The Lorentz Force Law And Kaluza Theories

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Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is inspiration for many modern attempts to develop new physical theories. The original theory has problems which may well be overcome, and thus Kaluza theory should be looked at again: it is a natural, if not necessary, geometric unification of gravity and electromagnetism. Here a general demonstration that the Lorentz force law can be derived from a range of Kaluza theories is presented. This is investigated via non-Maxwellian kinetic definitions of charge that are divergence-free and relate Maxwellian charge to 5D components of momentum. The possible role of torsion is considered as an extension. It is shown, however, that symmetric torsion components are likely not admissible in any prospective theory. As a result Kaluza's original theory is rehabilitated and a call for deeper analysis made.

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1 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [1][2][3][4] using a fifth wrapped-up spatial dimension gives a taste of unification of electromagnetism with gravity in a way that is generally considered incomplete and is widely believed to be untenable. Nevertheless all sorts of variants and modern versions have been constructed [1][5]. The underlying aim was, and remains, particularly promising in terms of explanatory power due to the natural unification of electromagnetism with geometry. The phrase 'Kaluza miracle' used to be used more often than now to express the stunning coincidence of Kaluza's original theory. Under certain circumstances the stress-energy tensor of electromagnetism could be derived from Kaluza's relatively straight forward and geometrically appealing assumptions. Such is often the nature of the explanatory power of pure mathematics in theoretical physics [6]. The philosophy of this investigation lies in the notion that such explanatory power is not only a guide to the theoretical physicist, when dealing with fundamental physics, but necessary. Though discussion of such a post-reductionist philosophical view is not the aim of this paper it is the personal belief of the author that this is the case [7][8][9], and that such an approach in theory design is being obfuscated by the conceptual difficulties of quantum mechanics. The approach here is classical.

Naturally Kaluza's theory may be a part of a larger and still more explanatory theory, for example a full unification of all four known forces. So here mathematical generality is sought for future compatibility and applicability.

Kaluza theories naturally define the electromagnetic 4-potential out of 4 components of the 5D metric that can be carried over to the 4D embedding that represents Einsteinian space-time [1], and from which Maxwell's laws can be derived in the natural way. It naturally embeds general relativity [10][11][12] using the so-called cylinder condition. The cylinder condition states that partial derivatives in the Kaluza direction are vanishing. Charge conservation however is not usually precise in the Maxwellian sense due to the derivation of the field equations of Kaluza theory in terms of a scalar field, that also arises from the metric. This typically prevents Maxwellian charge from being fully divergence free. The Lorentz force law [10] is not usually dealt with by such theories, perhaps because it can be derived from the stress-energy tensor of electromagnetism [10]. However, this assumes the usual stress-energy tensor is always follows. The scalar field means this is not the case.

The Lorentz force law is the most enigmatic and conceptually unsatisfying physical law within current classical theory in the author's opinion. A study of the problems of its derivation from the usual electromagnetic stress-energy tensor [10] justify this concern. The Lorentz force law, however, is but the relativistic form of Coulomb's law. In this sense it is as simple and fundamental as the inverse square law of gravity. It is central to the understanding of charge and electromagnetism. It is in this vein that derivation of the Lorentz force law independently of the usual electromagnetic stress-energy tensor is undertaken. By making it independent of the field equations it becomes independent to some larger extent of the particular theory being used. Details provided by the text.

The Lorentz force law derived elsewhere [13] in Kaluza theory usually requires a constant scalar field, where the scalar field is a consequence of the way space-time is embedded in the 5D Kaluza space. This however places constraints on admissible solutions, actually quite tight ones, that prevent the full range of electromagnetic fields that are physically required. Reintroducing them requires more degrees of freedom. Previously this led the author to consider variants of Kaluza theory, rightly or wrongly, that at the time of writing this paper found little interest or criticism [14][15][16]. Not using these previous works as a dependency is therefore important. Nevertheless these works were an influence, an essential part of the process leading to this work, and this work is in many ways a refined compilation and corrected culmination of the various analyses of those previous drafts. The Internet age also provides us with tools to better record and reference such otherwise private and likely flawed notes. Some of the issues need reiteration:

Definition 1.0.1: 'Nullish' electromagnetic fields satisfy: $F_{ab}F^{ab} = 0$. Null electromagnetic fields have the nullish property plus the following condition, where the star is the Hodge star operator: $F_{ab}(*F^{ab}) = 0$.

Kaluza's original theory [1] prohibits non-nullish solutions (or even near non-nullish solutions) where the metric defines a constant scalar field. Nullishness is too tight to admit important electromagnetic fields, in particular the essential electrostatic fields. That electrostatic or near-electrostatic fields are non-nullish, and therefore a problem in any theory that omits them, can be seen by comparing definition (1.0.1) with the following well-known fact from special relativity. That is, by considering the special relativistic limit:

$$F_{ab}F^{ab} = 2(B \cdot B - E \cdot E) \tag{1.0.2}$$

.

Thus the previous works [14][15][16] were aimed at increasing degrees of freedom in different ways to allow for such non-nullish electromagnetic fields.

The aim here is mathematically more general, and should therefore be of more interest to more researchers. It is not assumed that the scalar field is constant. A range of possibilities are allowed such as whether or not to use torsion. The work here is independent, within the limits defined, of how Kaluza theory may ultimately be embedded in more far-reaching theories. The work here, therefore, has a higher value as a resource. Thus we make reference to both Kaluza and Kaluza-Cartan theories (see shortly).

The killer criticism of Kaluza theories more generally is the problem of stability [17]. Essentially the wrapped up fifth dimension tends to collapse under positivity of curvature. The analogy with mass-energy and the energy conditions that loosely define this positivity (that are related to causality in general relativity [18]) lead inevitably to the failure of Kaluza theory. This is the most important objection to Kaluza theories. Alternative approaches were explored in [14][15][16] which are discussed again here if for no other reason than to point out, simply, that the stability problem arises from assumptions regarding matter models and curvatures and are not necessarily true in all Kaluza theories. Again this leads to the need to generalise the derivation of the Lorentz force law so that the widest range of possible alternatives may be permitted under alternative assumptions. In this way the correct theory, if such exists, need not be identified here. But mathematical generality should be sought. The need to avoid mathematical and logical dependencies on previous works that are not fully peer-reviewed is maintained throughout.

All things considered, this work re-establishes Kaluza theories in the form originally envisaged by Kaluza (and by Einstein [19]): as unifications of electromagnetism and gravity. We might loosely term Kaluza theories that use torsion Kaluza-Cartan theories - as already done by the author in [14] and [15]. Where possible the present results have been extended to include torsion, but do not depend on torsion.

2 A Development Note

At first the objective of the research undertaken here was to try to discount torsion [20][21][22][23][24][11] as a source of needed degrees of freedom, since its lack of presence is geometrically an obvious assumption in many physical theories. This is analogous to Euclid's fifth postulate. The assumption of Euclid's fifth postulate is an addition, and its removal enabled geometric theories like general relativity to be possible. Perhaps the same might be true for torsion? Whilst few would consider it necessary or even a good idea to investigate such an assumption, that was the original program. This might be called a post-reductionist approach [7][8][9] in that the widest possible explanatory simplicity of the whole is sought, trying to glean more than the sum of the parts. Practically this meant showing that a sufficient range of electromagnetic fields could be obtained (without torsion) from existing Kaluza theory. That program failed at first and the result was therefore the exact opposite: to then try to explicitly allow torsion to obtain the extra degrees of freedom required. This itself was unsatisfactory [14][15] in that some of the postulates seem arbitrary. A little more detail follows.

Kaluza theory depends on electromagnetic fields in curved space-time being defined in 5D by having 5D Ricci flat curvature. This curvature can be defined differently with different connections. Thus degrees of freedom can be added to Kaluza theory in 5D vacuum by allowing the Ricci flatness to be defined in terms of, say, a torsion connection. The result of this was the rather unwieldy theory presented in [14] since the degrees of freedom presented in Kaluza theory with constant scalar field and without torsion was simply inadequate. This theory presented a number of further unsatisfactory characteristics. One being the problem of the need for symmetric components of torsion that were curtailed by an order of magnitude constraint that appears arbitrary. The next step was to attempt to omit the symmetric torsion terms altogether [15]. However the resultant theory also contained arbitrariness in a similar manner, albeit hidden within different postulates.

So the next step was [9], an attempt to go back to the original ideas of a Kaluza theory strictly without torsion, but this time by not being too strict on the Ricci flatness requirement. This of course leads to interpretational difficulties: how do you distinguish electromagnetic fields from matter models? However, this paper presents no such difficulties.

Arguments based on four dimensional theories have been made against the use of symmetric torsion components [25]. It is interesting that this issue arises again in this paper. It is also the reason why [15] was developed out of [14]. Nevertheless it also seems that torsion is a natural extension of general relativity required by the presence of classical spin, or point sources of classical angular momentum [26][21] - these considerations seem to lead naturally to Einstein-Cartan theory in 4 dimensions, as a necessary extension, whether for fundamental or modelling purposes. Einstein-Cartan theory appears to be an ω-consistent extension of general relativity. It should also be the case in Kaluza theories that torsion is a useful extension. The mathematics in this paper therefore tries to maximise generality.

The scalar field is here allowed to vary, and torsion is included with such generality that you can bolt it on or remove it (at least completely antisymmetric torsion) as required. Getting the generalisation right for the derivation of the Lorentz force law in the presence of a scalar field is the important content of this work. This finally resolves both the problem of degrees of freedom and the arbitrariness of assumptions present in the previous torsion-based variants. To do this we make careful use of limits and orders of magnitude estimates.

In all the previous research, as well as in this one, kinetic charges are defined in terms of 5th-dimensional components of momentum. This was briefly outlined in [13] under very limited conditions. A Lorentz force law follows in many cases. As momentum the kinetic charge has a divergence law via the Einstein tensor. It approximates Maxwellian charge. The definition of charge used throughout this work references the Levi-Civita connection, and is in no way determined by torsion. Maxwellian charge also has a vector potential and thus local conservation, but kinetic charge being covariant with respect to the Levi-Civita connection is the more fundamental in five dimensions. These issues are expounded in the text proper.

3 A Note on Stability, Causality and Matter Models

The killer criticism of Kaluza theories is the problem of stability [17]. Essentially the wrapped-up fifth dimension tends to collapse under evolution over time. The analogy with mass-energy and the energy conditions that loosely define curvature non-negativity, and are perhaps essential for causality in general relativity [18], lead inevitably to this failure of Kaluza theory. This is the usual reason to consider Kaluza theories untenable - we are alternatively forced to resort to 'exotic matter'.

However, it's possible to get around such issues with a little tolerance for unknowns.

The phrase 'exotic matter' has connotations of arbitrariness and empirical unphysicality. But that is a 4D consideration. What is essentially needed in 5D is a different approach to both the positivity of matter-energy, and to causality. Getting around the stability problem follows from simply not extending the energy conditions into 5D, but instead using a different approach. We still need a classical causal limit and 4D positivity (or similar) for mass and energy, and in particular real-life observable particles. But the extra dimension allows for the possibility of 5D exotic curvature that does not correspond necessarily with 4D exotic matter. Astronomical observations suggesting a cosmological constant also complicate the discussion - the cosmological constant is usually implemented via an addition to the Einstein tensor, but it could equally be an arbitrary factor in the definition of the energy conditions. In a sense there is no 'correct' energy condition: they are applied as required. However the problem with that is, consequently, there is no definitive, no uniformly applicable, definition of the underlying positivity of matter-energy in general relativity. The lack of global energy content for gravitational waves further raises questions regarding matter-energy in general relativity. Whilst all this may pose few problems for the working physicist who is modelling particular observed phenomena, a post-reductionist approach demands more: whatever the local and quasi-local resolutions may be to the gravitational wave problem, a more natural interpretation is suggested by the Bel-Robinson tensor [11], and this may be taken as a leading suggestion.

The Bel-Robinson tensor is but one example of what are called super-energy tensors [27][28][29]. They always have positivity in a well-defined and intuitively appealing sense. This makes them particularly appealing as alternatives to the Einstein tensor. It is proposed here, as in [14][15][16] (and presumably elsewhere, since the idea is quite obvious and seems to be behind much of the mathematical development of super-energy tensors) that they may hold the solution to this conundrum.

Here's how it could work: The vanishing of the divergence of super-energy tensors is linked with the causality [27] of the underlying tensor with which the super-energy tensor is associated. This is known [27]. It is modelled on similar reasoning to the conservation theorem [18] in general relativity. Whilst more work needs to be done to clarify this, there is sufficient case presented in [27] to support the argument here.

The generalised Bel tensor [28][29] is associated with the Riemannian curvature (where all contributing tensors, connections and operators are defined without reference to torsion), and similarly the generalised Bel-Robinson tensor is associated with the Weyl tensor. The word 'generalised' is used in the literature to indicate n-dimensional definitions, rather than just the usual four. That clarification is dropped here. Interestingly the Bel-Robinson tensor is only necessarily symmetric in 4 and 5 dimensions, exactly those of interest. The 'causality' thus proven [27] in the case of vanishing divergence of an arbitrary super-energy tensor (actually the condition that it be vanishing is tighter than necessary [27]) is not as clear a conception of causality as ideally desired, but it's a good start. For starters the causality of the Riemannian curvature doesn't necessary imply the causality of the metric. Further, as with Cauchy-Kowalevsky type theorems, it is only a local result and does not make for a well-posed theory. But remember, real physics isn't causal. It has causal features of course (even in quantum mechanics), but real physics need not demand a well-posed theory in the sense general relativists assume [11][18]. So maybe the 'causality' (in the sense of [27]) of certain 5D super-energy tensors is all that is experimentally, (ie actually) required? Further constraints in any case may be added in 5D to tighten the geometrical constraints further, and produce determined and over-determined Cauchy problems.

The original Kaluza theory imposed Ricci flatness and derived a limited subset of electromagnetic fields (when the consequent scalar field was set constant) from that. A trick to derive all electromagnetic fields is possible, the scalar field can be set large [1]: but that is as arbitrary as any other fix previously presented. The idea is to here allow the scalar field to vary more naturally in Ricci flat Kaluza space (ie the 5D space) and derive a Lorentz force law anyway, and to do this independently of the hypothesis (or not) of torsion. Matter models are then just non-Ricci flat parts of the 5D Kaluza space, or regions where the 5D Einstein tensor is not Ricci flat. This latter point, Ricci flatness outside of matter models, suggests we might look at Ricci scalar flat spaces for matter models too. This could be an example of a tightening geometrical constraint that we may be allowed in 5D, if we choose, that would be unreasonable in 4D. Campbell's embedding theorem [30] suggests such a constraint on the Kaluza space could be reasonable in 5D. So this is just one example of adding further geometrical constraints (on matter models in this case) to impose further control over such properties as causality in Kaluza theories - fine tuning of this can await application, further development and/or empirical data.

A super-energy tensor is in some sense a measure of the square of its underlying tensor, thus it is interesting to note that if the vanishing of the divergence of the Bel tensor is taken to be the (in some sense) correct energy condition, it would not prohibit negative mass-energy. But it would make the proximity of negative and positive mass-energies expensive. The results in 4D would consequently appear approximately similar to the positivity of the Einstein tensor. Or so it can be argued. This potentially opens the door to 4-geon [31][32][33][34] and 5-geon topological structures for particles. Could it help deal with any outstanding cosmological anomalies too? The question of stability of Kaluza theories therefore is still open: the case is not closed.

Whatever the outcome, the stability issues of the original Kaluza theory, with simple energy conditions following general relativity, cease to apply. The killer objection to Kaluza theories is simply not valid without assumptions that need not in any case be made.

A further point about n-D geometry is worth making: when the Riemannian curvature is harmonic [35][36][37], it follows that the Bel tensor is divergence free [28][29]. 5D Harmonic matter models may therefore be quite natural. Further, Ricci scalar flat harmonic matter models [38][35][36][37] generalise Ricci flatness (or actually Einstein spaces [38]) in that both Bel and Bel-Robinson super-energy tensors then have vanishing divergence. Using such constraints the 5D geometry may quickly become over-determined. Causality becomes the least of the problems - finding exact solutions to model practical situations becomes an impractical theoretical requirement, although some nice properties such as real analicity result too. Although this may make Kaluza theories too difficult for immediate practical use, that isn't the issue here.

The weakest reasonable assumption would be to impose divergence of the Bel tensor on 5D matter models, in second place (for simplicity) followed by the probably slightly tighter condition: the harmonicity of the Riemann tensor. For maximum generality of this work no further postulates are made regarding the divergence laws of super-energy tensors. A weakness in previous attempts [14][15][16] was in trying to prematurely make such requirements explicit.

Whether or not analogous approaches apply to super-energy tensors that involve torsion in their definitions is also superfluous to the objectives of this paper. The aim of this section is simply to show that objections to Kaluza theories are not mathematically founded without additional assumptions which would in any case be unwarranted given the preceding discussion.

4 Conventions

The following conventions are adopted unless otherwise specified. Though unfamiliar in places these are necessary for following the multiple systems used and need to be constantly referred to to avoid confusion.

Five dimensional metrics, tensors and pseudo-tensors and operators are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. Lower case indices can either be 4D or generic for definitions depending on context. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope. Space-time is given signature $(-, +, +, +)$, Kaluza space $(-, +, +, +, +)$ in keeping with [10]. The Minkowski metric therefore has a determinant of -1. Under the Wheeler et al [10] nomenclature the sign conventions used here as a default are $[+, +, +]$. The first dimension (index 0) is time and the 5^{th} dimension (index 4) is the topologically closed Kaluza dimension. Time and distance are not geometrized so c is the speed of light and **G** the gravitational constant. The scalar field component is labelled ϕ^2 as in the literature. It may also be labelled Φ if the index gets in the way. The matrix of g_{cd} can be written as $|g_{cd}|$. The Einstein summation convention may be used without special mention.

Connection coefficients with torsion will take the form: Γ^c_{ab} or Γ^{abc}. The metric with a torsion tensor defines a unique metric connection. Therefore two unique connections for a given metric are one with and one without torsion, though they may coincide when no torsion is considered. The unique Levi-Civita connection (ie defined without torsion, even when there may be torsion considered in the system) is written as: $\mathit{\Gamma}^c_{ab}$, and the covariant Levi-Civita derivative operator (ie without torsion): $\triangle_a$, when torsion is also being considered (though this need not apply if torsion is not being considered at all in that the two connections become identical). So we have in terms of the Levi-Civita connection:

$$F_{ab} = \triangle_a A_b - \triangle_b A_a = \partial_a A_b - \partial_b A_a \text{ equally } F = dA \tag{4.0.1}$$

In order to distinguish tensors constructed using torsion G_{ab} and R_{ab} (i.e. where the Ricci tensor is defined in terms of Γ^c_{ab}) from those that do not use torsion (ie that are defined in terms of $\mathit{\Gamma}^c_{ab}$), the torsionless case uses cursive: $\mathcal{G}_{ab}$ and $\mathcal{R}_{ab}$. On any given manifold with torsion, both these parallel systems of connection coefficients and dependent tensors can be used. That is, the Ricci tensor (with torsion), R_{ab}, and the Ricci tensor, $\mathcal{R}_{ab}$, are both defined and are in general different on the same manifold. Further each of these can have hats on or hats off, giving: $\hat{R}_{AB}$ and $\hat{\mathcal{R}}_{AB}$. It is a potentially confusing part of this work that all four systems may be used simultaneously. But it gets better/worse! We also need multiple systems of metrics, whether 4D or 5D, whether with torsion or without torsion. For example when two metrics are related by a conformal transformation. To manage this we would most easily use different colour, or, document technology being limited at the time of writing, the use of a simple font variation may suffice.

Torsion introduces non-obvious and unfamiliar conventions in otherwise established and common definitions, thus leading to much premature hair loss. The order of the indices in the connection coefficients actually matters, and this includes in the Ricci tensor definition and the definition of the connection coefficient symbols themselves:

$$\nabla_a w_b = \partial_a w_b - \Gamma^c_{ab} w_c \tag{4.0.2}$$

Some familiar defining equations consistent with [1] define the Ricci tensor and Einstein tensors in terms of the connection coefficients along usual lines, noting that with torsion the order of indices can no longer be interchanged:

$$R_{ab} = \partial_c \Gamma^c_{ba} - \partial_b \Gamma^c_{ca} + \Gamma^c_{ba}\Gamma^d_{dc} - \Gamma^c_{da}\Gamma^d_{bc} \tag{4.0.3}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \beta_g T_{ab} \tag{4.0.4}$$

For convenience we will define $\alpha_g = 1/\beta_g$ which might typically be set to $\alpha_g = c^4/8\pi\mathbf{G}$. Analogous definitions can also be used with the Levi-Civita connection to define $\mathcal{R}_{ab}$ and $\mathcal{G}_{ab}$ in the obvious way.

The appendices have an important role in outsourcing definitions and mathematics that would otherwise interrupt the flow of the text. The appendices are designed to be independent of the text, but the text is not independent of the appendices. The appendices work as a sort of library of sub-routines for the text. The appendices are however dependent on previous sections of the appendices and in that sense need to be read in parallel with the main text.

Other conventions may be used as noted in the text.

5 Kaluza's Original Theory

Kaluza's 1921 theory of gravity and electromagnetism [2][3][4] using a fifth wrapped-up spatial dimension is at the heart of many modern attempts to develop new physical theories [1][5]. From supersymmetry to string theories topologically closed small extra dimensions are used to characterize the various forces of nature. It is therefore inspiration for many modern attempts and developments in theoretical physics. However it has a number of foundational problems and is often considered untenable. This paper looks at these problems from a purely classical perspective and attempts to dispel them.

5.1 The Metric

The original Kaluza theory assumes a (1,4)-Lorentzian Ricci flat manifold to be the 5D metric, split as shown below (and for interest this can be compared to the later ADM formalism [6]). A_a is to be identified with the electromagnetic potential, ϕ^2 is to be a scalar field, and g_{ab} the metric of 4D space-time. Note that a scaling factor k is present, it is mathematically arbitrary, but physically relates to units. By inverting this metric as a matrix (readily checked by multiplying $\hat{g}_{AB}$ by $\hat{g}^{BC}$) we get raised indices.

Definition 5.1.1: The 5D Kaluza and Kaluza-Cartan metric.

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix}$$

$$\hat{g}^{AB} = \begin{bmatrix} g^{ab} & -kA^a \\ -kA^b & \frac{1}{\phi^2} + k^2 A_i A^i \end{bmatrix} \tag{5.1.1}$$

Maxwell's law are automatically satisfied, using (4.0.1) to define F with respect to the potential: dF=0 follows from dd = 0. We are then free to define the charge current via d*F= 4π*J (or similar). Regardless of the factor this leads to d*J=0 via dd=0 [10], which is consistent with local conservation of charge.

In order to write the metric in this form there is a subtle assumption, that g_{ab}, which will be interpreted as the usual four dimensional space-time metric, is itself non-singular. This will always be the case for moderate or small values of A_x which will here be identified with the electromagnetic 4-vector potential. The raising and lowering of this 4-vector are defined in the obvious way in terms of g_{ab}. The 5D metric can be represented at every point on the Kaluza manifold in terms of this 4D metric g_{ab} (when it is non-singular), the vector potential A_x, and the scalar field ϕ^2. We have also assumed that topology is such as to allow the Hodge star operator and Hodge duality of forms to be well-defined (see [10] p.88). This means that near a point charge source the above interpretation of the charge (Maxwell charge) and therefore Maxwell charge conservation breaks down as the potential may cease to be well-defined. Whereas the kinetic charge defined in the appendices (12.3.1) does not have this problem. So two different definitions of charge are used: the Maxwellian, and the kinetic charge. It is the kinetic charge that will obey a more general conservation law, per force - it's part of the Einstein tensor.

With values of ϕ^2 around 1 and relatively low 5-dimensional metric curvatures we need not concern ourselves with this assumption beyond stating it on the basis that physically these parameters encompass tested theory. Given this proviso A_x is a vector and ϕ^2 is a scalar - with respect to the tensor system defined on any 4-dimensional submanifold (or region of a submanifold) that can take the induced metric g.

Herein lies a further reason why setting the scalar field large to obtain the usual electromagnetic stress-energy tensor seems arbitrary. We also need a weak field limit to link the two forms of charge used in this work.

5.2 Kaluza's Cylinder Condition And The Original Field Equations

Kaluza's cylinder condition is that all partial derivatives in the 5th dimension i.e. ∂_4 and $\partial_4\partial_4$ etc... of all metric components and of all tensors and their derivatives are zero. A perfect 'cylinder'. This leads to constraints on g_{ab} given in [1] by three equations, the field equations of the original Kaluza theory, where the Einstein-Maxwell stress-energy tensor can be recognised embedded in the first equation. Beware in particular that the conventions are as used by the referenced author and not those used in this paper. The field equations are derived by simply setting the torsionless Ricci curvature to 0. $\Box$ represents the 4D D'Alembertian [1].

$$G_{ab} = \frac{k^2\phi^2}{2}\left\{\frac{1}{4}g_{ab}F_{cd}F^{cd} - F^c_aF_{bc}\right\} - \frac{1}{\phi}\{\nabla_a(\partial_b\phi) - g_{ab}\Box\phi\} \tag{5.2.1}$$

$$\nabla^a F_{ab} = -3\frac{\partial^a\phi}{\phi}F_{ab} \tag{5.2.2}$$

$$\Box\phi = \frac{k^2\phi^3}{4}F_{ab}F^{ab} \tag{5.2.3}$$

Note that there is both a sign difference and a possible factor difference with respect to Wald's [11] and Wheeler's [10] Einstein-Maxwell equation. The field equations give (without torsion [1]) nullish solutions under the original Kaluza cylinder condition and constant scalar field, such that $G_{ab} = -\frac{k^2}{2}\,F_{ac}F^c_b$. Compare this with [11] where we have $G_{ab} = 2F_{ac}F^c_b$ in geometrized units for ostensibly the same fields. The units would need to be agreed between the two schemes by adjusting k, and other constants, and adjusting for the sign difference. The sign difference appears to be due to the mixed use of metric sign conventions in [1].

These equations will be referred to as the first, second and third torsionless field equations, or Kaluza's original field equations. Kaluza did not include torsion in his definition of the Ricci tensor. They are valid only in Kaluza vacuum, that is, when $\mathcal{R}_{ab} = 0$ and when torsion is vanishing or not relevant.

We might interpret this as outside of matter and charge models, if we define such to be when the Ricci curvature is not 0. Though this is a little misleading as Maxwell charges, however small, are present in these equations. This however is due to the small difference introduced by the approximate identification of Maxwell and kinetic charges. Kaluza theories should take the kinetic charge (definition 12.3.8) as the truly divergence-free form.

5.3 The Foundational Problems

An issue addressed in this paper is the variety of electromagnetic solutions that are a consequence of Kaluza theory, whilst maintaining the Lorentz force law. A sufficient variety of electromagnetic fields must be available, and the Lorentz force law should be explicitly derivable. The missing solutions are the non-nullish solutions and include the important electrostatic fields. So they include some really important fields! The other usual objection to Kaluza theories, stability, is addressed elsewhere in the text.

One inadequate and arbitrary fix in standard Kaluza theory is to set the scalar field term large to ensure that the second field equation (5.2.2) is approximately zero despite scalar fluctuations. This approach will not be taken here as it is contrived. The stress-energy tensor under scalar field fluctuations is different from the Einstein-Maxwell tensor [10][11] and the accepted derivation of the Lorentz force law (for electrovacuums [10]) can not be assumed. A variable scalar field as required by the third field equation for non-nullish fields (5.2.3) also implies non-conservation of Maxwell charge via the second field equation (5.2.2), and problems also arise with respect to the Lorentz force law in the case of a variable scalar field. Thus in most Kaluza theories, including the original the scalar field is in effect fixed, and the non-nullish solutions then need reintroducing by increasing the available degrees of freedom.

This could be attempted via the introduction of torsion [15][16]. The electromagnetic field devoid of matter and charge sources will then be characterized by $\hat{R}_{AB} = 0$ instead of $\hat{\mathcal{R}}_{AB} = 0$, providing a Lorentz force law still results. It can also be attempted by reintroducing a variable scalar field, but again by making sure that, given certain constraints, this still leads to a Lorentz force law.

As components of momentum, the kinetic charge is of necessity locally conserved, provided there are no irregularities in the topology of the Kaluza 5th dimension. See Postulates K1-K3 in the appendices for well-behaved topological requirements. Note that conservation of Maxwellian charge (which will be shown to be identifiable with kinetic charge) is locally guaranteed by the existence of the potential and the exterior derivative, but breaks-down under curvature. The two definitions are to be related, but the kinetic charge deemed more fundamental as it admits a curvature-independent local divergence-free law via the (torsionless) Einstein tensor.

Another foundational issue of Kaluza theory is that even with a scalar field it does not have convincing sources of mass or charge built in. The second field equation (5.2.2) has charge sources, but it's unlikely that realistic sources are represented by this equation. They appear as ghosts. The better interpretation is that real matter and charge sources must be defined as being when $\hat{\mathcal{R}}_{AB} \neq 0$ in Kaluza's original theory. Analogously by identifying Kaluza fields with $\hat{R}_{AB} = 0$ (with torsion) we would presumably have to identify matter and charge sources now with $\hat{R}_{AB} \neq 0$. However the mass-energy conservation law remains by definition in terms of $\hat{\mathcal{G}}_{AB}$ - i.e. the torsionless Einstein tensor, and generally only with respect to the Levi-Civita connection. This is extended to the completely antisymmetric torsion connection case in the appendices (11.1.15) and (11.1.18).

This then suggests rather that the Kaluza fields remain when the torsionless Einstein tensor is vanishing, or equivalently when the torsionless Ricci tensor is vanishing, as in Kaluza's original theory. We then have matter-charge models and spin models being defined in the obvious way in terms of the torsionless Einstein tensor and the antisymmetric components of the torsion Einstein tensor respectively. Noting however that this line of reasoning can only be fully satisfactory when torsion is completely antisymmetric.

5.4 A Solution?

The torsionless Einstein tensor remains the matter-charge source in any case. But it's nice to note that spin conservation also arises in the completely antisymmetric torsion case. Both are presented by (11.1.18). The scalar field rather than torsion will however be used to obtain the full range of required electromagnetic fields, thus correcting the attempts made in [14][15]?

A departure from previous works that considered torsion [14][15] will now be made.

Definition 5.4.1: The *Kaluza vacuum* is a Ricci flat region of a Kaluza space with respect to the torsionless definition of the Ricci tensor, ie $\hat{\mathcal{R}}_{AB} = 0$. The *Kaluza vacuum* in the presence of torsion further requires that $\hat{V}_{AB} = 0$.

This equates to vanishing matter-charge sources and vanishing spin sources respectively, which defines a clear demarcation between matter-charge-spin models and the classical fields of the Kaluza vacuum. At the completely antisymmetric limit it follows that $\hat{G}_{[AB]} = 0$ in Kaluza vacuum as shown in the appendices (11.1.18).

6 A Complete Set Of Postulates

In this section a complete set of postulates is given that is used in this paper to investigate a range of different Kaluza theories. How is this possible? First there are three core postulates common to all variants. Additional postulates that can be interpreted as forming conditions necessary for a classical *weak field limit* limit, and which link Maxwellian and kinetic charge definitions together, then follow. Postulate L2 need only apply when torsion is admitted. Postulate

L3 need only be present when non-antisymmetric torsion terms are admitted. When such terms are not admitted they are in any case trivially satisfied.

Subsequently two variant geodesic postulates are considered. These are two possible variant models, or options, for matter-charge model kinetics. They are not exhaustive. In particular spin is not considered. By providing two very different options here the analysis can at least try to cover a range of possibilities.

A careful balancing act is needed with respect to the scalar field induced by (5.1.1). Postulate B1 is added, and is different from that assumed in previous works [14][15][16]. It allows for a limited scalar field. It essentially defines the limit where the scalar field fluctuations are small relative to electromagnetism and gravity, but not vanishing unless additionally specified. Postulate B1 is compatible with the field equations by inspection.

Definition (5.4.1) is so important as a defining characteristic of the field equations that it is listed below. However it is not strictly necessary as a postulate. It is interesting to note that whilst torsion effects matter-charge-spin models, it does not here effect the symmetric torsionless part of the Kaluza vacuum curvature. The Kaluza vacuum remains defined in terms of the torsionless Ricci tensor. Torsion simply adds a new conserved (and in this case vanishing) tensor to that definition. To state it explicitly: the Kaluza vacuum satisfies the original Kaluza field equations.

Finally it is understood that further constraints in the form of energy or super-energy conditions are needed physically, but that these are not dealt with here. The broad issues are however briefly discussed elsewhere in the text.

6.1 Core Geometric Postulates

Core Postulates K1, K2 and K3 (including the famous cylinder condition) are given in the appendices (12.1.1). These define the geometry and topology common to all Kaluza variants considered here.

6.2 A Weak Field Limit

The deviation from the 5D-Minkowski metric is given by a tensor $\hat{\mathtt{h}}_{AB}$. This tensor belongs to a set of small tensors that we might label $O(h)$. Whilst this uses a notation similar to orders of magnitude, and is indeed analogous, the meaning here goes further. This is the weak field approximation of general relativity using a more flexible notation. Partial derivatives, to whatever order, of metric terms in a particular set $O(x)$ will be in that same set at any such limit. In principle we are doing more than following the weak field limit procedure [10] of general relativity. In the weak field approximation of general relativity, terms that consist of two $O(h)$ terms multiplied together get discounted and are treated as vanishing at the limit. We might use the notation $O(h^2)$ to signify such terms. There is the weak field approximation given by discounting $O(h^2)$ terms. But we might also have a less aggressive limit given by, say, discounting $O(h^3)$ terms, and so on. We can talk about weak field limits (plural) that discount $O(h^n)$ terms and are therefore of order $O(h^{n-1})$ for $n > 1$, but they are based on the same underlying construction. This is an upper-bound of significance of any term in the sense that $O(h^{n-1}) \subset O(h^n)$.

LIMIT POSTULATE (L1): The metric can be written as follows in terms of the 5D Minkowski tensor and $\hat{h} \in O(h)$:

$$\hat{g}_{AB} = \hat{\mu}_{AB} + \hat{\mathtt{h}}_{AB}$$

Torsion will also be considered a weak field under normal observational conditions, similarly to L1. Torsion is defined in terms of the Christoffel symbols. Christoffel symbols are in part constructed from the partial derivatives of the metric and that part is constrained by L1 to be $O(h)$. The contorsion term being the difference. See [20]. The contorsion (and therefore the torsion) will be treated as $O(h)$ accordingly.

LIMIT POSTULATE (L2): The contorsion and torsion are $O(h)$ terms.

One further constraint is required at the weak field limit. Its use will be minimized (both the application of the antisymmetry and the allowance for some small symmetry terms), but it will nevertheless be important. In L3, symmetric parts of the torsion and contorsion tensor (and their derivatives) are treated as particularly 'small' in that they are small relative to any antisymmetric parts of the torsion and contorsion tensor, torsion already assigned to $O(h)$ by L2. The torsion tensor will be given the following limit: It is to be weakly completely antisymmetric - *a weak antisymmetric limit*. Thus the symmetric parts of the contorsion and torsion tensors will be $O(h^2)$ at the weak field limit. All derivatives thereof follow the same rule:

LIMIT POSTULATE (L3): The symmetric parts of the contorsion and torsion tensors will be $O(h^2)$ at the weak field limits.

L1 and L2 are natural postulates for a weak field limit. L3 is not so natural and seems arbitrary. L3 is trivially satisfied in the case of completely antisymmetric torsion. L3 is used to maintain maximum generality of the results of this work, but the difficulties of allowing non-anti-symmetric torsion components recurs throughout the work.

6.3 Geodesic Options

The two kinetic postulates under consideration are detailed and discussed in the appendices: Postulate G1 is (11.2.1) and Postulate G2 is (11.2.2). They are options to be selected and then applied to the kinetics of ideal point particles. In any experimental reality, under the hypotheses here, any torsion or spin presence would likely alter the kinetics. Such variants are not explored here, and treated as in any case likely small effects.

6.4 Weak Scalar Field

Above and beyond LIMIT POSTULATE (L1) for metric components, we apply the specific tighter constraint:

LIMIT POSTULATE (B1): ϕ^2 is $O(h^2)$ over the region of interest. The scalar field results from the the decomposition of the Kaluza metric into 4D metric, potential vector and scalar field. It is contained within the metric explicitly in (5.1.1).

This is compatible with the break-down of the metric and the original field equations by inspection. Considering such it is arguably necessary for consistency for the other fields to be L1.

6.5 On Non-Nullish Electromagnetic Fields

Postulate B1 is sufficiently weak to allow for the non-nullish electromagnetic fields which are missing if the scalar field is set constant. In setting the scalar field to be vanishing, then, the problems that led to the previous works [14][15][16] arise. Taking Kaluza theory at face value is here argued to be the best approach.

6.6 A Quick Reference List Of Postulates

Thus we have: K1, K2, K3, L1, L2, L3, G1, G2, B1 and definition (5.4.1).

K1, K2, K3 always apply
L1 applies
L2 applies if there is torsion considered
L3 applies if there is non-completely anti-symmetric torsion considered
G1 and G2 can be selected options as required to study particle kinetics.
B1 applies

Further constraints in the form of energy or super-energy conditions are needed physically but are not needed or defined in this work.

Of these L3 is the least favourable with regards to physicality, the most likely not to be necessary. In such cases the stronger postulate that there are no symmetric components of torsion at all is assumed. Definition (5.4.1) defines the Kaluza vacuum, and therefore the field equations of such a ‘vacuum’. Energy and/or super-energy conditions, or similar constraints are not needed here or considered.

7 Geometrized Charge

7.1 Maxwell Charge

Maxwell charge density is defined, in keeping with the second original Kaluza field equation and Maxwell's equations in S.I. units, as follows:

$$\mu_0 {J_M}^a = \triangle_c F^{ac}$$
$$Q_M = {J_M}^a (c^{-1}, 0, 0, 0)_a \tag{7.1.1}$$

Where k in the metric becomes a conversion factor between geometrical quantities and the physical units for 4-potential, here implied by μ_0, the permeability of vacuum.

7.2 Identifying Kinetic Charge and Maxwell Charge

Now to investigate the relationship between kinetic charge and Maxwell charge. For this we need the $O(h)$ weak field limit defined by L1, and the cylinder condition. Discounting $O(h^2)$ terms using an arrow:

$$\hat{\mathcal{G}}^{a4} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}\hat{g}^{a4}\hat{\mathcal{R}} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}(-kA^a)\hat{\mathcal{R}} \to \hat{\mathcal{R}}^{a4}$$

$$\hat{\mathcal{R}}^{a4} = \partial_C \hat{\digamma}^{C4a} - \partial^4 \hat{\digamma}^{C\ a}_{\ C} + \hat{\digamma}^{Cab}\hat{\digamma}^{D}_{\ DC} - \hat{\digamma}^{C\ a}_{\ D}\hat{\digamma}^{Db}_{\ \ C}$$
$$\hat{\mathcal{G}}^{a4} \to \hat{\mathcal{R}}^{a4} \to \partial_c \hat{\digamma}^{c4a} \tag{7.2.1}$$

With reference to the appendices for the Christoffel symbols, we get:

$$\hat{\mathcal{G}}^{a4} \to \frac{1}{2}k\partial_c F^{ca} = -\frac{1}{2}k\partial_c F^{ac} \tag{7.2.2}$$

Similarly,

$$\hat{\mathcal{G}}_{a4} \to +\frac{1}{2}k\partial_c {F_a}^c \tag{7.2.3}$$

And so by definition of kinetic current charge density and λ, the Kaluza length (12.3.8):

$$J^{*a} \to +\frac{\alpha_g \lambda k}{2}\partial_c F^{ac} \tag{7.2.4}$$

Apply L1 again,

$$J^{*a} \to +\frac{\alpha_g \lambda k}{2}\mu_0 {J_M}^a \tag{7.2.5}$$

And using (12.3.7) in the appropriate space-time frame and Kaluza atlas frame:

$$Q^* \to +\frac{\alpha_g \lambda k}{2}\mu_0 Q_M \tag{7.2.6}$$

This has units of momentum-density times length.

So kinetic and Maxwell charge and current densities are related by a simple formula. The right hand side being Maxwell's, the left-hand side kinetic. This correlates the two definitions of charge at the required limit. A dependence on a possibly variable Kaluza length is however present.

This subsection did not require L2 or L3. With the inconsequential exception of equation (7.2.3) B1 can be omitted. The cylinder condition can even be weakened to allowing $O(h^2)$ terms and it teh derivation will still work. It is a very general result.

7.3 A Lorentz-Like Force Law

The Christoffel symbols are as follows in both G1 and G2 in the case of either completely antisymmetric torsion or no torsion at all:

$$\begin{aligned}
&\hat\Gamma^c_{(4b)} = \tfrac{1}{2}g^{cd}(\delta_4\hat g_{bd} + \delta_b\hat g_{4d} - \delta_d\hat g_{4b}) + \tfrac{1}{2}\hat g^{c4}(\delta_4\hat g_{b4} + \delta_b\hat g_{44} - \delta_4\hat g_{4b}) = \\
&\tfrac{1}{2}g^{cd}[\delta_b(\phi^2 kA_d) - \delta_d(\phi^2 kA_b)] + \tfrac{1}{2}g^{cd}\delta_4\hat g_{bd} + \tfrac{1}{2}\hat g^{c4}\delta_b\hat g_{44} = \\
&\tfrac{1}{2}\phi^2 g^{cd}[\delta_b kA_d - \delta_d kA_b] + \tfrac{1}{2}g^{cd}kA_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}kA_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat g_{bd} + \tfrac{1}{2}\hat g^{c4}\delta_b\phi^2 = \\
&\tfrac{1}{2}\phi^2 kF^c_b + \tfrac{1}{2}g^{cd}kA_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}kA_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat g_{bd} + \tfrac{1}{2}\hat g^{c4}\delta_b\phi^2 = \\
&\tfrac{1}{2}\phi^2 kF^c_b - \tfrac{1}{2}g^{cd}kA_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat g_{bd} = \tfrac{1}{2}\phi^2 kF^c_b \quad - \tfrac{1}{2}g^{cd}kA_b\delta_d\phi^2 \qquad (7.3.1)
\end{aligned}$$

$$\hat\Gamma^c_{44} = \tfrac{1}{2}\hat g^{\mathrm{cD}}(\delta_4\hat g_{4D} + \delta_4\hat g_{4D} - \delta_D\hat g_{44}) = \text{-}\ \tfrac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2 \qquad (7.3.2)$$

$$\begin{aligned}
&\hat\Gamma^c_{(ab)} = \tfrac{1}{2}g^{cd}(\delta_a g_{db} + \delta_b g_{da} - \delta_d g_{ab}) \\
&+\tfrac{1}{2}g^{cd}(\delta_a(\phi^2k^2A_dA_b) + \delta_b(\phi^2k^2A_aA_d) - \delta_d(\phi^2k^2A_aA_b)) \\
&+\tfrac{1}{2}\hat g^{c4}(\delta_a\hat g_{4b} + \delta_b\hat g_{4a} - \delta_4\hat g_{ab}) \\
&= \Gamma^c_{(ab)} \quad +\tfrac{1}{2}g^{cd}(\delta_a(\phi^2k^2A_dA_b) + \delta_b(\phi^2k^2A_aA_d) - \delta_d(\phi^2k^2A_aA_b)) \\
&\qquad -k^2A^c(\delta_a\phi^2A_b + \delta_b\phi^2A_a) \qquad (7.3.3)
\end{aligned}$$

In the case of non-completely antisymmetric torsion we must add $O(h^2)$ error terms. The error terms being delimited by L3.

So, for a coordinate system within the maximal atlas:

$$\begin{aligned}
&0 = \tfrac{d^2x^a}{d\tau^2} + \hat\Gamma^a_{(BC)}\tfrac{dx^B}{d\tau}\tfrac{dx^C}{d\tau} \\
&= \tfrac{d^2x^a}{d\tau^2} + \hat\Gamma^a_{(bc)}\tfrac{dx^b}{d\tau}\tfrac{dx^c}{d\tau} + \hat\Gamma^a_{(4c)}\tfrac{dx^4}{d\tau}\tfrac{dx^c}{d\tau} + \hat\Gamma^a_{(b4)}\tfrac{dx^b}{d\tau}\tfrac{dx^4}{d\tau} + \hat\Gamma^a_{44}\tfrac{dx^4}{d\tau}\tfrac{dx^4}{d\tau} \\
&= \tfrac{d^2x^a}{d\tau^2} + \hat\Gamma^a_{(bc)}\tfrac{dx^b}{d\tau}\tfrac{dx^c}{d\tau} + (\phi^2kF^a_b - g^{ad}kA_b\delta_d\phi^2)\tfrac{dx^b}{d\tau}\tfrac{dx^4}{d\tau} - \tfrac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\tfrac{dx^4}{d\tau}\tfrac{dx^4}{d\tau} \quad (7.3.4)
\end{aligned}$$

Taking $\phi^2 = 1$, and the L3 error terms to be vanishing, and the charge-to-mass ratio to be:

$$Q'/m_{k0} = \frac{dx^4}{d\tau} \qquad (7.3.5)$$

We derive a Lorentz-like force law:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -(Q'/m_{k0})kF^a_b\frac{dx^b}{d\tau} \tag{7.3.6}$$

Putting arbitrary L3 error terms back in, and variable ϕ^2, we have:

$$\frac{d^2x^a}{d\tau^2} + (\hat{\Gamma}^a_{(bc)} + Err)\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} =$$
$$-(Q'/m_{k0})(\phi^2 kF^a_b - g^{ad}kA_b\delta_d\phi^2 + Err)\frac{dx^b}{d\tau} - (-\frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2 - Err)\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{7.3.7}$$

Note that the same error order of magnitude also exists if we allow partial derivatives in the Kaluza direction of $O(h^2)$ - that is, loosening the usual Kaluza cylinder condition.

This can be reduced to the following by taking only the most significant subterms in each term. Further, the 5D Christoffel symbol can be approximated with the 4D Christoffel symbol given L1, L2, L3, B1 and either the cylinder condition proper or the same to $O(h^2)$. That is, approximately, only one $O(h^2)$ error term remains as not dominated by a greater term:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} =$$
$$-(Q'/m_{k0})kF^a_b\frac{dx^b}{d\tau} + (\frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2 + Err)\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{7.3.8}$$

Where the error to $O(h^2)$ is:

$$Err \to -\tfrac{1}{2}\hat{g}^{\mathrm{aD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D}) + \hat{T}_{44}{}^a \to -\hat{g}^{\mathrm{ad}}\delta_4\hat{g}_{4d} + \hat{T}_{44}{}^a = k\delta_4A^a + \hat{T}_{44}{}^a \tag{7.3.9}$$

This makes sense as a force law under both G1 and G2, even when the same is applied to the 4D space-time rather than the 5D Kaluza space, provided that 4D torsion is inherited by space-time from the 5D Kaluza space in the obvious manner. The derivation of this force law has not required even the full cylinder condition, but just an $O(h^2)$ approximation. What has been derived here is a Lorentz-like force law independently of the Kaluza field equations. It can be noted that under the stronger constraints of the full cylinder condition and vanishing symmetric components of torsion that the error term is exactly zero. L3 also ensures that to the accuracy sought we do not need to worry about torsion-normal coordinates varying from normal coordinates.

This is not, however, yet, the 4D force law sought.

7.4 No Symmetric Torsion Components

A further argument can now be provided against the admittance of symmetric torsion components and therefore the need to tighten L3 to 'no symmetric torsion components'. It is also an argument for the cylinder condition (ie not the loose version used in the preceding).

In order to derive the Lorentz force law (not just a Lorentz-like force law) a conformal transformation of the metric is going to be made such that the new scalar field that results is set to constant identity. This will set the $\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$ term to zero in the event that the cylinder condition is fully satisfied and there are no symmetric torsion components. That this is necessary to derive cleanly the full Lorentz force law is an argument for the vanishing of the error terms.

There are, in any case, other ways to make the error terms vanish, and constraints that could be added to the geometry to force the Lorentz force law. There are, in any case, other ways to make $\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$ term vanish. So there is no dogmatism here. Nevertheless the cleanest way to get the usual Lorentz force law cleanly derivable from the aforementioned is to apply the cylinder condition fully and to ensure that the non-completely antisymmetric torsion terms (ie with symmetric components) are either vanishing or all but vanishing.

Under such a conformal transformation, and under such constraints, we need to rederive everything using the new 5D $\hat{h}$ and new 4D h metrics, i.e. a new Kaluza space and a new space-time. Note that each of these can appear in other systems depending on which tensor at any one time is the reference metric (some subtlety is required with now a total of 4 over-lapping systems of metrics, with 8 different systems of connections if we include torsion). Bold is used to indicate, and here define, tensors for which the reference metric is the new metric:

$$\hat{h}_{AB} \equiv \phi^{-2}\hat{g}_{AB} = \begin{bmatrix} \phi^{-2}g_{ab} + k^2 A_a A_b & kA_a \\ kA_b & 1 \end{bmatrix}$$

$$= \begin{bmatrix} h_{ab} + k^2 A_a A_b & kA_a \\ kA_b & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{h_{ab}} + \mathrm{k}^2 \mathbf{A_a A_b} & k\mathbf{A_a} \\ k\mathbf{A_b} & 1 \end{bmatrix} \equiv \mathbf{\hat{h}_{AB}}$$

$$\phi^2 \hat{g}^{AB} = \begin{bmatrix} \phi^2 g^{ab} & -\phi^2 kA^a \\ -\phi^2 kA^b & 1 + \phi^2 k^2 A_i A^i \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{h^{ab}} & -\phi^2 kA^a \\ -\phi^2 kA^b & 1 + \phi^2 k^2 A_i A^i \end{bmatrix} = \begin{bmatrix} \mathbf{h^{ab}} & -k\mathbf{A^a} \\ -k\mathbf{A^b} & 1 + k^2 \mathbf{A_i A^i} \end{bmatrix} \equiv \mathbf{\hat{h}^{AB}} \tag{7.4.1}$$

The bottom line appears to be that the components do not change more than a small $O(h^2)$ amount under any such conformal transformation. And indeed this is also true for the Christoffel symbols in the geodesic equations. We might therefore be tempted to say that such small transformations do not affect the geodesics significantly and so we are free to add back in other small $O(h^2)$ terms such as the torsion and other errors.

This however is unlikely as the limits we have selected also limit the charge models that can be hypothesised. In effect the $\frac{dx^4}{d\tau}$ terms. Such charge models would represent small charges assigned to large masses, slow rotation in the

Kaluza dimension producing little in the way of charge. In some situations this model could be useful, say when large spherical capacitors possess small charge. It is unlikely to be sufficient for fundamental particles, such as the electron, that have very strong fields up close (ie in any case contrary to the weak field postulates), and nevertheless are known to obey the Lorentz force law.

As discussed elsewhere we could add compensatory negative masses to any such model. This would be a tethering of a primary mass-charge to a secondary negative mass with no charge to tweak the mass-charge ratio. But we would be looking at large compensations. This may or may not be one way to do it. But it doesn't look promising.

Another route is suggested here: using the properties of conformal transformations. That is, the subset of 5D conformal transformations that preserve the cylinder condition and orders of magnitude of terms generally.

Let the charge model instead follow a 5D null geodesic, or a near null geodesic. As before, secondary negative or positive masses can be added to construct any mass-charge ratio we require. But this time we can not reinstate the torsion or other error terms as they have no correlation with the conformal transformation. The trick is to note that conformal transformations preserve null geodesics. And to note that equations, (7.4.1), derived from a conformal transformation, represent an alternative Kaluza space satisfying exactly the same mathematics as the original, but with constant scalar field. Thus the old and the new Kaluza space must have the same null geodesics. The same paths are followed. The same Lorentz-like force law applies.

Therefore we can omit the last term in (7.3.8) - at least provided our primary mass-charge is modelled by very fast motion around the Kaluza dimension. Any secondary negative masses required are likely to be considerably less than with the slow model. This is therefore a superior model.

Note that the negative masses need not become 4D physical negative masses. They are part of the 5D modelling.

7.5 A Lorentz Force Law For Certain Charge Models

With the null geodesic approach to charge models, allowing for compensatory negative and positive masses as required, disallowing symmetric torsion terms, and strictly enforcing the cylinder condition, we are led to the following improved Lorentz-like law in a coordinate system in the 5D maximal atlas:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -(Q'/m_{k0})kF^a_b\frac{dx^b}{d\tau} \tag{7.5.1}$$

That this is possible for certain prospective charge models is rather promising.

8 The Lorentz Force Law

It is necessary to confirm that equation (7.5.1) not only looks like the Lorentz force law formally, but is indeed the Lorentz force law in 4D. Multiplying both sides of (7.5.1) by $\frac{d\tau}{d\tau'}\frac{d\tau}{d\tau'}$, where τ' is an alternative affine coordinate frame, and setting $\tau = \tau^*$, the proper Kaluza time in a 5D frame following the geodesic, gives:

$$\frac{d^2x^a}{d\tau'^2} + \Gamma^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k\frac{d\tau}{d\tau'}(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau'} \tag{8.0.1}$$

Given $\frac{Q^*}{\rho_0\lambda} = \frac{Q'}{m_0}\frac{d\tau}{d\tau^*}$ by (12.3.3) in the selected original frame, and therefore $\frac{m_{k0}}{\rho_0\lambda}Q^* = Q'\frac{d\tau}{dt_0}$ by definition, we can set the new frame such that $\tau' = t_0$ via the projected 4D space-time frame of the charge. And a Lorentz force is derived:

$$\frac{d^2x^a}{d\tau'^2} + \Gamma^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k(Q^*/\rho_0\lambda)F^a_b\frac{dx^b}{d\tau'} \tag{8.0.2}$$

In order to put the correct Lorentz force law in a prefered form, this can be rewritten as follows, using the antisymmetry of $F^a_b = -F^a{}_b$:

$$= k(Q^*/\rho_0\lambda)F^a{}_b\frac{dx^b}{d\tau'} \tag{8.0.3}$$

Using (7.2.6), at the L1 limit, this can be rewritten again in terms of the Maxwell charge density:

$$\to +k((\frac{\alpha_g k}{2}\mu_0 Q_M)/\rho_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{8.0.4}$$

Thus the Lorentz force law (allowing for small variations in the Kaluza length λ) is, using the usual q and relativistic rest mass m:

$$\frac{d^2x^a}{d\tau'^2} + \Gamma^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} \to (\frac{\alpha_g k^2}{2}\mu_0)(q/m)F^a{}_b\frac{dx^b}{d\tau'} \tag{8.0.5}$$

This can be seen to be the Lorentz force law proper in the event that we fix $c = 1$, $\mathbf{G} = 1$, $\mu_0 = 1$ and $\alpha_g = 8\pi$ as given on page 474 of [10] by setting $k = \pm 1/\sqrt{4\pi}$.

9 The Kaluza Length λ

Some Kaluza theories might be able to allow the fixing of λ by postulate. Here however that will not be done since this may present problems for conservation laws. It is not the most general approach. Thus we need to investigate under what circumstances we may entertain small order variations of $O(h)$ in the following term:

$$\frac{\partial_x\lambda}{\lambda} \tag{9.0.1}$$

Here the Kaluza length must be allowed to vary. But it must also be constrained to ensure that the Lorentz force law is obeyed.

This limited capacity for variation follows from straight forward orders of magnitude considerations using equation (12.3.8) as follows:

$\hat{\triangle}_A \hat{\mathcal{G}}^{AB} = 0$ in 5D, so to $O(h)$, in both 4D and 5D, we have $\partial_a \hat{\mathcal{G}}^{aB}$is $O(h^2)$.

This follows from L1 and the cylinder condition. From this we can deduce in 4D the following equation:

$$\triangle_a \hat{\mathcal{G}}^{aB} \approx \; O(h^2) \tag{9.0.2}$$

Similarly we can deduce, see eqn(12.3.5), that 4D kinetic charge density J^{**a} is approximately divergence-free also in 4D space-time. Though this also follows from the field equations, or equally Maxwell's equations at the L1 and electromagnetic limit.

Now, eqn(12.3.6) needs to be similarly conserved to prove the point. But of course this is a measure of the conserved quantity $\hat{\mathcal{G}}^{a4}$ (in 5D) around a Kaluza loop. The sum of the divergence around the loop must also be vanishing. A similar quantity (analogous to J^{**}) can be posited in relation to $\hat{\mathcal{G}}^{ab}$. In both cases we can not compress or rarefy the Kaluza dimension without creating a source or a sink of $\hat{\mathcal{G}}^{ab}$ in 4 dimensions.

Though only a heuristic argument, these considerations effectively limit how much the Kaluza dimension can be compacted or rarefied at the L1 limit.

10 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is inspiration for many modern attempts to develop new physical theories. However for a number of reasons it is generally considered untenable. Here a range of arguments has been made to rehabilitate Kaluza theory as originally intended.

It becomes apparent through the text that non-antisymmetric components of torsion (in the event torsion is allowed) remain problematic. But simply allowing completely anti-symmetric torsion (in 5D) is very natural and has no ill-effects on the arguments presented. The one possible exception to this is spin-spin coupling in the event that this may effect the Lorentz force law of charged particles. Of course in any case we may assume that, at the classical limit in bulk matter, such a field has yet to be detected and its effects may be negligible.

Classical electrodynamics is rederived in the spirit of Kaluza's original theory. Gravity and electromagnetism are unified in a way not fully achieved by general relativity. A Lorentz force law is proven, and no problem exists with missing electromagnetic fields as usually implied by setting the scalar field to unity. The scalar field is here, contrarily, allowed to vary but can be scaled out of the calculations by a conformal transformation. In an experimental situation this would appear as a non-electromagnetic massive scalar field accompanying matter, not unlike so-called dark matter.

Orders of magnitude limits were invoked which limit the intensity of the fields allowed in order to derive the Lorentz force law and other features of the theory. This means that there is a well-defined limit where the general relativistic limit necessarily breaks down. A further break-down must also occur approaching a quantum limit when objects hypothesised in general relativity to have intense fields, or even just close to charged particles, lead to singularities in the induced space-time metric even when the 5D Kaluza metric has not yet become singular.

One outstanding issue is that realistic charge models have not been defined. This would require objects outside of the proposed magnitude limits as well as complicated exact solutions. There is further a constraint on such charge models for them to obey the Lorentz force law in the presence of a scalar field: that they be closely related to null geodesics travelling in the Kaluza dimension. Barring this failure to readily provide realistic charge models, which poses theoretical challenges to the 5D theory, the postulates currently required are straight forward. It is in a certain sense a simple theory. In effect all we have is a 5D manifold with a cylinder condition on one spatial dimension with certain well-defined weak fields and limits. Interpretation of many of the postulates can be made in physically appealing terms. However, many of the consequences are really very complicated.

Super-energy was here introduced to resolve problems of causality, time evolution and stability. This replaces the need to worry too much about energy conditions as in general relativity, but is not unique in its contribution to the theory. It was only presented to show that possible alternatives exist to the current emphasis on energy conditions, and in particular to point out that proofs of instability of Kaluza theory are based on unecessary assumptions regarding the well-behaved nature of energy.

Why go to all this effort to unify electromagnetism and gravitation and to make electromagnetism fully geometric? Because experimental differences could be detectable given sufficient technology on the one hand, and, on the other, simply because such an attempt at unification might be right or lead in the right direction. Such an attempt may widen the search. This theory differs from both general relativity and Einstein-Cartan theory. But it remains essentially the same theory that Kaluza proposed in 1921, despite the addition of torsion to allow for Kaluza-Cartan theories and the suggested but under-substantiated use of super-energy. Is this a problem? No. Why should it be? The case here is that a deeper analysis of Kaluza's theory has been too long over-looked. That case has been made. Promising potential ways forward have here been given.

Further, attempting to extend classical theory significantly prior to a unification with quantum mechanics may be a necessary step in a future unification, whether the Kaluza/Kaluza-Cartan theory presented here turns out to be the right way or not. It may be that current attempts at unification of general relativity with quantum mechanics are more difficult than necessary as neither classical nor quantum theory may yet have been framed correctly. Perhaps attempts at unification are premature? That would be a very good reason to develop further, and even independently, both the classical and quantum limits. It may even be that the attempts of Klein [1] to add quantum fields to Kaluza theory were premature, or not quite, or not yet, the right approach.

It is often asserted that the true explanation for gravitational theory and space-time curvatures will most likely, by reductionist logic, emerge out of its constituent quantum phenomena. Such an approach has merit, but is overly optimistic, and does not optimize the search [8]. Before constituent quantum parts can be properly defined and subdivided the larger scale whole must have been present initially to then be so divided. Something of the context is evidently missing from quantum mechanics, general relativity or both on account of the difficulty of squaring the two. The dividing and putting together of parts assumes a context, and a context assumes a whole [7]. Implicitly reductionism assumes contextual knowledge. There is paradoxically an implicit non-reductionist assumption within reductionism. Generally we may take our conception of such a whole for granted, but we should bear in mind that this is a limited approach, speaking more of our limitations and need for easy or familiar concepts than of reality. Taking a global, more 'synthetic' perspective can be more difficult but may also be more insightful. A more holistic (in the sense of non-reductionist or post-reductionist, but nevertheless empirical) approach may be required at both the large and small scales. Such considerations are further justification for the approach attempted here to better unify gravity with electromagnetism, and to deepen the analysis of Kaluza theory. The approach here takes a more top down post-reductionist view, where the methodology is to try to understand how electromagnetism and gravitation must be unified together first, before trying to understand how this may integrate with the more counter-intuitive world of quantum mechanics. By tracing all the difficulties in creating a working theory it is hoped that a unique way forward naturally presents itself. Since it is impossible to trace all possibilities and difficulties (even in the event that such may ultimately lead to a unique theory) empirical testing once again necessarily remains the arbiter of scientific truth. Combining theoretical analysis with experiment is extremely potent. Therefore in addition to theoretical investigation empirical methods must be sought.

There are many ways to elaborate this research further using observational data. For example: could the scalar field be responsible for the appearance of so-called dark-matter? Whilst cosmological models present difficulties that laboratory experiments do not present, the application of Kaluza theories to cosmology may provide the easiest route to provisional testing. More direct evidence one way or the other may be available in intense fields. Unlike general relativity, Kaluza theory as presented here breaks-down prior to singularity formation. What happens to the geometry in extremis could at some point be investigated via experimental research. Laboratory experiments may be possible on the confines of the classical and the quantum scales, especially under intense fields of various forms. This however is not easy to pass comment on: something completely new would probably be required. On the other hand the complexity of observations of astronomical phenomena might otherwsie provide the most satisfying route forward. It is difficult to say. That such methods seem far off, however, suggests that further theoretical work may be first be necessary in order to reach a point where experiment or observational analysis may be feasible.

11 Appendix I: Torsion, Spin and Particles

The objective of this appendix is to provide supporting and additional information in a stand-alone way that can simply be referred to by the main text. Dependencies within this appendix are in sequential order.

11.1 Introducing The Geometry Of Torsion

5D Cartan torsion is here admitted. It is noted that Einstein-Cartan theory, that adds torsion to the dynamics of relativity theory is most probably a minimal ω-consistent extension of general relativity [26][21] and therefore the use of torsion is not only natural, but arguably a necessity on philosophical and physical grounds. That argument may also be applied here. What we have defined by this addition can be called Kaluza-Cartan theory as it takes Kaluza's theory and adds torsion. The torsion connection is the unique metric connection for any torsion tensor.

For both 5D and 4D manifolds (i.e. dropping the hats and indices notation for a moment), torsion will be introduced into the connection coefficients as follows, using the notation of Hehl [20].

$$\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) = S_{ij}{}^k \tag{11.1.1}$$

This relates to the notation of Kobayashi and Nomizu [12] and Wald [11] as follows:

$$T^i{}_{jk} = 2S_{jk}{}^i \equiv \Gamma^i_{jk} - \Gamma^i_{kj} \tag{11.1.2}$$

We have the contorsion tensor $K_{ij}{}^k$ [20] as follows, and a number of relations [20]:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kd}(\partial_i g_{dj} + \partial_j g_{di} - \partial_d g_{ij}) - K_{ij}{}^k = \mathit{\Gamma}^k_{ij} - K_{ij}{}^k \tag{11.1.3}$$

$$K_{ij}{}^k = -S_{ij}{}^k + S_j{}^k{}_i - S^k{}_{ij} = -K_i{}^k{}_j \tag{11.1.4}$$

Notice how the contorsion is antisymmetric in the last two indices.

With torsion included, the auto-parallel equation becomes [20]:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{(ij)}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{11.1.5}$$

$$\Gamma^{k}_{(ij)} = {\not\Gamma}^{\,k}_{ij} + S^{k}{}_{(ij)} - S_{(j}{}^{k}{}_{i)} = {\not\Gamma}^{\,k}_{ij} + 2S^{k}{}_{(ij)} \tag{11.1.6}$$

Only when torsion is completely antisymmetric is this the same as the extremals [20] which give the path of spinless particles and photons in Einstein-Cartan theory: extremals are then none other than geodesics with respect to the Levi-Civita connection.

$$\frac{d^2x^k}{ds^2} + {\not\Gamma}^{\,k}_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{11.1.7}$$

With complete antisymmetry we have many simplifications such as:

$$K_{ij}{}^{k} = -S_{ij}{}^{k} \tag{11.1.8}$$

Stress-Energy And Conservation Laws

Inspired by the Belinfante-Rosenfeld procedure [12][22], by defining the torsionless Einstein tensor in terms of torsion bearing components, yields what can be interpreted as extra spin-spin coupling term $\hat{X}_{AB}$:

$$\hat{\mathcal{G}}_{AB} = \hat{G}_{AB} + \hat{V}_{AB} + \hat{X}_{AB} \tag{11.1.9}$$

$$\hat{V}_{AB} = -\frac{1}{2}\hat{\nabla}^{C}(\hat{\sigma}_{ABC} + \hat{\sigma}_{BAC} + \hat{\sigma}_{CBA}) \tag{11.1.10}$$

Where σ is defined as the spin tensor in Einstein-Cartan theory. However, here we do not start with spin (and some particle Lagrangians), but with the torsion tensor. So instead the spin tensor is defined in terms of the torsion tensor using the Einstein-Cartan equations. Here spin is explicitly defined in terms of torsion:

$$\hat{\sigma}_{ABC} = 2\hat{S}_{ABC} + 2\hat{g}_{AC}\hat{S}_{BD}{}^{D} - 2\hat{g}_{BC}\hat{S}_{AD}{}^{D} \tag{11.1.11}$$

This simplifies definition (11.1.10):

$$\hat{V}_{AB} = -\frac{1}{2}\hat{\nabla}^{C}(\hat{\sigma}_{CBA}) = -\hat{\nabla}^{C}(\hat{S}_{CBA} + \hat{g}_{CA}\hat{S}_{BD}{}^{D} - \hat{g}_{BA}\hat{S}_{CD}{}^{D}) \tag{11.1.12}$$

By considering symmetries and antisymmetries we get a divergence law:

$$\hat{\nabla}_{B}\hat{V}^{AB} = 0 \tag{11.1.13}$$

The Case Of Complete Antisymmetry

Note that the mass-energy-charge divergence law for the torsionless Einstein tensor is in terms of the torsionless connection, but the spin source divergence law here is in terms of the torsion-bearing connection. However, for completely antisymmetric torsion we have:

$$\hat{\nabla}_C \hat{\mathcal{G}}_{AB} = \hat{\triangle}_C \hat{\mathcal{G}}_{AB} + \hat{K}_{CA}{}^{D} \hat{\mathcal{G}}_{DB} + \hat{K}_{CB}{}^{D} \hat{\mathcal{G}}_{AD}$$

So,

$$\hat{\nabla}^A \hat{\mathcal{G}}_{AB} = 0 + 0 + \hat{K}^{A}_{B}{}^{D} \hat{\mathcal{G}}_{AD} = -\hat{K}_B{}^{AD} \hat{\mathcal{G}}_{AD}$$
$$= -\hat{K}_B{}^{AD} \hat{\mathcal{G}}_{DA} = +\hat{K}_B{}^{DA} \hat{\mathcal{G}}_{DA} = +\hat{K}_B{}^{AD} \hat{\mathcal{G}}_{AD} = 0 \tag{11.1.14}$$

$$\hat{\nabla}^A (\hat{G}_{AB} + \hat{X}_{AB}) = 0 \tag{11.1.15}$$

And so there is a stress-energy divergence law with respect to the torsion connection also, at least in the completely antisymmetric case.

Further, still assuming complete antisymmetry of torsion, by definition of the Ricci tensor:

$$\hat{R}_{AB} = \hat{\mathcal{R}}_{AB} + \hat{K}_{DA}{}^{C} \hat{K}_{BC}{}^{D} - \partial_C \hat{K}_{BA}{}^{C} - \hat{K}_{BA}{}^{C} \hat{\Gamma}^{D}_{DC} + \hat{K}_{DA}{}^{C} \hat{\Gamma}^{D}_{DC} - \hat{K}_{DB}{}^{C} \hat{\Gamma}^{D}_{AC}$$
$$= \hat{\mathcal{R}}_{AB} - \hat{K}_{AD}{}^{C} \hat{K}_{BC}{}^{D} - \hat{\nabla}^C \hat{S}_{ABC} \tag{11.1.16}$$

$$\hat{G}_{[AB]} = \hat{R}_{[AB]} = -\hat{\nabla}^C \hat{S}_{ABC} = -\hat{V}_{AB} \tag{11.1.17}$$

$-\hat{V}_{AB}$ is the antisymmetric part of $\hat{G}_{AB}$ at this limit. And $\hat{X}_{AB}$ is a symmetric spin-torsion coupling adjustment - again only in the case of completely antisymmetric torsion.

The net result:

$$\hat{\nabla}^A \hat{\mathcal{G}}_{AB} = 0$$
$$\hat{\nabla}^A \hat{G}_{[AB]} = 0 \tag{11.1.18}$$

These divergence laws function as 5D Kaluza-Cartan 'conservation' laws, given complete antisymmetry, a well-behaved topology, though they may require positivity conditions or similar.

Torsion-Normal Coordinates

By using the same argument, verbatim, as in Wald [11] p. 41-42 normal coordinates can be defined about any point also in the presence of torsion using the auto-parallel equation instead of the geodesic equation. Completely antisymmetric torsion yields the same normal coordinates as without torsion, the paths varying only due to non-completely anti-symmetric terms.

The introduction of a postulate is needed in the presence of non-completely anti-symmetric torsion terms so that local normal and local torsion-normal coordinates will be comparable. Such a postulate must ensures that any symmetric terms are of low significance.

The seeming arbitrariness of any such postulate suggests perhaps that non-antisymmetric terms should be avoided in prospective theories.

11.2 Geodesic Motion, An Assumption

This work assumes that some sort of particle model of matter and charge is possible, that it can be added to the original theory without significantly changing the ambient space-time solution. No lagrangians are needed, and this is a departure from standard practice. The approach here is purely geometrical. There are further complications such as the lack of an explicit matter-charge model, and the allowance for the possible presence of torsion. Charge is identified with 5D momentum components. We might imagine that what has been described is a particle whizzing around the fifth dimension like a roller coaster on its spiralled tracks. However we might better consider a 'solid' ring, locked into place around the 5th dimension, rotating at some predetermined proper Kaluza velocity. An exact solution could perhaps involve changes in the size of the 5th dimension and exotic mass-energy. Complicated oscillations, and so on. Such realistic models are likely to be very difficult to construct.

In Einstein-Cartan theory geodesics, or extremals, are followed by spinless particles in 4D Einstein-Cartan theory [20]. Other particles follow different paths when interaction with torsion is present. Auto-parallels and extremals are two torsion analogs of geodesics, but neither of which in the most general case need determine the paths followed by all particles in Einstein-Cartan theory. Note that spinless particles according to [20] follow extremals. Extremals coincide with auto-parallels when torsion is completely antisymmetric. Particles with spin may interact in other ways. So the assumption is that torsion-spin coupling does not significantly effect the path of the particle, at least to some approximation. The simplest choice is to use auto-parallels. Exactly how sensitive this assumption is would require further research. However, it can be packaged into a single clean assumption.

AUTOPARALLEL POSTULATE: **Geodesic Option 1**. That any particle-like model, that is to be identified with a charge, approximately follows 5D auto-parallels.

(11.2.1)

But we might also try Levi-Civita geodesics (ie extremals):

L-C GEODESIC POSTULATE: **Geodesic Option 2**. That any particle-like model, that is to be identified with a charge, approximately follows 5D geodesics where the geodesic equation is the usual one defined in terms of the Levi-Civita connection.

(11.2.2)

These are the two main choices that are evidently available. Other possibilities, when considered, specificially if spin-spin coupling via torsion is involved, would need to be shown to satisfy a Lorentz force law independently of these two options.

12 Appendix II: The Cylinder Condition, Topology, And Some Consequences

This section defines the standard cylinder condition, some basic definitions and terminology, some axiomatic structure common to all variant theories under consideration, and some common consequences. This appendix has dependencies on previous appendices, but not on the main text.

12.1 The Cylinder Condition And Common Topological Assumptions of Kaluza Theories

The following define some common postulates that define Kaluza space, the 5D space in which space-time is then interpreted as being embedded, in particular the famous cylinder condition [1]:

POSTULATE (K1): **Geometry**. The geometry, the Kaluza space, under consideration is a 5D smooth Lorentzian manifold.

POSTULATE (K2): **Well-behaved**. Kaluza space is assumed regionally hyperbolic in the sense that there exists through each point in a considered region a 4D spatial cauchy surface, plus time, such that the 4D hypersurface is a simply connected 3D space extended around a 1D loop topologically in the simplest manner. Kaluza space is globally oriented and time-oriented.

POSTULATE (K3) **Cylinder condition**. One spatial dimension is topologically closed and 'small', the Kaluza dimension, the 1D loop. This is taken to mean that there are global unit vectors that define this direction, the Kaluza direction. The partial derivatives ∂_4 of all tensors in this Kaluza direction are taken to be zero in some coordinate system.

(12.1.1)

These postulates can be applied to any Kaluza theory that is considered in this work.

12.2 The Cylinder Condition And Charts

The cylinder condition by construction allows for an atlas of charts wherein the Kaluza dimension (defined by the cylinder condition in the obvious sense) is naturally presented by the fourth index. The atlases that are compliant are restricted. This means that the cylinder condition can be represented by a subatlas of the maximal atlas. The set of local coordinate transformations that are compliant with this atlas (called a Kaluza atlas) is non-maximal by construction. A further reduction in how the atlas might be interpreted could also be implied by setting c=1, and fixing **G** numerically. That said, any consistent choice of units suffices. This doesn't prevent reflection of an axis however, and indeed reflection of the Kaluza dimension is here equivalent to a (kinetic) charge inversion. However, given orientability and an orientation we can remove even this ambiguity. We can further reduce a Kaluza atlas by removing boosts in the Kaluza dimension. Space-time is taken to be a subframe within a 5D frame within a Kaluza subatlas of a region wherein uncharged matter can be given a rest frame via a 4D Lorentz transformation. Boosting uncharged matter along the Kaluza axis will give it kinetic charge. The Kaluza atlas represents the 4D view that kinetic charge is 4D covariant. Rotations into the Kaluza axis can likewise be omitted. This results in additional constraints on the connection coefficients associated with charts of this subatlas. The use of this subatlas does not prevent the theory being generally covariant, but simplifies the way in which we look at the Kaluza space through a 4D physical limit.

Definition 12.2.1: A *Kaluza atlas* is:

(i) A subatlas (possibly just over a region) of the maximal atlas of Kaluza-Cartan space where boosts and rotations into the Kaluza dimension (as defined by the cylinder condition) are explicitly omitted.

(ii) All partial derivatives in the Kaluza direction are vanishing.

(iii) Inversion in the Kaluza direction and rescalings can also be omitted so as to establish units and orientation.

(iv) For each point on the Kaluza atlas a chart exists with 'torsion-normal' coordinates where index 4 is the Kaluza dimension. This simply defaults to normal coordinates when torsion is completely antisymmetric.

12.3 Kinetic Charge

Kinetic charge is defined as the 5D momentum component in terms of the 5D Kaluza rest mass of a hypothesised particle: ie (i) its rest mass in the 5D Lorentz manifold (m_{k0}) and (ii) its proper Kaluza velocity ($dx_4/d\tau^*$) with respect to a frame in the maximal atlas that follows the particle. And equally it can be defined in terms of (i) the relativistic rest mass (m_0), relative to a projected frame where the particle is stationary in space-time, but where non-charged particles are stationary in the Kaluza dimension, and in terms of (ii) coordinate Kaluza velocity (dx_4/dt_0):

Prov. Definition 12.3.1: kinetic charge: $Q^* = m_{k0}dx_4/d\tau^* = m_0 dx_4/dt_0$

This provisional definition (refined below) makes sense because mass can be written in fundamental units (i.e. in distance and time). And the velocities in question defined relative to particular frames. It is not a generally covariant definition but it is nevertheless mathematically meaningful. This kinetic charge can be treated in 4D space-time and the Kaluza atlas as a scalar: the first equation above is covariant with respect to the Kaluza atlas. It can be generalized to a 4-vector, and it is also conserved as shown shortly. In general relativity at the special relativistic Minkowski limit the conservation of momenergy can be given in terms of the stress-energy tensor [6]. This is approximately true at a weak field or special relativistic limit and can be applied equally to Kaluza theory, via the Levi-Civita (ie torsionless) connection. We have a description of conservation of momentum including in the 5th dimension ($4 \neq j \neq 0$):

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x_i} = 0 \ , \ \frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x_i} = 0 \text{ and } \frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x_i} = 0 \qquad (12.3.2)$$

We also have term i=4 vanishing by the cylinder condition. Thus the conservation of kinetic charge becomes (when generalized to different space-time frames) the property of a 4-vector current, which we know to be locally conserved: $\partial_0 \hat{T}^{04} + \partial_1 \hat{T}^{14} + \partial_2 \hat{T}^{24} + \partial_3 \hat{T}^{34} = 0$.

To make sense of this in 5D we need to change the provisional definition above and make it density-based as follows (imagine a ring rather than a particle). The alternative definition can be made in terms of the mass density ρ_0, coupled with the Kaluza dimension's size or Kaluza length λ. In this way we do not presuppose that the rest mass we observe in space-time is necessarily the m_0 above: what happens for example to the apparent rest mass in 4D if the Kaluza distance changes and the density compressed or rarefacted? m_0 makes most sense as a definition of rest mass in 4D when this does not happen. Generalization demands the following definition, replacing m_0 as follows (and maintaining throughout the identity the 5d mass $m_x = \lambda q_x$ which for a small 3D volume v of uniform charge $= \lambda v \rho_x$. This gives q_0 and ρ_0 a truly 4D flavour and usual mass and mass-density units - noting that mass-density units are the same in 4d and 5d):

Definition 12.3.3: 5D (ring) kinetic charge density: $Q^* = \lambda \rho_{k0} dx^4 / d\tau^*$ $= \lambda \rho_0 dx^4 / dt_0$

This leads to a density-slice definition of 4D kinetic charge as follows:

Definition 12.3.4: 4D (slice) kinetic charge density: $Q^{**} = \rho_{k0} dx^4 / d\tau^*$ $= \rho_0 dx^4 / dt_0$

4D (slice) kinetic charge current density, however, is also definable as the 4-vector, induced from 5D Kaluza space as follows (using the Kaluza atlas to ensure it is well-defined as a 4-vector) in terms of the stress-energy tensor:

$$J^{**a} = -T^{a4} = -\alpha_g \hat{\mathcal{G}}^{a4} \qquad (12.3.5)$$

And a measure of the total current density can be give as:

$$J^{*a} = -\alpha_g \lambda \hat{\mathcal{G}}^{a4} \qquad (12.3.6)$$

Using Wheeler et al [10] p.131, and the appropriate space-time (or Kaluza atlas) frame following the particle, we have (converting from stress-energy density to momentum density units in the case of Q^{**}):

$$Q^{**} = J^{**a}(c^{-1}, 0, 0, 0)_a$$

$$Q^{*} = J^{*a}(c^{-1}, 0, 0, 0)_a = \lambda Q^{**} \tag{12.3.7}$$

So we have a scalar, then a vector representation of relativisitic invariant charge current, and finally a 2-tensor unification with conserved mass-energy via the Einstein tensor. It follows that the vanishing of the divergence of kinetic charge in 4D is only approximate, in 5D it is exact. The sign is arbitrary and can be chosen for later convenience. It toggles the sign of charge.

Definition 12.3.8: *Kinetic charge current density* is defined to be the 4-vector $J^{*a} = -\alpha_g \lambda \hat{\mathcal{G}}^{a4}$, with respect to the Kaluza atlas that represents this total conserved 5D charge current. It is conserved also when projected onto 4D unlike the slice version which must be summed or integrated over the Kaluza dimension. The units are stress-energy density times distance. Stress-energy density having the same units in 4D and 5D. Note the divergence of the Einstein tensor:

$$\hat{\triangle}_A \hat{\mathcal{G}}^{AB} = 0 \text{ and } \hat{\triangle}_A \hat{\mathcal{G}}^{A4} = 0 \approx \hat{\triangle}_a \hat{\mathcal{G}}^{a4}$$

In the case of complete antisymmetry the above also holds for the torsion connection covariant derivative. The approximation requires a weak field or special relativistic limit.

12.4 Consistency With Special Relativity

Some concerns with regards to the special relatistic limit are dealt with here, showing that Kaluza theory still makes sense when identifying components of the momentum in the Kaluza dimension with charge. In this section set $c = 1$ for convenience.

How Relativistic Rest Mass Is Related To Kaluza Rest Mass

Kinetic charge is identified with divergence-free 5D momentum components represented in a space-time rest frame . It can be treated as a 4-vector by virtue of the cylinder condition approaching the flat limit.

That this is consistent with special relativity can be tentatively investigated via a simplistic analysis: the 5D relativistic mass resulting from momentum in the 5th dimension is the source also of the relativistic rest mass.

The additions of velocities in special relativity is not obvious. Assume a flat 5D Kaluza space (i.e without geometric curvature or torsion, thus analogously to special relativity at a flat space-time limit, a 5D Minkowski limit). Space-time can be viewed as a 4D slice (or series of parallel slices) perpendicular to the 5th Kaluza dimension that minimizes the length of any loops that are perpendicular to it. This requires the usual cylinder condition of Kaluza theory [1]. Taking a particle and an inertial frame, the relativistic rest frame where the particle is stationary with respect to space-time but moving with velocity u in the 5th dimension, and a second frame where the charge is now moving in space-time at velocity v, but still with velocity u in the 5th dimension, then the total speed squared of the particle in the second frame is according to relativistic addition of orthogonal velocities:

$$s^2 = u^2 + v^2 - u^2 v^2 \tag{12.4.1}$$

The particle moving in the Kaluza dimension with velocity u, but stationary with respect to 4D space-time, will have a special relativistic 4D rest mass (m_0), strictly this remains a 5D quantity for now. It is normally greater than its 5D Kaluza rest mass (m_{k0}). This may be a difficulty in that it restricts realistic matter-charge models. Though this could be adjusted, for example by exotic structures. We can see that the Kaluza rest mass, ie the mass m_{k0} of the particle as a 5D object in a 5D geometry is consistent with the orthogonal addition of velocities as follows:

$$m_0 = \frac{m_{k0}}{\sqrt{(1-u^2)}} \text{ where } u = \tanh[\sinh^{-1}(Q'/m_{k0})] \tag{12.4.2}$$

$$m_{rel} = \frac{m_0}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2)}} \times \frac{1}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2-v^2+u^2v^2)}} \tag{12.4.3}$$

By putting $u = \tanh[\sinh^{-1}(Q'/m_{k0})]$ (keeping the hyperbolics to recall the conversion between unidirectional proper and coordinate velocities) into the definition of relativistic rest mass in terms of Kaluza rest mass and solving, we get that charge, whether positive or negative, is related to the relativistic rest mass according to the following formula:

$$\begin{aligned}\cosh[\sinh^{-1}(Q'/m_{k0})] = m_0/m_{k0} = \frac{dt_0}{d\tau^*} \\ = \sqrt{(Q'/m_{k0})^2 + 1}\end{aligned} \tag{12.4.4}$$

Observed electrons have static charge, angular momentum, a magnetic moment, and a flavor. The main thing distinguishing the electron from the muon is the flavor. The mass difference between the muon and the electron is about 105 MeV, perhaps solely due to this difference in flavor. The issue of modeling particles within a classical theory is, not surprisingly, a difficult one! Thus at this stage the idealized hypothetical charges used here, and real particles, can only be tentatively correlated.

It is possible to proceed without concern for the difficult issues of such charge models, instead simply developing the mathematics 'as is' and seeing where it leads without prejudice.

Proper Kaluza Velocity As A Scalar

This section shows that the proper velocity $\mathbf{W}$ (written as a vector) with only one component in the Kaluza dimension is invariant under 4D space-time boosts orthogonal to it. The proper Kaluza velocity therefore is a constant with respect to *local* coordinate changes within a Kaluza atlas. It could be claimed that this result should follow in any case from the definition of proper velocity if the local coordinate transformation is only in the 4 dimensions of space-time, however this is not true for rotations - a rigorous proof is always better. The result here simply says that with respect to the Kaluza atlas the value is a scalar.

$W_4 = dx_4/d\tau$ proper velocity in a particle-following space-time frame

$$U_4 = \frac{W_4}{\sqrt{1+{W_4}^2}} \text{ coordinate velocity using proper velocity}$$

Using orthogonal addition of coordinate velocities formula to boost space-time frame by orthogonal coordinate velocity $\mathbf{V}$:

$$\mathbf{V} = (V, 0, 0, 0)$$
$$\mathbf{U} = (0, 0, 0, U_4)$$

Coordinate velocity vector in new frame, using the orthogonal velocity addition formula:

$$\overline{\mathbf{U}} = \mathbf{V} + \sqrt{1-V^2}\ \mathbf{U}$$

So,

$$\overline{U}_4 = \sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}}$$

Define proper velocity in new frame: $\overline{\mathbf{W}}$, using proper velocity definition:

$$\overline{W}_4 = \frac{\overline{U}_4}{\sqrt{1-V^2-{\overline{U}_4}^2}}$$

$$= \frac{\sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}}}{\sqrt{1-V^2-(\sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}})^2}}$$

$$= \frac{W_4}{\sqrt{1+W_4^2}\sqrt{1-(\frac{W_4}{\sqrt{1+W_4^2}})^2}}$$

$$= \frac{W_4}{\sqrt{1+W_4^2-W_4^2}} = W_4 \tag{12.4.5}$$

$\overline{W_4} = W_4$ is the result required

13 Appendix III: Mathematical Appendix

An appendix of mathematical working independent of the text, but possibly dependent on previous appendices.

13.1 The Christoffel Symbols And Connection Coefficients

Here we assume the usual definitions of the Christoffel symbols, and the cylinder condition.

$$\begin{aligned}
2\hat{\Gamma}^{A}_{BC} &= \textstyle\sum_d \hat{g}^{AD}(\partial_B \hat{g}_{CD} + \partial_C \hat{g}_{DB} - \partial_D \hat{g}_{BC}) \\
&= \textstyle\sum_d \hat{g}^{Ad}(\partial_B \hat{g}_{Cd} + \partial_C \hat{g}_{dB} - \partial_d \hat{g}_{BC}) \\
&\quad + \hat{g}^{A4}(\partial_B \hat{g}_{C4} + \partial_C \hat{g}_{4B} - \partial_4 \hat{g}_{BC})
\end{aligned}$$

$$\begin{aligned}
2\hat{\Gamma}^{A}_{bc} &= \textstyle\sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) \\
&+ \textstyle\sum_d \hat{g}^{Ad}(\partial_b k^2\phi^2 A_c A_d + \partial_c k^2\phi^2 A_d A_b - \partial_d k^2\phi^2 A_b A_c) \\
&+ \hat{g}^{A4}(\partial_b k\phi^2 A_c + \partial_c k\phi^2 A_b - \partial_4 g_{bc} - \partial_4 k^2\phi^2 A_b A_c)
\end{aligned}$$

$$2\hat{\Gamma}^{A}_{4c} = \textstyle\sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 k^2\phi^2 A_c A_d + \partial_c k\phi^2 A_d - \partial_d k\phi^2 A_c) + \hat{g}^{A4}\partial_c \phi^2$$

$$2\hat{\Gamma}^{A}_{44} = 2\textstyle\sum_d \hat{g}^{Ad}\partial_4 k\phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d \phi^2 + \hat{g}^{A4}\partial_4 \phi^2 \tag{13.1.1}$$

$$\begin{aligned}
2\hat{\Gamma}^{ABC} &= \partial^B \hat{g}^{CA} + \partial^C \hat{g}^{AB} - \partial^A \hat{g}^{BC} \\
&= \textstyle\sum_d (\hat{g}^{Bd}\partial_d \hat{g}^{CA} + \hat{g}^{Cd}\partial_d \hat{g}^{AB} - \hat{g}^{Ad}\partial_d \hat{g}^{BC})
\end{aligned}$$

$$2\hat{\Gamma}^{A4C} = \textstyle\sum_d (-kA^d \partial_d \hat{g}^{CA} + \hat{g}^{Cd}\partial_d \hat{g}^{A4} - \hat{g}^{Ad}\partial_d \hat{g}^{4C})$$

$$\begin{aligned}
2\hat{\Gamma}^{a4c} &= \textstyle\sum_d (-kA^d \partial_d g^{ca} - g^{cd}\partial_d kA^a + g^{ad}\partial_d kA^c) \\
&= -\textstyle\sum_d (kA^d \partial_d g^{ca}) + kF^{ac}
\end{aligned} \tag{13.1.2}$$

The Electromagnetic Limit $\phi^2 = 1$

Now putting in $\phi^2 = 1$, and for convenience $k = 1$:

$$
\begin{aligned}
2\hat{\Gamma}^A_{bc} &= \sum_d \hat{g}^{Ad}(\partial_b g_{cd}+\partial_c g_{db}-\partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b A_c A_d+\partial_c A_d A_b-\partial_d A_b A_c)\\
&\quad + \hat{g}^{A4}(\partial_b A_c + \partial_c A_b - \partial_4 g_{bc} - \partial_4 A_b A_c)\\
2\hat{\Gamma}^A_{4c} &= \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c)\\
\hat{\Gamma}^A_{44} &= \sum_d \hat{g}^{Ad}\partial_4 A_d
\end{aligned}
\tag{13.1.3}
$$

Simplifying...

$$
\begin{aligned}
2\hat{\Gamma}^a_{bc} &= 2\Gamma^a_{bc} + \sum_d g^{ad}(A_b F_{cd} + A_c F_{bd}) + A^a\partial_4 g_{bc} + A^a\partial_4 A_b A_c\\
2\hat{\Gamma}^4_{bc} &= -\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) - \sum_d A^d(A_b F_{cd} + A_c F_{bd})\\
&\quad - (1+\sum_i A_i A^i)(\partial_4 g_{bc} + \partial_4 A_b A_c) + (\partial_b A_c + \partial_c A_b)\\
2\hat{\Gamma}^a_{4c} &= \sum_d g^{ad}(\partial_4 g_{cd} + \partial_4 A_c A_d) + \sum_d g^{ad}F_{cd}\\
2\hat{\Gamma}^4_{4c} &= -\sum_d A^d(\partial_4 g_{cd} + \partial_4 A_c A_d) - \sum_d A^d F_{cd}\\
\hat{\Gamma}^a_{44} &= \sum_d g^{ad}\partial_4 A_d\\
\hat{\Gamma}^4_{44} &= -\sum_d A^d\partial_4 A_d
\end{aligned}
\tag{13.1.4}
$$

The Scalar Limit $A_i = 0$

The scalar limit is similarly defined, and for convenience $k = 1$:

$$
\begin{aligned}
2\hat{\Gamma}^A_{bc} &= \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{ab} - \partial_d g_{bc}) - \hat{g}^{A4}\partial_4 g_{bc}\\
2\hat{\Gamma}^A_{4c} &= \sum_d \hat{g}^{Ad}\partial_4 g_{cd} \qquad + \hat{g}^{A4}\partial_c\phi^2\\
2\hat{\Gamma}^A_{44} &= -\sum_d \hat{g}^{Ad}\partial_d\phi^2 \qquad + \hat{g}^{A4}\partial_4\phi^2
\end{aligned}
\tag{13.1.5}
$$

Simplifying...

$$
\begin{aligned}
\hat{\Gamma}^a_{bc} &= \Gamma^a_{bc}\\
2\hat{\Gamma}^4_{bc} &= -\tfrac{1}{\phi^2}\partial_4 g_{bc}\\
2\hat{\Gamma}^a_{4c} &= \sum_d g^{ad}\partial_4 g_{cd}\\
2\hat{\Gamma}^4_{4c} &= \tfrac{1}{\phi^2}\partial_c\phi^2\\
2\hat{\Gamma}^a_{44} &= -\sum_d g^{ad}\partial_d\phi^2\\
2\hat{\Gamma}^4_{44} &= \tfrac{1}{\phi^2}\partial_4\phi^2
\end{aligned}
\tag{13.1.6}
$$

14 References

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