Relating spontaneous and explicit symmetry breaking in the presence of the Higgs mechanism

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Abstract

One common way to define spontaneous symmetry breaking involves necessarily explicit symmetry breaking. We add explicit symmetry breaking terms to the Higgs potential, so that the spontaneous breaking of a global symmetry in multi-Higgs-doublet models is a particular case of explicit symmetry breaking. Then we show that it is possible to study the Higgs potential without assuming that the local gauge $SU(2)_L$ symmetry is spontaneously broken or not (it is known that gauge symmetries may not be possible to break spontaneously). We also discuss the physical spectrum of multi-Higgs-doublet models and the related custodial symmetry.

We review background symmetries: these are symmetries that despite already explicitly broken, can still be spontaneously broken. We show that the CP background symmetry is not spontaneously broken, based on this fact: we explain in part a recent conjecture relating spontaneous and explicit breaking of the charge-parity (CP) symmetry; we also relate explicit and spontaneous geometric CP-violation.

1 Introduction

There are several definitions of spontaneous breaking of global symmetries [1, 2], all are related with the existence of disjoint[1] phases in a system. In the context of statistical mechanics [1], spontaneous symmetry breaking is often defined as a particular case of explicit symmetry breaking via an external source.

Let $\mathcal{A}$ be an algebra of operators, let G be a group of global transformations $\mathcal{A} \to \mathcal{A}$.

The system's expectation value $\omega_{J,N}$ is a positive linear functional $\omega_{J,N} : \mathcal{A} \to \mathbb{R}$, $J \geq 0$ is the intensity of an external source breaking the symmetry G, while N is the size of the system.

[1] disjoint in the sense that a system cannot go through a phase transition by physically realizable operations

For finite size N, the system is well behaved with continuous expectation values[2] as a function of J, i.e. for any operator $A \in \mathcal{A}$ and any symmetry $\beta \in G$:

$$\begin{cases} \omega_{J,N}(A - \beta(A)) = 0 & \text{if } J = 0 \\ \lim_{J\to 0} \omega_{J,N}(A - \beta(A)) = 0 \end{cases}$$

Definition 1 (In statistical mechanics)**.** The spontaneous symmetry breaking of G happens when there are finite expectation values breaking the symmetry G, for an arbitrarily small explicit symmetry breaking, i.e.

$$\lim_{J\to 0}\{\lim_{N\to\infty} \omega_{J,N}(A - \beta(A))\} \neq 0$$

for some $A \in \mathcal{A}$ and some $\beta \in G$.

The non-null limit is possible since the (pointwise) limit of a convergent sequence of continuous functions is not necessarily continuous.

Other definitions in the context of statistical mechanics do not consider an external source and coherently are not based on the existence of expectation values that explicitly break the symmetry (since that would not be possible by definition of the system's expectation value with $J = 0$), but are based instead on a long-range order parameter which is the expectation value of a G-symmetric function $f(A)$ (e.g. the modulus $f(A) = |A|$) of an operator A which is translation invariant and breaks G; or on a conditional expectation value of some operator A given some condition $C = 0$ that breaks the symmetry; or on a two-point correlation function with the points at an infinite distance from each other (related with boundary conditions[3]) [1]. It is widely accepted that these definitions should be all equivalent to Def. 1 (e.g. in the Ising model [1]), although it does not seem easy to prove it because the systems with or without external source are physically different [3].

When it comes to quantum non-abelian gauge field theories, the theories themselves lack a non-perturbative mathematical definition [4], so it is even more difficult to relate these different definitions. By analogy with statistical mechanics, we expect that they are related—since the correlation functions of quantum field theory can be defined as the Wick-rotation of correlation functions of a statistical field theory [5]. In the presence of the Higgs mechanism, there is yet another definition of spontaneous symmetry breaking, most common in the context of perturbation theory of the Electroweak interactions:

Definition 2 (Electroweak symmetry breaking)**.** After a suitable perturbative non-abelian gauge fixing, the vacuum expectation value (vev) of the Higgs field is determined (up to quantum

[2]It is not strictly required that the expectation values are continuous for finite N to have spontaneous symmetry breaking [2], but the systems with local interactions (e.g. the Ising model or gauge theories) share this property. If already for finite N the expectation values would not be continuous, we could study the symmetry breaking in the finite system which may be a much simpler problem (certainly a very different problem).

[3]There are yet more definitions based on an Hamiltonian formulation of statistical field theory [2] with the boundary conditions such as the initial time playing a key role. It is widely accepted that the definitions based on boundary conditions should be equivalent to Def. 1 [2, See Sec. 10.C].

corrections) by one of the possible minima of the Higgs potential. The symmetries broken by the Higgs vev are the spontaneously broken symmetries.

The Def. 2 does not seem to be physically (not to mention mathematically) equivalent to Def. 1 in the context of the quantum Electroweak theory, since spontaneous symmetry breaking is a non-perturbative phenomenon of the entire system which in Def. 2 is drastically simplified to a classical problem of minimization of a polynomial [6]. Note that in Def. 2 the Higgs vev explicitly breaks the symmetry and therefore Def. 2 should be compared with Def. 1. The comparison with other definitions that do not involve explicit symmetry breaking vevs would be more troublesome[4], but assuming that such alternative definitions are all compatible with Def. 1, it suffices for our purposes to compare definitions 1 and 2. One reason why Def. 2 involves explicit symmetry breaking vevs is that perturbation theory can only deal with small perturbations of the Higgs field, which is only guaranteed if the Higgs vev is non-null[5].

However, the fact is that the perturbative predictions from the Electroweak theory seem to be a very good approximation to the existing experimental data in high-energy physics[7], and the (non-perturbative) lattice simulations so far agree with this picture [8–10] (also for two-Higgs-doublet models [11]). Therefore, for consistency these definitions should be related. While we cannot give a solid proof that this is so, we can check in concrete models that the perturbative definition 2 is consistent with the non-perturbative definition 1. The consistency is not merely formal, but also phenomenological since non-perturbative lattice simulations [12] and the functional renormalization group [13] are becoming increasingly relevant in the studies of Electroweak physics and beyond, and are well established in Flavour physics and QCD.

There is a further ingredient to take into account [14]: a spontaneous breaking of local gauge symmetry without gauge fixing may be impossible in a gauge theory such as the Electroweak theory. The argument is based on the fact that local gauge transformations affect only a small sized system near each space-time point and so the non-commutativity of the limits seen above does not apply (under some assumptions on the analiticity of $\omega_{J,N}$). It can be argued that the Higgs mechanism avoids the presence of Nambu-Goldstone bosons precisely because the local gauge symmetry is not spontaneously broken [15, 16]. Many non-perturbative studies support this picture [17–20]. Moreover, there is a group-theory correspondence between gauge-invariant composite operators and the gauge-dependent elementary fields in the Electroweak theory [16, 21] (also for two-Higgs-doublet models [22]).

The above discussion implies that there must exist specific relations between the gauge-dependent minima of the Higgs potential and the gauge-invariant operators appearing in the Lagrangian, for consistency reasons. That is, relations between explicit and spontaneous symme-

[4]The alternative definitions (e.g. boundary conditions) involve assumptions beyond the Lagrangian and so they are more dependent on the particular Quantum Field Theory framework (e.g. perturbative/continuum or non-perturbative/UV-cutoff, scattering processes or bound-states), and we want to use several frameworks for phenomenology studies.

[5]When considering superselection sectors, we don't deal perturbatively with a null Higgs vev, we deal with a statistical ensemble of systems each with a non-null Higgs vev corresponding to one superselection sector and we study each system perturbatively.

try breaking. Some of these relations were noted recently for the CP (charge-parity) symmetry in multi-Higgs-doublet models and were summarized in the form of a conjecture [23], previous relations in the same context were found earlier for a specific three-Higgs-doublet model [24].

In this paper we address four problems in the context of multi-Higgs-doublet models which as we will see are related:

- check that the non-perturbative Def. 1 of spontaneous symmetry breaking is compatible with the usual assumptions of perturbation theory (Def. 2);
- how to study the Higgs potential and its phenomenological consequences without assuming spontaneous symmetry breaking of the gauge symmetry $SU(2)_L$;
- why the custodial symmetry is accidentally conserved in the Higgs potential of the Standard Model, and its relation with the physical spectrum;
- the relations mentioned above between explicit and spontaneous CP symmetry breaking [23, 24].

In Sec. 2 we state the assumptions we will make throughout the paper, they are an extension to multi-Higgs-models of the perturbative assumptions of Ref. [21] and do not imply that $SU(2)_L$-gauge symmetry is or is not spontaneously broken. In Sec. 3 we review background symmetries: these are symmetries that despite they are already explicitly broken, can be still spontaneously broken. In Sec. 2 we add explicit symmetry breaking terms to the Higgs potential, so that Def. 1 of spontaneous symmetry breaking applies. Our assumptions and framework are compatible with the usual assumptions of Electroweak symmetry breaking (Def. 2), as we show in Sec. 5. We also show that explicit symmetry breaking implies that spontaneous symmetry breaking is allowed, in Sec. 6. In Sec. 5 we discuss the physical spectrum of multi-Higgs-doublet models and the related custodial symmetry. Finally in Sec. 8, we show that the CP background symmetry is not spontaneously broken. Based on this fact we relate explicit and spontaneous geometric CP-violation, and in Sec. 9 we explain part of the conjecture mentioned above [23]. We conclude in Sec. 10.

2 Higgs potential and minima

We consider a G-invariant Higgs potential. The $SU(2)_L$-gauge is a normal subgroup of G. We also assume that $G/SU(2)_L$ is a group of global transformations[6]. In analogy with Def. 2, to study the Higgs potential and in particular the global symmetries which are spontaneously broken or not, we assume that:

[6] Since the $U(1)_Y$ gauge symmetry is abelian and there are no Gribov-Singer ambiguities for abelian gauge fixing (unlike for a non-abelian gauge symmetry such as $SU(2)_L$), we can unambiguously fix the local gauge with a gauge-fixing local term and deal only with the $U(1)_Y$ global symmetry

- we fix the local gauge and assume that the vevs of $SU(2)_L$-invariant operators are given by the usual perturbative expansion, which is an expansion in: 1) the Higgs field around one constant point of the $SU(2)_L$-orbit for which the potential has an absolute minimum; and 2) in the couplings of the interactions[7];
- whenever there are two or more $SU(2)_L$-orbits minimizing the Higgs potential related by $G/SU(2)_L$, there is spontaneous symmetry breaking of the global symmetry $G/SU(2)_L$: we add a small $G/SU(2)_L$-symmetry breaking term to the Higgs potential such that only one of the $SU(2)_L$-orbits is the absolute minimum of the modified Higgs potential, the perturbation expansion then implies that in the limit that the term goes to zero there are finite vevs breaking $G/SU(2)_L$;
- the corrections due to the interactions are calculated with usual perturbation theory (including loops), in particular the interactions do not change which global symmetries are spontaneously broken or not, beyond .

These are non-trivial assumptions. In the Standard Model, the $SU(2)_L$-gauge orbit minimizing the Higgs potential is unique and therefore there is no experimental evidence in the context of Electroweak physics, that these assumptions relating spontaneous symmetry breaking with non-unique $SU(2)_L$-gauge orbits are valid. Moreover these assumptions imply that the Golstone's theorem applies for global symmetries [2], and there is theoretical evidence that also for global symmetries there are exceptions to the Goldstone's theorem[8]. Thus more non-perturbative studies and/or experimental data are required to support these assumptions [11, 22].

The electromagnetic symmetry $U(1)_{em}$ is the representation of the $U(1)_Y$ gauge symmetry in the $SU(2)_L$-invariant operators (coincides with the $U(1)_{em}$ symmetry in the usual perturbative formulation, see Sec. 5) and we can treat it as a global symmetry after $U(1)_Y$ local gauge fixing. Therefore, under our assumptions the $U(1)_{em}$ symmetry can also be spontaneously broken like all other global symmetries if there are two $SU(2)_L$-gauge orbit minimizing the Higgs potential

[7]The usual perturbative expansion is an expansion in the couplings with the mass of the W boson kept finite ($M_W = gv/2$) therefore it is also an expansion for large Higgs vev. The only difference with respect to the usual perturbative expansion is that we only evaluate vevs of $SU(2)_L$-invariant operators so we do not assume that $SU(2)_L$ is spontaneously broken.

The vevs of the $SU(2)_L$-gauge-invariant operators are the physical observables if the $SU(2)_L$ gauge symmetry is not spontaneously broken, as it seems to be the case [15, 16, 21]. In the context of the perturbative formulation of Electroweak theory, there are already studies of the (multi-)Higgs potential based on $SU(2)_L$-invariant bilinears of the Higgs field [25–28].

The local gauge fixing is perturbative with a local term and in a suitable gauge [21, 29] (such as the usual gauges used in perturbation theory), we assume that the (non-perturbative) Gribov-Singer ambiguities do not affect our results. The reference point is constant in the chosen $SU(2)_L \times U(1)_Y$ local gauge.

[8]The standard perturbative expansion is based on the $\lambda\phi^4$ quantum theory (mexican hat potential), but in the $\lambda\phi^4$ quantum theory the (non-perturbatively) renormalized coupling λ is necessarily null (trivial) [5, 30]—so the prototype of the Golstone's theorem in the quantum field theory context does not lead to spontaneous symmetry breaking and is itself an exception to the Goldstone's theorem.

In general, the full quantum theory may change which symmetries are spontaneously/explicitly broken/conserved, e.g. due to a significantly different effective potential (like in $\lambda\phi^4$), some symmetry is not only global but is also a local gauge symmetry, the confinement mechanism (which does not seems an exclusive of QCD [31]), spontaneous symmetry breaking via non-Higgs fields, unknown mechanisms of symmetry breaking [32], etc.

related by a $U(1)_{em}$ transformation[9].

Under these assumptions, we have to solve a classical (but still non-perturbative) problem of minimization of a polynomial invariant under a group of symmetries [6]. It is required that the Higgs potential (just the Lagrangian is not enough) explicitly breaks the spontaneously broken symmetries, so that the choice of one of the possible reference points related by G uniquely depends on the explicit symmetry breaking term up to $SU(2)_L$ transformations[10]. If the reference point is not unique up to G transformations, then we still make an arbitrary choice but not one which breaks a symmetry—we will try to avoid this case, but it is possible that two different arbitrary choices lead to compatible numerical results and it is not easy to check *a priori* that the reference point is unique up to G transformations.

3 Background symmetries

Let $\mathcal{A}$ be an algebra of operators, let G be a group of global transformations $\mathcal{A} \to \mathcal{A}$, with G_b and $G_f \subset G_b$ normal subgroups of G[11].

Consider a G_f-symmetric functional $\omega : \mathcal{A} \to \mathbb{R}$, by definition all the symmetries conserved by ω are explicitly conserved by all correlation functions, independently of whether the symmetries are spontaneously broken or not. That is, $\omega(A) = \omega(\beta(A))$ for all $A \in \mathcal{A}$ and all $\beta \in G_f$.

The G_f-invariant operators are the representation space of the group G/G_f—we have the homomorphism $G \to G/G_f$ where G_f is the kernel of the homomorphism.

Consider now the functional ω_B depending on a G_f-invariant background field[12] B. In analogy with Def. 1, we say that G/G_f is a background symmetry of ω_B when $\omega_B(A) = \omega_{\beta(B)}(\beta(A))$ for all G_f-invariant $A \in \mathcal{A}$ and all $\beta \in G/G_f$, i.e. ω_B is G/G_f-invariant up to

[9] We are not dependent on these assumptions to determine what would happen if the $SU(2)_L$-gauge orbit minimizing the Higgs potential breaks the $U(1)_{em}$ generator: the photon would become massive due to the abelian Higgs mechanism—there are theoretical arguments [33] and also experimental evidence from superconductivity where the abelian Higgs mechanism also happens. It would not depend on the $U(1)_Y$ gauge-fixing and would not imply spontaneous breaking of the local gauge $U(1)_{em}$ [2]. The $U(1)_Y$ gauge-fixing merely allows us to simplify the study by treating the $U(1)_Y$ symmetry and the remaining global symmetries in the same consistent way, which is particularly useful to interpret the results of non-perturbative lattice studies where $U(1)_Y$ is not a local gauge symmetry (reducing computation time) [8–10].

[10] It makes sense to choose the absolute minimum but that is not *a priori* mandatory. What is mandatory is that the choice of the Higgs reference point from those related by G is uniquely determined by the explicit symmetry breaking term up to $SU(2)_L$ transformations, otherwise it would be possible to have vevs breaking the symmetry $G/SU(2)_L$ even when the explicit symmetry breaking term is exactly null which would be inconsistent with Def. 1.

[11] b/f stand for background/functional for reasons which will become clear in this section.

[12] A spurion or (non-dynamical) background field enters in the definition of the Lagrangian but it is not a variable of the Lagrangian. When calculating the observables, the background fields are replaced by numerical values. It is a representation of a group of background symmetries of the Lagrangian, but there are no Noether's currents associated with such background symmetries if the numerical values are non-trivial. The observables are invariant under the action of the group of the background symmetries. See Ref. [34] for details and related studies; in the particular case that the group of true symmetries is a normal subgroup of the background symmetries (which we is the case we are considering), then the transformations of the group of background symmetries were called equivalence transformations in the literature [35].

transformations of the background fields[13].

As consequence of the isomorphism theorems [36], the following groups are isomorphic $G/G_b \simeq (G/G_f)/(G_b/G_f)$ and the homomorphism $G \to G/G_b$ can be achieved in two steps: first $G \to G/G_f$ and then $G/G_f \to (G/G_f)/(G_b/G_f)$. This is important since we can build operators invariant under the background group G_b using only the operators invariant under the group of symmetries G_f that we constructed in a first step.

Soft symmetry breaking The soft symmetry breaking terms are very useful for phenomenological applications [37]. These are quadratic terms of the Higgs potential, the corresponding parameters can be promoted to background fields, such that the symmetry which is softly broken is a background symmetry. Therefore we can study spontaneous symmetry breaking with the procedure described in Sec. 4 and Sec. 7, in the context of softly broken symmetries.

4 Procedure to add explicit symmetry breaking terms to the Higgs potential

The study of the spontaneous breaking of the background symmetry G/G_b in a potential V which has $SU(2)_L$-gauge symmetry and G background symmetry, under the assumptions of Sec. 2 follows the procedure:

- We replace the background fields by its particular numerical values;
- We find the absolute minimum of the potential and the correspondent constant reference point $\frac{v}{\sqrt{2}}\phi_0$ is chosen ($\phi_0^\dagger\phi_0 = 1$);
- We define the $SU(2)_L$-invariant projector on the $SU(2)_L$-orbit of ϕ_0: $P_0 = \int_{SU(2)_L} dg\, g\phi_0\phi_0^\dagger\, g^\dagger$, with the normalization of the Haar measure of the global group $SU(2)_L$ such that $P_0\phi_0 = \phi_0$.
- We modify the Higgs potential $W = V + \epsilon U$, where $\epsilon > 0$ is arbitrarily small and $U = -v^2\phi^\dagger P_0\phi + (\phi^\dagger\phi)^2$;
- Since the absolute minima of U is also an absolute minima of V, then the absolute minima of $W = V + \epsilon U$ is the absolute minima of U: which is the $SU(2)_L$-orbit of $\frac{v}{\sqrt{2}}\phi_0$;
- We reestablish the background fields and promote the reference operator $\frac{v^2}{2}P_0$ to a background field.

[13]There is a related definition: G/G_f is a background symmetry of ω_B when for any $\beta \in G$ there is some $h \in G_b$ such that $\omega_B(A) = \omega_{h\beta^\dagger B}(\beta h^\dagger(A))$ for all operator A. This definition is worse, since G/G_f is not a group when acting on G_f variant operators and a symmetry should be described mathematically by a group. We can make these two definitions equivalent by choosing an appropriate set of background fields.

The modified Higgs potential W is $SU(2)_L$-gauge-invariant and it has a G background symmetry, but it depends on one more background field ($\frac{v^2}{2}P_0$) than V.

The reference operator is a background field $\frac{v^2}{2}P_0$ whose numerical value depends on the numerical value of the remaining background fields. Since the potential V is G-invariant then $V_B(\phi) = V_{gB}(g\phi)$ so its minima (thus also the reference operator $V_B(\frac{v}{\sqrt{2}}\phi_0) = V_{gB}(g\frac{v}{\sqrt{2}}\phi_0)$) transforms under G covariantly with respect to the remaining background fields (represented by B) [35]. Therefore, the vacuum functional ω_{B,P_0} has a background symmetry G, i.e. $\omega_{B,P_0}(A) = \omega_{gB,gP_0g^\dagger}(g^\dagger A)$ for any $g \in G$ and operator A.

So the question is if the vevs of the operators are invariant under a G-transformation of the numerical values of the reference operator. By construction the operator P_0 breaks $g \in G$ if and only if g is explicitly broken by W. If $g \in G$ conserves $P_0 = gP_0g^\dagger$, then $\omega_{B,P_0}(A) = \omega_{B,g^\dagger P_0 g}(A)$ for all A. If $g \in G$ does not conserve P_0, then $\omega_{B,P_0}(P_0) \neq \omega_{B,g^\dagger P_0 g}(P_0)$. In conclusion, a transformation $g \in G$ is conserved by the vevs (i.e. it is not spontaneously broken) if and only g is explicitly conserved by the modified Higgs potential W.

We can write the vev of A as function of the numerical value of the reference operator $f_A(P_0) = \omega_{B,P_0}(A)$. We say that the background symmetry G_b is conserved by the vev of A when for any $h \in G_b$ we have $f_A(P_0) = f_A(h^\dagger P_0 h)$, i.e. $\omega_{B,P_0}(A) = \omega_{B,h^\dagger P_0 h}(A)$. We then say that the background symmetry G/G_b is conserved when for any $g \in G$ we have $\omega_{B,P_0}(A) = \omega_{B,g^\dagger P_0 g}(A)$ for all operator A whose vev conserves G_b.

Note that since the Higgs potential W has a background symmetry G, then the parameters of the Higgs potential W are background fields, transforming under G covariantly with ϕ. Under the assumptions of Sec. 2 the interactions do not change which symmetries are spontaneously broken or not, so that the modified Higgs potential W determines the spontaneously broken symmetries. Therefore, considering vevs of background operators, i.e. operators involving only background fields, suffices to determine the spontaneously broken symmetries.

This procedure and associated assumptions, were implicitly used before in the study of global symmetries in two-Higgs-doublet models, in the context of lattice simulations [11, 22].

5 Compatibility with Electroweak symmetry breaking

In the procedure described in Sec. 4, we are not assuming that the $SU(2)_L$-gauge symmetry is spontaneously broken, but we are not assuming that it is not spontaneously broken either; the assumptions made in Sec. 2 are compatible with further assumptions on gauge symmetry breaking, and they are suitable for studies looking for evidence of the spontaneous breaking of the $SU(2)_L$-gauge symmetry—e.g. comparing perturbative predictions from vevs of gauge-invariant/dependent operators with experimental results and with non-perturbative studies.

After (perturbative) local gauge fixing and neglecting Gribov-Singer ambiguities, we can treat the $SU(2)_L$ as a global symmetry. Consider the most general Higgs potential V which has a background symmetry G (compact group), with G_b as a normal subgroup. Assuming spontaneous symmetry breaking of $SU(2)_L$, we have the Higgs vev[14] $\Omega_{B,\phi_0}(\phi) = \frac{v}{\sqrt{2}}\phi_0$, where Ω is the $SU(2)_L$-gauge-dependent vacuum functional. Then $\frac{v}{\sqrt{2}}\phi_0$ is a background field which transforms under G covariantly with the remaining background fields [35], so $SU(2)_L$ is a background symmetry of Ω_{B,ϕ_0}.

In analogy with Sec. 4, $g \in G/SU(2)_L$ is conserved by Ω_{B,ϕ_0} when $\Omega_{B,\phi_0}(A) = \Omega_{B,g\phi_0}(A)$ for all operator A whose vev conserves $SU(2)_L$. In Def. 2, the Higgs vev is enough to determine the spontaneously broken symmetries, then the $SU(2)_L$-invariant operators which depend linearly on the Higgs field are enough to determine the spontaneously broken transformations $g \in G/SU(2)_L$. That is, we have to evaluate the vevs of the $SU(2)_L$-invariant operators $\Psi^\dagger\phi$, where ϕ is the Higgs field and Ψ is a polynomial of the background fields and ϕ_0. The vev is $\Omega_{B,\phi_0}(\Psi^\dagger\phi) = \frac{v}{\sqrt{2}}\Psi^\dagger\phi_0$.

Since $\frac{v}{\sqrt{2}}\phi_0$ is the only background field which is not $SU(2)_L$-gauge invariant, then $\frac{v}{\sqrt{2}}\Psi^\dagger\phi_0$ depends on $\frac{v}{\sqrt{2}}\phi_0$ via P_0 only—the invariant tensors of $SU(2)_L$ are the Kronecker delta and the Levi-Civita tensor, and since the Levi-Civita is skew-symmetric it will not contribute for products of ϕ_0 with itself. Note that P_0 is a background field in the $SU(2)_L$-gauge-invariant vacuum functional ω_{B,P_0} of Sec. 4.

Therefore, a transformation $g \in G/SU(2)_L$ is broken (spontaneously or explicitly) in the gauge-dependent vacuum functional Ω if and only if it is broken in the gauge-invariant vacuum functional of Sec. 4.

We conclude that it is possible to study the Higgs potential without making assumptions on whether the $SU(2)_L$-gauge symmetry is conserved or spontaneously broken.

6 Explicit symmetry breaking implies that spontaneous symmetry breaking is allowed

6.1 Non-renormalizable potential

In this subsection we assume that the Higgs potential is a polynomial of arbitrary order. From the point of view of the classical problem of minimization there is no reason to limit the order of the potential—of course that the parameters must be adjusted to reproduce the experimental data and thus the parameters of the potential will be such that they approach a fourth order potential for sufficient small energy scale [38]. When taking into account the quantum effects, then we are working in the framework of an effective field theory, without making assumptions about the ultra-violet completion of the theory, which given the present experimental situation seems appropriate [13, 39].

In the following proposition, a G-invariant Higgs potential is a polynomial of the Higgs field ϕ with an arbitrary order and of the background fields.

[14] For instance, we can add the explicit symmetry breaking term $U = -\frac{v}{\sqrt{2}}\phi^\dagger\phi_0 + \phi^\dagger\phi$ to the Higgs potential $V + \epsilon U$, so that the minimum of the potential is unique and given by $\frac{v}{\sqrt{2}}\phi_0$.

Proposition 3. *Consider a G-invariant Higgs potential $V(\phi)$, with an absolute minimum at $\phi = \frac{v}{\sqrt{2}}\phi_0$.*

There is spontaneous symmetry breaking of $g \in G$ for the Higgs potential V if and only if there is a G-invariant Higgs potential W which is not invariant under a transformation $g \in G$ of the Higgs field at $\phi = \frac{v}{\sqrt{2}}\phi_0$ (i.e. $W(\frac{v}{\sqrt{2}}\phi_0) \neq W(\frac{v}{\sqrt{2}}g\phi_0)$ with the background fields fixed).

Proof. Consider a G-invariant potential W_B, such that $W_B(\frac{v}{\sqrt{2}}\phi_0) \neq W_B(g\frac{v}{\sqrt{2}}\phi_0)$ at $\phi = \frac{v}{\sqrt{2}}\phi_0$ and some $g \in G$, where B are the background fields.

If there is such Higgs potential W, then we can replace all Higgs fields by the minima $\phi = \frac{v}{\sqrt{2}}\phi_0$ and the vev of such operator $\omega_{B,P_0}(W) = W_B(\frac{v}{\sqrt{2}}\phi_0) \neq W_B(g\frac{v}{\sqrt{2}}\phi_0) = \omega_{B,gP_0g^\dagger}(W)$ will break $g \in G$ for the potential V. The reference operator P_0 is the projector on the $SU(2)_L$ orbit of ϕ_0 as in Sec. 4.

Let $p(\frac{v}{\sqrt{2}}\phi_0) = \omega_{B,P_0}(p)$ be a real G-invariant polynomial term in P_0 and in the background fields evaluated at $\phi = \frac{v}{\sqrt{2}}\phi_0$.

If there is no such Higgs potential W, then the operator $p(\frac{v}{\sqrt{2}}\phi_0)$ must also be invariant under $g \in G$, otherwise the term $p(\phi)$ could appear in a G-invariant Higgs potential and so W would exist. Therefore if W does not exist, there is no spontaneous symmetry breaking of $g \in G$. □

In the remaining of the paper we will only consider a polynomial of a fixed order m (which can be order $m = 4$ and thus a renormalizable potential). So the above proposition is not valid, since it was assumed that terms of order larger than m could enter in the potential.

6.2 Not necessarily non-renormalizable potential

In the following proposition, a G-invariant Higgs potential is a polynomial of the Higgs field ϕ with order m (where $m = 4$ or $m > 4$) and of the background fields. G includes Z_2, and Z_2 is a symmetry of the potential (not only a background symmetry).

Proposition 4. *If there is a G-invariant Higgs potential W which is not invariant under a transformation $g \in G$ of the Higgs field only, then there is a G-symmetric Higgs potential U which spontaneously breaks $g \in G$.*

Proof. Consider a G-invariant potential W_B, such that $W_B(\frac{v}{\sqrt{2}}\phi_0) \neq W_B(g\frac{v}{\sqrt{2}}\phi_0)$ for some $\phi = \frac{v}{\sqrt{2}}\phi_0$ and some $g \in G$, where B are the background fields.

We multiply the terms of W by $a\phi^\dagger\phi$ with appropriate constants a as many times as necessary to get only terms of the same order m as the (maximum) order of W in ϕ, such that its symmetry is still G and not larger. We further modify W adding a term proportional to $(\phi^\dagger\phi)^{m/2}$ such that the resulting polynomial V verifies $V(\phi) > 0$ for any $\phi \neq 0$.

Consider the Higgs potential: $U = -y\phi^\dagger\phi + V(\phi)$, and we adjust y such that $\frac{v}{\sqrt{2}}\phi_0$ is the absolute minimum, i.e. $y\frac{v}{\sqrt{2}} = \phi_0^\dagger \frac{\partial V}{\partial \phi^\dagger}(\phi = \frac{v}{\sqrt{2}}\phi_0)$. Note that since V is a term of order m in ϕ, we get[15] $V(\phi) = \frac{m}{2}\phi^\dagger \frac{\partial V}{\partial \phi^\dagger}(\phi)$ so $\frac{v}{\sqrt{2}}\phi_0$ is an absolute minimum of U.

[15] We are assuming a complex field ϕ, for a real field ϕ we get $V(\phi) = m\phi^\dagger \frac{\partial V}{\partial \phi^\dagger}(\phi)$.

Therefore U is an example of a G-symmetric Higgs potential, with an absolute minimum $\frac{v}{\sqrt{2}}\phi_0$ breaking $g \in G$. $\square$

The fact that a Higgs potential with a background symmetry G can in principle be necessarily invariant under a transformation $g \in G$ of the Higgs field only, is related with the fact that not all symmetries are realizable in the Higgs potential [40] and so accidental symmetries may appear.

Note that we can play with the particular values of the parameters so to avoid spontaneous symmetry breaking, so the fact that spontaneous symmetry breaking is allowed, does not imply that it necessarily happens—as we will see in Sec 7 for the case of the $U(1)_{em}$ global symmetry.

Also, the potential U may imply that some subgroup of G (e.g. $U(1)_Y$ or $SU(2)_L$) is also broken by its absolute minimum, this is important if we require that $U(1)_{em}$ is not spontaneously broken; or if we naively try to evaluate vevs of $SU(2)_L$-gauge-dependent operators.

6.3 Perturbative renormalizability

We now present a simple example showing that the converse of the above proposition—i.e. if all G-invariant Higgs potential is invariant under a transformation $g \in G$ of the Higgs field only, then there is no G-symmetric Higgs potential U which spontaneously breaks $g \in G$—is not valid for any group G, in case we limit the G-invariant potentials to fourth order.

Consider a one-dimensional complex field ϕ and the potential $V(\phi) = -\phi^\dagger\phi + \frac{1}{2}(\phi^\dagger\phi)^2$. Then the phase $\phi = \phi_0$ is arbitrary at the minimum of the potential. The potential above is the most general potential invariant under the $G_b = Z_6$ group generated by the transformation $\phi \to e^{i\pi/3}\phi$ and we can check that it is also invariant under $G = U(1)$ generated by the transformation $\phi \to e^{i\theta}\phi$ with θ arbitrary. But the operator ϕ^6 is Z_6-invariant so we could explicitly break $U(1)$ without breaking $G_b = Z_6$ if ϕ^6 would be allowed to enter in the Lagrangian multiplied by an infinitesimal term. The vev of the background operator ϕ_0^6 would spontaneously break $G = U(1)$ while conserving $G_b = Z_6$.

This example can be modified to include the $U(1)_Y \times SU(2)_L$ symmetry, for instance in a two-Higgs-doublet model, with $Z_6 \subset SO(2)$ being a subgroup of the rotations of the two doublets.

7 Custodial symmetry and the physical spectrum

As it was mentioned in Sec. 2, we can consider the $U(1)_Y$ gauge symmetry as a global symmetry. In this section, we will not consider the $U(1)_Y$ gauge symmetry at all, local or global. We also consider the Higgs field as a real representation space which enlarges the possible symmetries, we can do it since the potential is a real function [27, 41, 42]. There are good reasons for this: for efficiency, the lattice simulations of Electroweak physics usually do not include the photons; the custodial accidental symmetry (which includes the global $U(1)_{em}$ and CP) plays an important role in Electroweak physics, despite that it is broken by the $U(1)_Y$ gauge field; this apparent redundancy is important to predict correctly the number of pseudo-Goldstone bosons after spontaneous symmetry breaking in the multi-Higgs doublet models [25–27]; moreover, we want to study in detail the CP transformation, which necessarily affects the generator of the $U(1)_Y$ gauge symmetry anyway[16].

The group of background symmetries G includes $SU(2)_L$ as a normal subgroup, but the outer automorphisms of $SU(2)_L$ are trivial so $G = (G/SU(2)_L) \times SU(2)_L$. Therefore $(G/SU(2)_L)$ conserves the generators of the $SU(2)_L$ gauge group, which form an algebra of quaternions.

The maximal background group G_m is then $G_m = (Sp(n)/Z_2) \times SU(2)_L$ where $Sp(n)$ is the compact symplectic group and n is the number of Higgs doublets [27, 43][17]. The maximal background symmetry which conserves the reference operator P_0 and thus it is not spontaneously broken[18] is $G_0 = Sp(n-1) \times SO(3)$, it is a normal subgroup of $G_m/SU(2)_L$.

The elements $g \in Sp(n)$ are given by the $n \times n$ real matrices Z_0 (skew-symmetric) and Z_j (symmetric), such that $g = \exp(Z_0 1 + Z_j i\sigma_j)$ where $i\sigma_j$ are Pauli matrices corresponding to the generators of the custodial group $SU(2)_R$. Since there are $n(n-1)/2$ skew-symmetric and $n(n+1)/2$ symmetric real matrices, the total number of generators is $n(2n+1)$.

Then, the hermitian matrices are given by $(H_0 + S_j i\sigma_j)$, with H_0 (symmetric) and H_j (skew-symmetric), so we have $2n(n-1) + n = n(2n-1)$ linearly independent possibilities for Higgs bilinears appearing in the Higgs potential. Since the invariant tensors of $SU(2)_L$ are products of Kronecker deltas and Levi-civita tensors, then the Higgs potential only depends on the Higgs field via Higgs bilinears.

For one Higgs-doublet $G_m = (Sp(1) \times SU(2)_L)/Z_2 = (SU(2)_R \times SU(2)_L)/Z_2$. The only Higgs bilinear is $\phi^\dagger \phi$. Therefore under the assumptions of Sec 2, the custodial symmetry (thus also $U(1)_{em}$ and CP) cannot be explicitly broken or spontaneously broken since there is only one $SU(2)_L$-orbit.

On the other hand, for more than one Higgs doublet, there is by construction more than one $SU(2)_L$-orbit so we can have the Pauli matrices $i\sigma_j$ corresponding to the custodial $SU(2)_R$ generators appearing in the Higgs bilinears, breaking explicitly and completely the custodial symmetry. Therefore in the context of multi-Higgs-doublet models, according to Sec. 6 there may be also spontaneous symmetry breaking of $U(1)_{em}$ and/or CP, as is already well known [25–27].

We introduce a $SU(2)_L$-invariant unitary matrix $R : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$, related with the reference point as $(\mathcal{P}_0)_{kl} = (R^\dagger P_0 R)_{kl} = \delta_{k1}\delta_{1l}$ $(k, l = 1, ..., n)$, as a consequence $(\mathcal{P}_0)_{kl}$ commutes with the generators of the custodial group $i\sigma_j$—where P_0 is the projector in the reference $SU(2)_L$-orbit defined in Sec 4. Note that the group G_0 conserves $(\mathcal{P}_0)$.

We can then express the Higgs field as $\varphi = R^\dagger \phi$, so the unitary matrix defines a basis for the

[16] In a rigorous definition of a complex representation, the imaginary unit is not affected by any transformation. The Majorana representations [27] or the set of all present representations [35] which were used before to deal rigorously with the CP transformation are isomorphisms of real representations.

[17] In Ref. [27], the same group is called $Sp(2n)$ instead of $Sp(n)$.

[18] There are symmetries which are necessarily spontaneously broken, these were studied and called frustrated symmetries in Ref. [44].

space of Higgs doublets, we will call it the reference basis since it is related with the reference point $\varphi_0 = R^\dagger\phi_0$ such that only the first $SU(2)_L$ doublet is non-null $(\varphi_0)_k = \delta_{k1}(\varphi_0)_1$.

Suppose that the quadratic part of the G-invariant potential V is given by $\phi^\dagger Y\phi$ and the quartic part is $(\phi^\dagger \otimes \phi^\dagger)Z(\phi \otimes \phi)$ [19]. We have then the parameters $\mathcal{Y} = R^\dagger Y R$ and $\mathcal{Z} = (R^\dagger \otimes R^\dagger)Z(R \otimes R)$ and $\mathcal{P}_0$ (which appears in the modified potential W).

We assume now without loss of generality, that $i\sigma_j\varphi_0 = i\tau_j\varphi_0$ after gauge-fixing, where $i\tau_j$ are the $SU(2)_L$ generators. From the Higgs field φ we can also form $SU(2)_L$-invariant operators, in particular the operators $\varphi_1^\dagger\varphi_k$ and $\varphi_1^\dagger i\sigma_j\varphi_k$ expand to the elementary fields in leading order of the expansion around the reference point and therefore correspond to the physical Higgs bosons appearing in the spectrum of the model, except for the would-be goldstone bosons $\varphi_1^\dagger i\sigma_j\varphi_1$ since $\varphi_1^\dagger i\sigma_j\varphi_1$ is null due to the fact that the $i\sigma_j$ are skew-hermitian 4×4 real matrices. Also $\varphi_1^\dagger D_\mu i\sigma_j\varphi_1$ expands to the $SU(2)_L$ gauge fields and $\varphi_1^\dagger\Psi$ and $\varphi_1^\dagger i\sigma_j\Psi_L$ expand to an elementary left-handed fermion field, where D_μ is the covariant derivative involving the $SU(2)_L$ gauge fields and Ψ_L is a fermionic $SU(2)_L$ doublet in the reference basis[20].

8 Geometric CP-violation

Following Sec. 7, we now consider the global $U(1)_Y$ gauge symmetry as a symmetry of the potential. The maximal group which conserves $U(1)_{em}$ is $G_{em}/SU(2)_L = (PSU(n) \times U(1)_{em}) \rtimes Z_2$ since it must contain $U(1)_{em}$ as a normal subgroup, where Z_2 is the CP (charge-parity) transformation.

As we have seen, in Sec. 7 the $U(1)_{em}$ can be spontaneously broken. But since $U(1)_{em} \rtimes Z_2 \subset G_0$, there is no spontaneous symmetry breaking of the background symmetry $U(1)_{em} \rtimes Z_2$. The CP transformation Z_2 is given by $\varphi \to -i\sigma_1 i\tau_1\varphi$ and the generator of $U(1)_{em}$ given by $\varphi \to (i\sigma_3 - i\tau_3)\varphi$. Note that as in Sec 7, $i\sigma_j\varphi_0 = i\tau_j\varphi_0$ after $SU(2)_L$ gauge fixing, where $i\tau_j$ are the $SU(2)_L$ generators and φ_0 is the reference point. We have then the CP-even neutral elementary fields $(\varphi_0)_1^\dagger\varphi_k$, the CP-odd neutral elementary fields $(\varphi_0)_1^\dagger i\sigma_3\varphi_k$ and the (complex) charged elementary fields $(\varphi_0)_1^\dagger(i\sigma_1 - \sigma_2)\varphi_k$. So the imaginary unit corresponds to $i\sigma_3$ and CP acts as a complex conjugation.

Neutral vacuum Assuming now that $U(1)_{em}$ is a true symmetry in the reference basis, then $\mathcal{Y}_{kl}$ and $\mathcal{Z}_{km\ ln}$ and $(\mathcal{P}_0)_{kl}$ are complex tensors—i.e. they commute with $i\sigma_3$. But they are not CP-invariant unless the tensors are real because CP acts as the complex conjugation.

All phases in the reference basis come from background fields, since the Z_2 (complex conjugation in the reference basis) background symmetry is not spontaneously broken. In this sense, CP violation is always determined by the background symmetry G.

[19]For simplicity we consider a potential up to fourth order, but we could consider more orders here. See Sec. 6.1.

[20]See [22] for more details on how to add the $U(1)_Y$ gauge field and the Yukawa couplings, in the case of the two-Higgs-doublet model—the generalization for the n-Higgs-doublet model is straightforward in the reference basis (which is the Higgs basis in two-Higgs-doublet model).

The group G may imply that the numerical phases of the background fields are arbitrary, or that all numerical phases are restricted to a finite set of possibilities (see below), or even that all the numerical phases are null (see Sec. 9). So the question now is if CP is just a background symmetry or it is a true symmetry.then all phases in the reference basis are related and the CP background symmetry is not s

$\Delta(54)$-symmetric three-Higgs-doublet model Geometric CP-violation involves calculable phases [24, 35, 45–47].

The idea of spontaneous geometric CP-violation arose in a three-Higgs-doublet model, with a $\Delta(54)$-symmetric Higgs potential which is a polynomial of fourth order. However, it is now known that the $\Delta(54)$-symmetry is not enough to guarantee spontaneous geometric CP-violation in a non-renormalizable potential beyond fourth order [48][21]. Moreover, there is also explicit geometric CP-violation [49] which shows that the root of geometric CP-violation is not the process of minimization of a polynomial.

We describe it not as CP-violation, but as CP conservation up to a background phase. So we are dealing with CP as a background symmetry.

We consider a three-Higgs-doublet model, with explicit symmetry $G_f = SU(2)_L \times U(1)_{em} \times (\Delta(54)/Z_3)$. Promoting the parameters of the fourth order potential to background fields, we have a background symmetry $G = G_b \rtimes Z_2$, with $G_b = SU(2)_L \times U(1)_{em} \times (\Sigma(216 \times 3)/Z_3)$, $G_b/G_f = A_4$ and $G/G_b = S_4$ [50].

As we have seen above, there is no spontaneous symmetry breaking of the background symmetry $G/G_b = U(1)_{em} \rtimes Z_2$. In the reference basis, one absolute minima is necessarily real. Choosing such minimum as the reference point, then the phases are all in the background fields (since there is no spontaneous violation of the background CP symmetry). On the other hand, imposing an explicit CP symmetry to the numerical values of the background fields, it was shown [35] that the phases of the parameters of the Higgs potential are limited to a finite set. These are the calculable phases.

Note that if the parameters of Higgs potential are $g \in G_f$-invariant in some basis, then they are also in the reference basis with the representation of G_f transformed to $UgU^\dagger$, and U is the unitary matrix doing the basis change.

We conclude that the absence of explicit violation of the background symmetry G/G_b, implies that G/G_b cannot be spontaneously violated. Therefore all CP-phases are in the background fields which are constrained by the imposition of a explicit CP-symmetry in the Higgs potential.

We can then use the Higgs vev to modify the Higgs potential, using the procedure of Sec. 4. The modified Higgs potential constitutes an example of explicit geometric CP-violation, which is always associated to spontaneous geometric CP-violation, as consequence of the fact that spontaneous symmetry breaking can be defined as a particular case of explicit symmetry break-

[21]Despite that for a large parameter region, the geometrical CP violation still holds in a non-renormalizable potential.

ing. In the sense of Def. 1, explicit geometrical CP-violation was necessarily present in the first example of spontaneous geometrical CP-violation [24]. Note that there are other examples of explicit geometrical CP-violation which do not involve the Higgs potential [49].

9 Absence of explicit CP-violation

We assume now that there are no polynomial terms which can break CP explicitly, in a G_f-invariant potential without background fields, as it was studied in the conjecture of Ref. [23]. In all the examples considered in Ref. [23], the group of symmetries was given by $G = G_f \rtimes Z_2$, so there was always an order-2 CP transformation present[22]. We assume therefore that $G = G_f \rtimes Z_2$. We also assume that $U(1)_{em}$ is conserved by the minimum of the potential. This can be seen as a particular case of CP-conservation up to a finite set of numerical phases (i.e. geometric CP-violation), where the finite set is the empty set.

Proposition 1 If explicit breaking of $G = G_f \rtimes Z_2$ (thus CP-violation) is not possible in a G_f-symmetric Higgs potential, then there is no spontaneous CP-violation.

Proof. There is a basis where the Z_2 transformation is the complex conjugation. As in Sec. 8, if the parameters of the Higgs potential are $g \in G_f$-invariant in some basis, then they are also in the reference basis with the representation of G_f transformed to $UgU^\dagger$, and U is the unitary matrix doing the basis change. Since in the reference basis the Z_2 transformation is also the complex conjugation, then U commutes with the complex conjugation so it is a real matrix.

The absence of spontaneous symmetry breaking of the background symmetry $U(1)_{em} \rtimes Z_2$ (see Sec. 8), implies then that also in the reference basis the parameters of the Higgs potential are real. □

From the proposition of Sec. 6, it follows that if explicit CP-violation is possible in a G_f-symmetric Higgs potential, then spontaneous CP-violation is possible. However, we are not assuming the vacuum to conserve $U(1)_{em}$ here.

If we additionally assume that we will choose parameters of the Higgs potential such that $U(1)_{em}$ is conserved by its minimum, then the terms in the Higgs potential which will be null for a neutral vacuum correspond to Lagrange multipliers to minimize the potential [25–27]. It is easy to see in the proof of the proposition of Sec. 6, that these terms do not lead necessarily to spontaneous CP-violation.

Then we have to look for the terms which verify 3 conditions: break CP, conserve $U(1)_{em}$ and finally are non-null at the neutral minima. If such term exists then spontaneous breaking of CP is allowed.

[22] In the context of groups which are not finite and abelian, the author is not aware of any example where $G = G_f \rtimes Z_2$ does not hold, and no such example is discussed in Ref. [23]. Note that the particular case of finite abelian groups was discussed fully in Ref. [23] and will not be addressed here.

We can also consider a different potential, function only of neutral Higgs fields (the neutral Higgs sector as in Ref. [23]). For such potential we can adapt the above discussion to show that for $G = G_f \rtimes Z_2$: if explicit CP-violation is not possible in the neutral Higgs sector of a G_f-symmetric Higgs potential, then there is no spontaneous CP-violation for a neutral vacuum. Note however that this different potential is only used as an auxiliary function to evaluate spontaneous CP-violation, since such potential cannot appear in a $SU(2)_L$-invariant Lagrangian.

So we explained the conjecture of Ref. [23]), for groups including CP transformations of order 2. However, there are more general groups involving only CP transformations of order > 2 [51], in principle there may be models where the Proposition 1 is still valid despite that the assumption $G = G_f \rtimes Z_2$ does not hold. Better mathematical tools to deal with such CP groups may help, for instance an algebraic basis of CP-pseudoscalars is not yet known for general multi-Higgs-doublet models [51].

Note that if we assume a Higgs potential of arbitrary order, then Prop. 6.1 applies and the condition $G = G_f \rtimes Z_2$ is not necessary.

10 Conclusion

Dealing with concepts which are not rigorously defined (in the mathematical sense) can have advantages with respect to an approach where every concept is rigorously defined [52]. In the context of Electroweak physics that is necessarily the case since a rigorously defined non-abelian gauge Quantum Field Theory does not exist yet. Therefore, assumptions play a key role.

But after making assumptions some problems are still very complicated. That is the case of building extensions of the Standard Model[23], and in particular studying the Higgs potential (a symmetric polynomial of many variables [6]).

So we started with a very complicated and not rigorously defined problem (Electroweak physics), then we make some assumptions, and end up with a new but still very complicated problem (minimization of a symmetric polynomial) only this time it is a rigorously defined problem. We should be careful: making assumptions can be used to focus on the physical questions as much as it can be used to avoid the physical questions.

To study the Higgs potential, one option is to check what are the implications of alternative assumptions. Such as non-perturbative assumptions—such as the ones used in lattice gauge theory or in the functional renormalization group, which can produce complementary results [12, 13]. Or working with real representations of groups—which in a real polynomial makes sense [42]

[23] Using extensions of the Standard Model is a practical way to produce predictions for experiments. But like statistical inference [53], (new) physics is not just about producing numbers. E.g. accounting all reasonable extensions, we may have one prediction for each logical possibility [54], which is a kind of look-elsewhere effect. Producing predictions where such effect is consistently accounted for is a hard problem (even if we use subjective but still consistent criteria, e.g. assumptions on spontaneous symmetry breaking without enough experimental or theoretical support).

and it is necessary[24] to deal with the custodial symmetry of the Higgs potential [25–28]. In this paper we showed that such option does lead to progress, despite that the perturbative Electroweak expansion is a good approximation to the experimental results.

We added explicit symmetry breaking terms to the Higgs potential, so that the spontaneous breaking of a global symmetry in multi-Higgs-doublet models is a particular case of explicit symmetry breaking. Then we showed that it is possible to study the Higgs potential without assuming that the local gauge $SU(2)_L$ symmetry is spontaneously broken or not. We showed that explicit symmetry breaking implies that spontaneous symmetry breaking is allowed. We also discussed the physical spectrum of multi-Higgs-doublet models and the related custodial symmetry.

We reviewed background symmetries, which despite they are already explicitly broken can still be spontaneously broken. We showed that the CP background symmetry is not spontaneously broken. We then related explicit and spontaneous geometric CP-violation. We also explained in part a recent conjecture relating spontaneous and explicit breaking of the CP symmetry.

Our study of the CP symmetry benefited much from the insights of non-perturbative studies and of considering real representations of groups. There are yet many unsolved problems, for instance an algebraic basis of CP-pseudoscalars is not known for general multi-Higgs-doublet models [51].

Therefore, the phenomenology of multi-Higgs-doublet models is not yet well understood, assuming gauge symmetry breaking or not. Also, the usual assumption that considering the fields as complex representations of groups suffices to study Higgs models (i.e. real representations are not needed) is not valid, since in fact it does not even suffice to study the operation of complex conjugation (related with the CP pseudoscalars).

In conclusion, assuming gauge symmetry breaking or using only complex representations of groups is not sufficient to study the phenomenology of multi-Higgs-doublet models.

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