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Relations Between Tsallis and Kaniadakis Entropic Measures and Rigorous Discussion of Conditional Kaniadakis Entropy

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Abstract: Tsallis and Kaniadakis entropies are generalizing the Shannon entropy and have it as their limit when their entropic indices approach specific values. Here we show some relations existing between Tsallis and Kaniadakis entropies. We will also propose a rigorous discussion of the conditional Kaniadakis entropy, deduced from these relations.

Keywords: Entropy, Generalized Entropies

1. Introduction

In information theory, measures of information can be obtained from the probability distribution of some events contained in a sample set of possible events. These measures are the entropies.

In 1948 [1], Claude Shannon defined the entropy H (Greek letter Eta), of a discrete random variable X, as the expected value of the information content: H(X) $= \sum_i p(x_i) I(x_i) = -\sum_i p(x_i) \log_b p(x_i)$ [2]. In this expression, I is the information content of X, the probability of i-event is p_i and *b* is the base of the used logarithm. Common values of the base are 2, Euler's number e, and 10.

Besides Shannon entropy, several other entropies are used in information theory; here we will consider the generalized entropies of Tsallis and Kaniadakis (also known as κ-entropy) entropies [2-4]. We will show relations between them. We will also propose a rigorous discussion of the conditional Kaniadakis entropy, deduced from these relations.

2. The entropies

In the following formulas we can see how the abovementioned entropies (Shannon, Tsallis and Kaniadakis) are defined, with a corresponding choice of measurement units equal to 1:

$$(1) \quad \text{Shannon}: \quad S = -\sum_i p_i \ln p_i$$

$$(2) \quad \text{Tsallis}: T = T_q = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right)$$

$$(3) \quad \text{Kaniadakis } (\kappa\text{-entropy}):$$

$$K_\kappa = -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa}$$

In (2),(3) we have entropic indices q and κ, and:

$$\lim_{q \to 1} T = S; \quad \lim_{\kappa \to 0} K = S.$$

Let us consider the joint entropy H(A,B) of two independent subsystems A,B. We have:

$$(4) \quad S(A,B) = S(A) + S(B)$$

$$(5) \quad T(A,B) = T(A) + T(B) + (1-q)T(A)T(B)$$

$$(6) \quad K(A,B) = K(A)\Im(B) + K(B)\Im(A)$$

$$\text{with } \Im = \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2}$$

Note that for the generalized additivity of κ-entropy, we need another function containing probabilities (see [5] and references therein).

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3. Basic relations between K and T

Let us consider κ-entropy K. We have that:

$$(7) \quad K_\kappa = \frac{T_{1+\kappa} + T_{1-\kappa}}{2},$$

In (7) we used the Tsallis entropies:

$$T(q = 1+\kappa) = -\frac{1}{\kappa}\sum_i p_i^{1+\kappa} + \frac{1}{\kappa};$$

$$T(q = 1-\kappa) = \frac{1}{\kappa}\sum_i p_i^{1-\kappa} - \frac{1}{\kappa}$$

In fact, we have:

$$K = \frac{1}{2}\left\{-\frac{1}{\kappa}\sum_i p_i^{1+\kappa} + \frac{1}{\kappa} + \frac{1}{\kappa}\sum_i p_i^{1-\kappa} - \frac{1}{\kappa}\right\}$$

$$= -\frac{1}{2\kappa}\left\{\sum_i p_i^{1+\kappa} - \sum_i p_i^{1-\kappa}\right\}$$

Eq.(7) is a simpler form of an expression given in [6,7]. However, besides this relation, because of the generalized additivity possessed by the Kaniadakis entropy, we need also another relation, concerning function $\Im$. It is:

$$(8) \quad \Im_\kappa = \frac{\kappa}{2}\left(-T_{1+\kappa} + T_{1-\kappa} + \frac{2}{\kappa}\right)$$

In fact:

$$\Im = \frac{\kappa}{2}\left\{\frac{1}{\kappa}\sum_i p_i^{1+\kappa} - \frac{1}{\kappa} + \frac{1}{\kappa}\sum_i p_i^{1-\kappa} - \frac{1}{\kappa} + \frac{2}{\kappa}\right\}$$

$$= \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2}$$

In (7) and (8), we have the Kaniadakis functions expressed by the Tsallis entropy. However, we can also write T expressed by means of Kaniadakis entropy.

$$2K + \frac{2}{\kappa}\Im$$

$$= T_{1+\kappa} + T_{1-\kappa} + \left(-T_{1+\kappa} + T_{1-\kappa} + \frac{2}{\kappa}\right)$$

And then:

$$(9) \quad K_\kappa + \frac{1}{\kappa}\Im_\kappa = T_{1-\kappa} + \frac{1}{\kappa}$$

Let us have: $\kappa = 1 - q$.

$$(10) \quad T_q = K_{1-q} + \frac{\Im_{1-q} - 1}{(1-q)}$$

We can have also:

$$2K - \frac{2}{\kappa}\Im$$

$$= T_{1+\kappa} + T_{1-\kappa} - \left(-T_{1+\kappa} + T_{1-\kappa} + \frac{2}{\kappa}\right)$$

So that:

$$(11) \quad K_\kappa - \frac{1}{\kappa}\Im_\kappa = T_{1+\kappa} - \frac{1}{\kappa}$$

Let us have: $\kappa = q - 1$. We have again Eq.10.

4. Generalized additivity

Let us consider two subsystems A and B. We can find a relation between the joint Tsallis and Kaniadakis entropies. Using (10), we obtain:

$$(12) \quad T_q(A,B) = K_{1-q}(A,B) + \frac{\Im_{1-q}(A,B)}{1-q} - \frac{1}{1-q}$$

When the subsystems are independent, for Tsallis entropy we have Eq.5, and then:

$$(13)\quad T_q(A,B) = K_{1-q}(A,B) + \frac{\Im_{1-q}(A,B)}{1-q} - \frac{1}{1-q} = K_{1-q}(A) + \frac{\Im_{1-q}(A)}{1-q} - \frac{1}{1-q} + K_{1-q}(B) + \frac{\Im_{1-q}(B)}{1-q} - \frac{1}{1-q} + (1-q)\left(K_{1-q}(A) + \frac{\Im_{1-q}(A)}{1-q} - \frac{1}{1-q}\right)\left(K_{1-q}(B) + \frac{\Im_{1-q}(B)}{1-q} - \frac{1}{1-q}\right)$$

To continue, we can assume:

$$(14)\quad \Im_{1-q}(A,B) = \Im_{1-q}(A)\,\Im_{1-q}(B) + (1-q)^2 K_{1-q}(A)K_{1-q}(B)$$

This relation was proposed in [8], but it is clear that it can obtained from (13):

$$K_{1-q}(A,B) + \frac{\Im_{1-q}(A,B)}{1-q}$$
$$= (1-q)K_{1-q}(A)K_{1-q}(B) + K_{1-q}(A)\Im_{1-q}(B) + \Im_{1-q}(A)K_{1-q}(B) + (1-q)\frac{\Im_{1-q}(A)}{1-q}\frac{\Im_{1-q}(B)}{1-q}$$

Remembering that $K_{1-q}(A,B) = K_{1-q}(A)\Im_{1-q}(B) + \Im_{1-q}(A)K_{1-q}(B)$, we must have (14) to fulfil (13).

6. Conditional entropy

In a previous paper [9] we proposed, in the framework of a discussion on mutual entropy, an expression for the conditional Kaniadakis entropy. If the entropic index has a low value, the formula we find in [9] can be considered an approximation of the expression that we are here deducing. Let us start from the conditional Tsallis entropy [10]:

$$(15)\quad T_q(A\,|\,B) = \frac{T_q(A,B) - T_q(B)}{1 + (1-q)T_q(B)}$$

Then $T_q(A\,|\,B)[1 + (1-q)T_q(B)] = T_q(A,B) - T_q(B)$. From now on, we do not write the indices of Kaniadakis functions. However, remember that they exist and have value (1−q). Let us use (10), written for joint and conditional entropies:

$$(16)\quad \left[K(A\,|\,B) + \frac{\Im(A\,|\,B)}{1-q} - \frac{1}{1-q}\right]\left[(1-q)K(B) + \Im(B)\right] = K(A,B) + \frac{\Im(A,B)}{1-q} - K(B) - \frac{\Im(B)}{1-q}$$

We have:

$$(17)\quad K(A\,|\,B)(1-q)K(B) + K(A\,|\,B)\Im(B) + K(B)\Im(A\,|\,B) + \frac{\Im(B)\Im(A\,|\,B)}{1-q} - K(B) - \frac{\Im(B)}{1-q}$$
$$= K(A,B) + \frac{\Im(A,B)}{1-q} - K(B) - \frac{\Im(B)}{1-q}$$

From this equation we find:

$$(18)\quad K(A\,|\,B)(1-q)K(B) + K(A\,|\,B)\Im(B) + K(B)\Im(A\,|\,B) + \frac{\Im(B)\Im(A\,|\,B)}{1-q}$$
$$= K(A,B) + \frac{\Im(A,B)}{1-q}$$

We can divide (18) in two equations:

$$(19) \quad K(A|B)\Im(B) + K(B)\Im(A|B) = K(A,B) \rightarrow K(A|B) = \frac{K(A,B) - K(B)\Im(A|B)}{\Im(B)}$$

$$(20) \quad (1-q)^2 K(A|B)K(B) + \Im(B)\Im(A|B) = \Im(A,B) \rightarrow \Im(A|B) = \frac{\Im(A,B) - (1-q)^2 K(A|B)K(B)}{\Im(B)}$$

Let us note that (19) and (20) are generalizing (6) and (14). In both (19),(20) we have conditional functions $K(A|B), \Im(A|B)$. Then, the rigorous expression of Kaniadakis conditional entropy is:

$$(21) \quad K_\kappa(A|B) = \frac{K_\kappa(A,B) - K_\kappa(B)\Im_\kappa(A|B)}{\Im_\kappa(B)}$$

In the case we are using the entropic index with a low value, this expression is approximated by:

$$K_\kappa(A|B) = \frac{K_\kappa(A,B) - K_\kappa(B)\Im_\kappa(A)}{\Im_\kappa(B)}$$

The discussion in [9] can be proposed again using (21). For instance, if we consider the mutual entropy (without renormalization):

$$(22) \quad MK_\kappa(A;B) = K_\kappa(A) - K_\kappa(A|B) = \frac{K_\kappa(A)\Im_\kappa(B) - K_\kappa(A,B) + K_\kappa(B)\Im_\kappa(A|B)}{\Im_\kappa(B)}$$

When A and B are independent, $\Im(A|B) = \Im(A)$ and then the mutual information is zero. Moreover, when we have a small value of the entropic index, function $\Im$ is equal to 1.

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