EVALUATION OF THE INTEGRAL $\int_0^\infty r^{2-\lambda}\, j_0(k_1 r)\, j_0(k_2 r)\, j_\lambda(k_3 r)\, dr$

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ABSTRACT

The integral $\int_0^\infty r^{2-\lambda}\, j_0(k_1 r)\, j_0(k_2 r)\, j_\lambda(k_3 r)\, dr$ is evaluated for any real positive k_1, k_2, k_3 and $\lambda \geq 0$

1. Introduction

Integrals of the type $\int_0^\infty r^{2-\lambda}\, j_{\lambda_1}(k_1 r)\, j_{\lambda_2}(k_2 r)\, j_{\lambda_3+\lambda}(k_3 r)\, dr$ have been evaluated recently [1] when λ_1, λ_2 and λ_3 satisfied the triangular condition

$$|\lambda_2 - \lambda_1| \leq \lambda_3 \leq \lambda_1 + \lambda_2 \tag{1.1}$$

and k_1, k_2 and k_3 formed the side of a triangle, i.e.

$$|k_2 - k_1| \leq k_3 \leq k_1 + k_2. \tag{1.2}$$

In this paper, the closure relation

$$\int_0^\infty k^2\, j_L(kr)\, j_L(kr')\, dk = \frac{\pi}{2r^2}\, \delta(r - r'), \tag{1.3}$$

will be used to derive an integral over 3 spherical Bessel functions when k_1, k_2 and k_3 can be any positive real numbers.

2. Evaluating the Integral

Using eq. (1.3), it is easy to show that

$$\int_0^\infty r^{2-\lambda}\, j_0(k_1 r)\, j_0(k_2 r)\, j_\lambda(k_3 r)\, dr \;=\; \frac{2}{\pi} \int_0^\infty q^2\, dq$$
$$\times \left(\int_0^\infty r^2\, j_0(k_1 r)\, j_0(k_2 r)\, j_0(qr)\, dr \right) \left(\int_0^\infty r'^{2-\lambda}\, j_0(qr')\, j_\lambda(k_3 r')\, dr' \right). \tag{2.1}$$

Now [2]

$$\int_0^\infty r^2\, j_0(k_1 r)\, j_0(k_2 r)\, j_0(qr)\, dr \;=\; \frac{\pi\, \beta(\Delta)}{4 k_1 k_2 q}, \tag{2.2}$$

where

$$\begin{aligned} \beta(\Delta) \;&=\; 0,\; q < |k_2 - k_1| \\ &=\; \frac{1}{2},\; q = |k_2 - k_1|\; or\; q = k_1 + k_2 \\ &=\; 1,\; |k_2 - k_1| < q < k_1 + k_2, \end{aligned} \tag{2.3}$$

and [3]

$$\int_0^\infty r'^{2-\lambda}\, j_0(qr')\, j_\lambda(k_3 r')\, dr' \;=\; \frac{\pi}{2^\lambda}\, \frac{(k_3^2 - q^2)^{\lambda-1}}{k_3^{\lambda+1}\, (\lambda-1)!}\, \theta(k_3 - q), \tag{2.4}$$

for $\lambda > 0$ [$\lambda = 0$ case results in a delta function]. Hence, the final result is

$$\int_0^\infty r^{2-\lambda}\, j_0(k_1 r)\, j_0(k_2 r)\, j_\lambda(k_3 r)\, dr \;=\; \frac{\pi}{2^{\lambda+2}\, \lambda!\, k_1 k_2 k_3^{\lambda+1}}$$
$$\times\; \{\theta(k_3 - (k_1 + k_2))\, [\, (k_3^2 - (k_2 - k_1)^2)^\lambda \;-\; (k_3^2 - (k_1 + k_2)^2)^\lambda\,]$$
$$+\; \beta(\Delta)\, [k_3^2 - (k_2 - k_1)^2]^\lambda\, \}, \tag{2.5}$$

for $\lambda \geq 0$.

3. Conclusions

The closure relation for spherical Bessel functions is used to evaluate the integral $\int_0^\infty r^{2-\lambda}\, j_0(k_1 r)\, j_0(k_2 r)\, j_\lambda(k_3 r)\, dr$ for any real positive values of k_1, k_2, k_3 and $\lambda \geq 0$

4. References

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