 8, 9, 10, 11] introduced the concept of the *notoph* (the Kalb-Ramond field) which is constructed on the basis of the antisymmetric tensor "potentials". It represents itself the non-trivial spin-0 field. See also Refs. [12, 13, 14]. We try to discuss these problems on the basis of the generalized Bargmann-Wigner formalism [15, 16]. A field of the rest mass m and the spin $s \geq \frac{1}{2}$ is represented by a completely symmetric multispinor of the rank $2s$.

The spin-0 and spin-1 field particles can be constructed by taking the direct product of 4-spinors [15, 16]. One can choose the following definitions (p. 209 of Ref. [12])

$$\epsilon^\mu(\mathbf{0},+1) = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}\,,\;\epsilon^\mu(\mathbf{0},0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\,,\;\epsilon^\mu(\mathbf{0},-1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}\,, \quad (1)$$

and ($\widehat{p}_i = p_i/\mid \mathbf{p}\mid,\ \gamma = E_p/m$), p. 68 of Ref. [12],

$$\epsilon^\mu(\mathbf{p},\sigma) = L^\mu_{\ \nu}(\mathbf{p})\epsilon^\nu(\mathbf{0},\sigma)\,, \quad (2)$$

$$L^0_{\ 0}(\mathbf{p}) = \gamma\,,\; L^i_{\ 0}(\mathbf{p}) = L^0_{\ i}(\mathbf{p}) = \widehat{p}_i\sqrt{\gamma^2-1}\,, \quad (3)$$

$$L^i_{\ k}(\mathbf{p}) = \delta_{ik} + (\gamma-1)\widehat{p}_i\widehat{p}_k \quad (4)$$

for the 4-vector potential field:

$$A^\mu(x^\mu) = \sum_{\sigma=0,\pm1}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\left[\epsilon^\mu(\mathbf{p},\sigma)a(\mathbf{p},\sigma)e^{-ip\cdot x} + (\epsilon^\mu(\mathbf{p},\sigma))^c b^\dagger(\mathbf{p},\sigma)e^{+ip\cdot x}\right]\,. \quad (5)$$

After renormalizing the potentials, *e. g.*, $\epsilon^\mu \rightarrow u^\mu \equiv m\epsilon^\mu$ we come to the field functions in the momentum representation, which do not diverge in the massless limit. Thus, the change of the normalization can lead to the removal of divergent behaviour of classical polarization vectors in $m \to 0$. It is straightforward to find $\mathbf{B}^{(+)}(\mathbf{p},\sigma) = \frac{i}{2m}\mathbf{p}\times\mathbf{u}(\mathbf{p},\sigma)$, $\mathbf{E}^{(+)}(\mathbf{p},\sigma) = \frac{i}{2m}p_0\mathbf{u}(\mathbf{p},\sigma) - \frac{i}{2m}\mathbf{p}u^0(\mathbf{p},\sigma)$, and the corresponding negative-energy strengths for the $F_{\mu\nu}$ field operator (in general, complex-valued).

We begin with the Lagrangian, including, in general, mass term. Here we use the notation $A_{\mu\nu}$ for the AST due to different "mass dimension" of the fields. The massless ($m=0$) Lagrangian is connected with the Lagrangians used in the conformal field theories by adding the total derivative:

$$\mathcal{L}_{CFT} = \mathcal{L} + \frac{1}{2}\partial_\mu\left(A_{\nu\alpha}\partial^\nu A^{\mu\alpha} - A^{\mu\alpha}\partial^\nu A_{\nu\alpha}\right)\,, \tag{6}$$

$$\mathcal{L} = \frac{1}{4}(\partial_\mu A_{\nu\alpha})(\partial^\mu A^{\nu\alpha}) - \frac{1}{2}(\partial_\mu A^{\mu\alpha})(\partial^\nu A_{\nu\alpha}) - \frac{1}{2}(\partial_\mu A_{\nu\alpha})(\partial^\nu A^{\mu\alpha}) + \frac{1}{4}m^2 A_{\mu\nu}A^{\mu\nu}\,. \tag{7}$$

The Lagrangian leads to the equation of motion in the following form (provided that the appropriate antisymmetrization procedure has been taken into account):

$$\frac{1}{2}(\Box + m^2)A_{\mu\nu} + (\partial_\mu A^{\ ,\alpha}_{\alpha\nu} - \partial_\nu A^{\ ,\alpha}_{\alpha\mu}) = 0\,. \tag{8}$$

Following the variation procedure one can obtain the energy-momentum tensor:

$$\Theta^{\lambda\beta} = \frac{1}{2}\left[(\partial^\lambda A_{\mu\alpha})(\partial^\beta A^{\mu\alpha}) - 2(\partial_\mu A^{\mu\alpha})(\partial^\beta A^\lambda_{\ \alpha}) - 2(\partial^\mu A^{\lambda\alpha})(\partial^\beta A_{\mu\alpha})\right] - \mathcal{L}g^{\lambda\beta}\,. \tag{9}$$

One can also obtain that for rotations

$$\mathbf{J}^k = \frac{1}{2}\epsilon^{ijk}J^{ij} = \epsilon^{ijk}\int d^3\mathbf{x}\left[A^{0i}(\partial_\mu A^{\mu j}) + \ + A_\mu^{\ j}(\partial^0 A^{\mu i} + \partial^\mu A^{i0} + \partial^i A^{0\mu})\right]\,. \tag{10}$$

Now it becomes obvious that the application of the generalized Lorentz conditions (which are the quantum versions of the free-space dual Maxwell's equations) leads in such a formalism to the absence of electromagnetism in conventional sense. The resulting Kalb-Ramond field is longitudinal (helicity $h=0$). We agree with the previous authors, e. g., Ref. [17], see Eq. (4) therein, about the gauge *non*-invariance of the separation of the angular momentum of the electromagnetic field into the "orbital" and "spin" part (10). We proved again that for the antisymmetric tensor field $\mathbf{J} \sim \int d^3\mathbf{x}\,(\mathbf{E}\times\mathbf{A})$.

For the spin 1 one can also start from the Weinberg-Tucker-Hammer equation:

$$[\gamma_{\alpha\beta}p^\alpha p^\beta - Ap^\alpha p_\alpha + Bm^2]\Psi = 0\,, \tag{11}$$

where $p_\mu = -i\partial_\mu$, and $\gamma_{\alpha\beta}$ are the Barut-Muzinich-Williams covariantly-defined 6×6 matrices. One can recover the Proca theory from it:

$$\partial_\alpha\partial^\mu F_{\mu\beta} - \partial_\beta\partial^\mu F_{\mu\alpha} + m^2 F_{\alpha\beta} = 0\,, \tag{12}$$

as a particular case.

Bargmann and Wigner claimed explicitly that they constructed $(2s+1)$ states (the Weinberg-Tucker-Hammer theory has essentially $2(2s+1)$ components). Therefore, we now apply

$$\Psi_{\{\alpha\beta\}} = (\gamma^\mu R)_{\alpha\beta}(c_a m A_\mu + c_f F_\mu) + (\sigma^{\mu\nu}R)_{\alpha\rho}(c_A m(\gamma^5)_{\rho\beta}A_{\mu\nu} + c_F I_{\rho\beta}F_{\mu\nu})\,. \tag{13}$$

Thus, A_μ, $A_{\mu\nu}$ and F_μ, $F_{\mu\nu}$ have different mass dimension. The $\gamma^\mu R$, $\sigma^{\mu\nu}R$ and $\gamma^5\sigma^{\mu\nu}R$ are the *symmetrical* matrices. After the suitable choice of the dimensionless coefficients c_i, the Lagrangian density for the photon-notoph field can be proposed:

$$\mathcal{L} = \mathcal{L}^{Proca} + \mathcal{L}^{Notoph} = -\frac{1}{8}F_\mu F^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + + \frac{m^2}{2}A_\mu A^\mu + \frac{m^2}{4}A_{\mu\nu}A^{\mu\nu}\,. \tag{14}$$

The limit $m \to 0$ may be taken for dynamical variables, in the end of calculations only. In Ref. [18] a very interesting model has been proposed. It is based on gauging the Dirac field on using the coordinate-dependent parameters $\alpha_{\mu\nu}(x)$ in

$$\psi(x) \to \psi'(x') = \Omega\psi(x) \ , \ \Omega = \exp\left[\frac{i}{2}\sigma^{\mu\nu}\alpha_{\mu\nu}(x)\right] . \tag{15}$$

Thus, the second "photon" was introduced. The compensating 24-component field $B_{\mu,\nu\lambda}$ reduces to the 4-vector field as follows:

$$B_{\mu,\nu\lambda} = \frac{1}{4}\epsilon_{\mu\nu\lambda\sigma}a^{\sigma}(x) . \tag{16}$$

As readily seen after comparison of these formulas with those of Refs. [8, 9, 10] , the second photon is nothing more than the Ogievetskiĭ-Polubarinov *notoph* within the normalization.

Marques and Spehler obtained tensor equations [19]:

$$\frac{2}{m}\partial_\mu T^{\mu\nu}_{\kappa} = -G^{\nu}_{\kappa} \, , \tag{17}$$

$$\frac{2}{m}\partial_\mu R^{\mu\nu}_{\kappa\tau} = -F^{\nu}_{\kappa\tau} \, , \tag{18}$$

$$T^{\mu\nu}_{\kappa} = \frac{1}{2m}\left[\partial^\mu G^{\nu}_{\kappa} - \partial^\nu G^{\mu}_{\kappa}\right] \, , \tag{19}$$

$$R^{\mu\nu}_{\kappa\tau} = \frac{1}{2m}\left[\partial^\mu F^{\nu}_{\kappa\tau} - \partial^\nu F^{\mu}_{\kappa\tau}\right] . \tag{20}$$

As opposed to them we shall modify the formalism in the spirit of Ref. [14]. The field function is now presented as

$$\Psi_{\{\alpha\beta\}\gamma\delta} = \alpha_1(\gamma_\mu R)_{\alpha\beta}\Psi^{\mu}_{\gamma\delta} + \alpha_2(\sigma_{\mu\nu}R)_{\alpha\beta}\Psi^{\mu\nu}_{\gamma\delta} + \alpha_3(\gamma^5\sigma_{\mu\nu}R)_{\alpha\beta}\widetilde{\Psi}^{\mu\nu}_{\gamma\delta} \, , \tag{21}$$

with

$$\Psi^{\mu}_{\{\gamma\delta\}} = \beta_1(\gamma^\kappa R)_{\gamma\delta}G^{\mu}_{\kappa} + \beta_2(\sigma^{\kappa\tau}R)_{\gamma\delta}F^{\mu}_{\kappa\tau} + \beta_3(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{F}^{\mu}_{\kappa\tau} \, , \tag{22}$$

$$\Psi^{\mu\nu}_{\{\gamma\delta\}} = \beta_4(\gamma^\kappa R)_{\gamma\delta}T^{\mu\nu}_{\kappa} + \beta_5(\sigma^{\kappa\tau}R)_{\gamma\delta}R^{\mu\nu}_{\kappa\tau} + \beta_6(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{R}^{\mu\nu}_{\kappa\tau} \, , \tag{23}$$

$$\widetilde{\Psi}^{\mu\nu}_{\{\gamma\delta\}} = \beta_7(\gamma^\kappa R)_{\gamma\delta}\widetilde{T}^{\mu\nu}_{\kappa} + \beta_8(\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{D}^{\mu\nu}_{\kappa\tau} + \beta_9(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}D^{\mu\nu}_{\kappa\tau} . \tag{24}$$

Hence, the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ can be expressed as a sum of nine terms. The essential constraints can be found in Ref. [20]. One should contract the field function with six antisymmetric matrices in order to ensure symmetrization. As a discussion, we note that in such a framework we have the physical content because only certain combinations of field functions are equal to zero. Furthermore, from the new system of equations one obtains the *second*-order equation for the symmetric traceless tensor of the second rank ($\alpha_1 \neq 0$, $\beta_1 \neq 0$):

$$\frac{1}{m^2}\left[\partial_\nu\partial^\mu G^{\nu}_{\kappa} - \partial_\nu\partial^\nu G^{\mu}_{\kappa}\right] = G^{\mu}_{\kappa} . \tag{25}$$

After the contraction in indices κ and μ this equation is reduced to:

$$\partial_\mu G^{\mu}_{\kappa} = F_\kappa \, , \tag{26}$$

$$\frac{1}{m^2}\partial_\kappa F^\kappa = 0 \, , \tag{27}$$

i. e., to the equations which connect the analogue of the energy-momentum tensor and the analogue of the 4-vector field.

The most general relativistic-invariant Lagrangian for the symmetric 2nd-rank tensor is

$$\mathcal{L} = -\alpha_1(\partial^\alpha G_{\alpha\lambda})(\partial_\beta G^{\beta\lambda}) - \alpha_2(\partial_\alpha G^{\beta\lambda})(\partial^\alpha G_{\beta\lambda}) - \alpha_3(\partial^\alpha G^{\beta\lambda})(\partial_\beta G_{\alpha\lambda}) + m^2 G_{\alpha\beta}G^{\alpha\beta}\,. \quad (28)$$

in the Minkowski space. It leads to the equation

$$\left[\alpha_2(\partial_\alpha\partial^\alpha) + m^2\right] G^{\{\mu\nu\}} + (\alpha_1+\alpha_3)\partial^{\{\mu|}(\partial_\alpha G^{\alpha|\nu\}}) = 0\,. \quad (29)$$

In the case $\alpha_2 = 1 > 0$ and $\alpha_1 + \alpha_3 = -1$ it coincides with Eq. (25). The mass dimension of $G^{\mu\nu}$ is $[energy]^1$. We now present possible relativistic interactions of the symemtric 2nd-rank tensor. The simplest ones should be the following ones: $\mathcal{L}^{int}_{(1)} \sim G_{\mu\nu}F^\mu F^\nu$, $\mathcal{L}^{int}_{(2)} \sim (\partial^\mu G_{\mu\nu})F^\nu$, $\mathcal{L}^{int}_{(3)} \sim G_{\mu\nu}(\partial^\mu F^\nu)$. On using

$$V(F) = \lambda_1(F_\mu F^\mu) + \lambda_2(F_\mu F^\mu)(F_\nu F^\nu) \quad (30)$$

we can give the mass to the G_{00} component of the spin-2 field. This is due to the possibility of the analogue of the Higgs spontaneous symmetry breaking, $v^2 = (F_\mu F^\mu) = -\lambda_1/2\lambda_2 > 0$.

Next, since the interaction of fermions with notoph are that of the order $\sim e^2$ since the beginning (as opposed to the fermion current interaction with the 4-vector potential A_μ, which is proportional to e) in the Lagrangian, it is more difficult to observe the notoph field.

We considered the Bargmann-Wigner formalism in order to derive the equations for the AST fields, and for the symmetric tensor of the 2nd rank. This problem is connected with the problem of the observability of the gauge. We introduced the additional symmetric matrix in the Bargmann-Wigner expansion, namely $(\gamma^5\sigma^{\mu\nu}R)$, in order to account for the dual fields. The consideration was similar to Ref. [21]. The interaction may give the mass to spin-2 particles in the way which is reminiscent to the spontaneous-symmetry-breaking Higgs formalism. I acknowledge discussions with participants of recent conferences on Symmetries and Gravitation.

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