Super Conformal Group in $D = 10$ Space-time

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Abstract

In this present discussion we discussed the super Poincaré group in $D = 10$ dimensions in terms of the highest division algebra of octonions. We have construct the Poincaré group in $D = 8$ dimension then it's extension to conformal algebra of $D = 10$ has been discussed in terms of octonion algebra. Finally extension of the conformal algebras of $D = 10$ dimensional space to super conformal algebra of Poincaré group have been done in a consistent manner.

1 Introduction

The well known result of Haag, Sohnius and Lopuszanski [1] proved that supersymmetry algebra is the only graded lie algebra of symmetries of S- matrix consistent with relativistic quantum field theory. Also that supersymmetry arises as a symmetry which combines fermions and bosons in the representation or multiplet of the graded group of Poincaré algebra. It is well known that [2] the traditional field theories however realistic plagued with the problems of mass hierarchy which become unavoidable at higher energy ranges. In this regard it is a fascinating fact that supersymmetry provide a self consistent cancellation method which remove these problems. And also that the higher dimensional Supersymmetric theories are the most possible gauge theories in order to understand the theories of everything (TOE)[3]. The close connections exist between division algebras, Fiertz identities and super Poincaré groups led to the important conclusion that [4] the classical Green-Schwarz superstring and $N = 1$ super Yang-Mills (SYM) theories of a single vector and spinor can exist only in the critical dimensions associated with the division algebras.

On the other hand in view of Hurwitz theorem [5] there exists four normed division algebras $\mathbb{R}$,$\mathbb{C}$,$\mathbb{H}$ and$\mathcal{O}$ respectively named as the algebras of Real numbers, Complex numbers, Quaternions [6] and Octonions [7]. It is pointed out that by Kugo-Townsend that [8] the supersymmetric gauge theories are well examined for$D = 3, 4, 6, 10$ in terms of components of division algebra respectively associated with the algebra of real numbers $\mathbb{R}$ (for D=3), of complex numbers $\mathbb{C}$ (for $D = 4$), quaternions $\mathbb{H}$ (for $D = 6$) and octonions $\mathcal{O}$ ($D = 10$). The connection between higher dimensional supersymmetric field theories and division algebra has already been established by Kugo-Townsend[8] ,Lukereski-Topan[9] ,A. Anastasiou et. all[10], J. M. Evans [11]. Also the connection between super Poincaré groups and division algebras has been studied by Feza Gürsey [12] and M. Günaydin[13]. The explicit relation between octonion algebra and $SO(8)$ previously well established by A. Reit Dündarer, Feza Gürsey[14]. Attempt has been made by Corinne A. Manogue, Jörg Schray [15] for the algebraic description of finite Lorentz transformations of vectors in $10-$dimensional space by octonion formulation, where the non-associativity of octonion algebra plays a crucial role. The relation between $SO(9,1)$ and $SL(2,O)$ has been well established. Also $D=11$ $M-$algebra has been studied in terms of octonions[9].

Keeping in view the close connection between division algebras and supersymmetry in higher dimensions we attempt to study $D=8$ and $D=10$ Poincaré algebra in terms of octonions by first establishing relation between $SO(8)$ group and octonion algebra then extending the Poincaré algebra of $D=8$ to conformal algebra. Then extending of Poincaré algebra in $D=10$ to conformal algebra in $D=10$ space.. Then finally supersymmetrization has been done for $D=10$ dimensions.

2 Octonionic representation of $SO(8)$:

Let us write the vector x in $D=8$ space as

$$x^{\mu}=(x^{1},x^{2},x^{3},x^{4},x^{5},x^{6},x^{7},x^{8}) \tag{1}$$

The metric defined in $D=8$ space as $\eta_{\mu\nu}=(1,1,1,1,1,1,1,1)$. As for the case of quaternion generalization, the complex quantity of Pauli matrices has been extended [16] to the imaginary quaternion triplet $e_{j}(\forall j=1,2,3\; e_{j}\in\mathbb{H})$. Consequently the Pauli matrices were generalized to 2×2 quaternion hermitian matrices for description of $SL(2,\mathbb{H})$ group. Likewise, let us construct a mapping from the Euclidien eight dimensional space to the set of octonion valued 2×2 Pauli matrices such that a eight-vector is described as

$$x^{\mu}\rightarrow\rho(x^{\mu})=x^{\mu}\Gamma_{\mu} \tag{2}$$

where Γ_{μ} are 2×2 octonion matrices such as

$$\Gamma_{8}=\left(\begin{array}{cc} 0 & e_{0}\\ e_{0} & 0 \end{array}\right),\Gamma_{i}=\left(\begin{array}{cc} 0 & e_{i}\\ -e_{i} & 0 \end{array}\right)\quad(\forall i=1\,to\,7). \tag{3}$$

The octonion basis elements $e_{i}^{\prime}s$ satisfy the relation [17]

$$e_{i}e_{j}=-\,\delta_{ij}e_{0}+f_{ijk}e_{k}\;(\forall i,j,k=1\,to\,7). \tag{4}$$

The f_{ijk} is completely antisymmetric G_{2} invariant tensor having the $+1$ value for the following permutations of combinations of (ijk) $(123),(471),(257),(165),(624),(543),(736)$. The Γ_{μ} matrices satisfy the following relation of clifford algebra

$$\Gamma_{\mu}\Gamma_{\nu}+\Gamma_{\nu}\Gamma_{\mu}=2\eta_{\mu\nu}. \tag{5}$$

So by the equation (2) we have

$$\rho(x^{\mu})=x^{\mu}\Gamma_{\mu}=\left(\begin{array}{cc} 0 & x^{8}+e_{1}x^{1}+e_{2}x^{2}+e_{3}x^{3}+e_{4}x^{4}+e_{5}x^{5}+e_{6}x^{6}+e_{7}x^{7}\\ x^{8}-e_{1}x^{1}-ex^{2}-e_{3}x^{3}-e_{4}x^{4}-e_{5}x^{5}-e_{6}x^{6}-e_{7}x^{7} & 0 \end{array}\right). \tag{6}$$

The determinent of $\rho(x^{\mu})$ may be defined unambigously for hermitian 2×2 octonionic matrices [18]. Which leads us to the

$$det[\rho(x^{\mu})]=-\,(x^{8})^{2}-(x^{1})^{2}-(x^{2})^{2}-(x^{3})^{2}-(x^{4})^{2}-(x^{5})^{2}-(x^{6})^{2}-(x^{7})^{2}. \tag{7}$$

In another way we may write it as

$$det[\rho(x^{\mu})] = - x_{\mu}x^{\mu} = -\eta_{\mu\nu}x^{\nu}x^{\mu} \tag{8}$$

We may define the generators of $SO(8)$ in 2×2 octonion matrices. First the generators of angular momentum in $SO(8)$ as

$$\begin{aligned}\Sigma_{\alpha\beta} =& \frac{i}{4}[\Gamma_{\alpha},\Gamma_{\beta}] \quad (\forall\alpha,\beta = 1\, to\, 7)\\ \Sigma_{\alpha\beta} =& -\frac{i}{2}f_{\alpha\beta\gamma}\begin{pmatrix} e_{\gamma} & 0 \\ 0 & e_{\gamma}\end{pmatrix} \ (\forall\alpha,\beta,\gamma = 1\, to\, 7).\end{aligned} \tag{9}$$

But this generate only 7 independent Lorenz generators of rotation in dimensional greater than three there is no unique plane orthogonal to a given axis[15]. For example since $f_{123} = f_{471} = f_{165} = +1$ so $\Sigma_{47} = \Sigma_{65} = \Sigma_{23}$, there are three degenerate plains of rotations along any axis. Similarly for the other permutation of combinations for $f_{ijk}\,(\forall i,j,k = 1\, to\, 7)$. There are 7 other generators defined as

$$\begin{aligned}\Sigma_{8\alpha} =& \frac{i}{4}[\Gamma_{8},\Gamma_{\alpha}] \quad (\forall\alpha = 1\, to\, 7)\\ \Sigma_{8\alpha} =& \frac{i}{2}\begin{pmatrix} -e_{\alpha} & 0 \\ 0 & e_{\alpha}\end{pmatrix} \quad (\forall\alpha = 1\, to\, 7).\end{aligned} \tag{10}$$

There are 28 independent Lorentz generators of $SO(8)$. Where 14 generators are come from $\Sigma_{8\alpha}\,(\forall\alpha = 1\, to\, 7)$ and $\Sigma_{\alpha\beta}\,(\forall\alpha,\beta = 1\, to\, 7)$. The commutation rule followed by the 7 generators of rotation are defined by

$$\begin{aligned}[\Sigma_{\alpha\beta},\Sigma_{lm}] =& \left(-\frac{i}{2}\right)^{2} f_{\alpha\beta\gamma}f_{lmn}\begin{pmatrix} e_{\gamma} & 0 \\ 0 & e_{\gamma}\end{pmatrix}\begin{pmatrix} e_{n} & 0 \\ 0 & e_{n}\end{pmatrix} - \left(-\frac{i}{2}\right)^{2} f_{lmn}f_{\alpha\beta\gamma}\begin{pmatrix} e_{n} & 0 \\ 0 & e_{n}\end{pmatrix}\begin{pmatrix} e_{\gamma} & 0 \\ 0 & e_{\gamma}\end{pmatrix}\\ =& \left(-\frac{1}{2}\right) f_{\alpha\beta\gamma}f_{lmn}f_{\gamma nq}\begin{pmatrix} e_{q} & 0 \\ 0 & e_{q}\end{pmatrix}.\end{aligned} \tag{11}$$

By the following identities of octonions [14]

$$\begin{aligned}f_{lmn}f_{nq\gamma} =& g_{lmq\gamma} + \delta_{lq}\delta_{m\gamma} - \delta_{l\gamma}\delta_{mq}\\ f^{\alpha\beta\gamma}g_{lmq\gamma} =& f^{\alpha}_{ql}\delta^{\beta}_{m} + f^{\alpha}_{lm}\delta^{\beta}_{q} + f^{\alpha}_{mq}\delta^{\beta}_{l} - f^{\beta}_{lm}\delta^{\alpha}_{q} - f^{\beta}_{ql}\delta^{\alpha}_{m} - f^{\beta}_{mq}\delta^{\alpha}_{l}\end{aligned} \tag{12}$$

we have the following commutation relation

$$[\Sigma_{\alpha\beta},\Sigma_{lm}] = -i(\eta_{\alpha l}\Sigma_{\beta m} - \eta_{\alpha m}\Sigma_{\beta l} + \eta_{\beta m}\Sigma_{\alpha l} - \eta_{\beta l}\Sigma_{\alpha m}) - K_{\alpha\beta lm} \ \ (\forall\alpha,\beta,l,m = 1\, to\, 7). \tag{13}$$

Where the $K_{\alpha\beta lm}$ is a four rank tensor defined as

$$K_{\alpha\beta lm} = \frac{1}{2}\left[f_{\beta lm}\begin{pmatrix} e_{\alpha} & 0 \\ 0 & e_{\alpha}\end{pmatrix} - f_{\alpha lm}\begin{pmatrix} e_{\beta} & 0 \\ 0 & e_{\beta}\end{pmatrix} + f_{\alpha\beta l}\begin{pmatrix} e_{m} & 0 \\ 0 & e_{m}\end{pmatrix} - f_{\alpha\beta m}\begin{pmatrix} e_{l} & 0 \\ 0 & e_{l}\end{pmatrix}\right]. \tag{14}$$

The $K_{\alpha\beta lm}$ doesn't raise any ambiguity, since there is no unique plain of rotation along any one axis. This four rank tensor contain the elements of generators of rotation. The 7 generators of Lorentz boost satisfy the commutation rule as

$$[\Sigma_{8\alpha},\Sigma_{8\beta}] = - i\eta_{88}\Sigma_{\alpha\beta} \qquad (\forall\alpha,\beta = 1\, to\, 7) \tag{15}$$

The commutation relation between the generators of rotation $\Sigma_{\alpha\beta}$ and generators $\Sigma_{8\alpha}$ are defined by using (9,10) as

$$[\Sigma_{8\alpha},\Sigma_{\beta\gamma}]=-i(\eta_{\alpha\gamma}\Sigma_{8\beta}-\eta_{\alpha\beta}\Sigma_{8\gamma})-X_{\beta\gamma\alpha\delta} \tag{16}$$

where $X_{\beta\gamma\alpha\delta}$ is a four rank tensor defined as

$$X_{\beta\gamma\alpha\delta}=-\frac{1}{2}g_{\beta\gamma\alpha\delta}\begin{pmatrix}-e_{\delta} & 0\\ 0 & e_{\delta}\end{pmatrix}\quad(\forall\beta,\gamma,\alpha,\delta=1\,to\,7) \tag{17}$$

which may be neglected by the same argument for the previous derivation. Since $SO(8)$ must have 28 independent generators, 14 of which come from $\Sigma_{\alpha\beta}$and $\Sigma_{8\alpha}$ and the other 14 come from the derivative algebra of octonions (G_2) as

$$so(8)=L\oplus G_2 \tag{18}$$

where the L is the lie algebra of generators $\Sigma_{\mu\nu}$ ($\forall\mu,\nu=0\,to\,7$). For the elements of derivative algebra (G_2) of octonions we may defined as a mapping D on octonions itself

$$D(xy)=(Dx)y+x(Dy) \tag{19}$$

The linear mapping D on an element $x\in\mathcal{O}$defined as[19]

$$D_{a,b}(x)=[a,b,x]+\frac{1}{3}[[a,b],x]\qquad(\forall a,b,x\in\mathcal{O}). \tag{20}$$

The 14 other generators of $SO(8)$ may be defined as

$$\Xi_{\mu\nu}=-\frac{i}{2}\begin{pmatrix}D_{\mu\nu} & 0\\ 0 & D_{\mu\nu}\end{pmatrix}\qquad(\forall\mu,\nu\in\mathcal{O}) \tag{21}$$

where the $D_{\mu\nu}$ is an element of G_2 algebra defined as

$$D_{\mu\nu}(x)=[e_\mu,e_\nu,x]+\frac{1}{3}[[e_\mu,e_\nu],x] \tag{22}$$

$x\in\mathcal{O}$ and e_μ,e_ν ($\forall\mu,\nu\in 1\,to\,7$) are the basis elements of octonion algebra $\mathcal{O}$. The commutation relations of the 7 generators of angular momentum and 14 generators $D_{\mu\nu}$ is defined as

$$[\Xi_{\mu\nu},\Sigma_{\alpha\beta}]=-\frac{1}{4}f_{\alpha\beta\gamma}\begin{pmatrix}[D_{\mu\nu},e_\gamma] & 0\\ 0 & [D_{\mu\nu},e_\gamma]\end{pmatrix}. \tag{23}$$

We have the following commutation relation

$$[\Xi_{\mu\nu},\Sigma_{\alpha\beta}]=-\frac{4i}{3}[\eta_{\mu\alpha}\Sigma_{\nu\beta}-\eta_{\mu\beta}\Sigma_{\nu\alpha}+\eta_{\nu\beta}\Sigma_{\mu\alpha}-\eta_{\nu\alpha}\Sigma_{\mu\beta}]-K'_{\alpha\mu\nu\beta} \tag{24}$$

where the $K'_{\alpha\mu\nu\beta}$ has the form as

$$K'_{\alpha\mu\nu\beta}=\begin{pmatrix}Y_{\alpha\mu\nu\beta} & 0\\ 0 & Y_{\alpha\mu\nu\beta}\end{pmatrix}. \tag{25}$$

With $Y_{\alpha\mu\nu\beta}$, a four rank tensor having value

$$Y_{\alpha\mu\nu\beta} =\frac{2}{3}(f_{\alpha\mu\nu}e_\beta - f_{\beta\mu\nu}e_\alpha) + \frac{1}{6}(f_{\alpha\beta\nu}e_\mu - f_{\alpha\beta\mu}e_\nu). \tag{26}$$

Which should be again neglected for the same argument of above derivations Similarly the commutation relations of $\Xi_{\mu\nu}$ with $\Sigma_{0\alpha}$ may be evaluated as

$$[\Xi_{\mu\nu}, \Sigma_{8\alpha}] = -\frac{i}{3}(\eta_{\nu\alpha}\Sigma_{8\mu} - \eta_{\mu\alpha}\Sigma_{8\nu}) - Z_{\mu\nu\alpha r} \tag{27}$$

where the $Z_{\mu\nu\alpha r}$ is the four raank tensor defined as

$$Z_{\mu\nu\alpha r} =\frac{2}{3}g_{\mu\nu\alpha r}\begin{pmatrix} -e_r & 0 \\ 0 & e_r \end{pmatrix}. \tag{28}$$

So taking into cosideration the degeneracy of rotation plains and the permutation of combinations of structure constant we have the following Lie algebra of $SO(8)$:

$$\begin{aligned}
&[\Sigma_{\alpha\beta}, \Sigma_{lm}] = -i(\eta_{\alpha l}\Sigma_{\beta m} - \eta_{\alpha m}\Sigma_{\beta l} + \eta_{\beta m}\Sigma_{\alpha l} - \eta_{\beta l}\Sigma_{\alpha m}) \;\; (\forall \alpha, \beta, l, m = 1\,\text{to}\,7)\\
&[\Sigma_{8\alpha}, \Sigma_{8\beta}] = -i\eta_{88}\Sigma_{\alpha\beta} \qquad (\forall \alpha, \beta = 1\,\text{to}\,7)\\
&[\Sigma_{8\alpha}, \Sigma_{\beta\gamma}] = -i(\eta_{\alpha\gamma}\Sigma_{8\beta} - \eta_{\alpha\beta}\Sigma_{8\gamma}) \quad (\forall \alpha, \beta, \gamma = 1\,\text{to}\,7)\\
&[\Xi_{\mu\nu}, \Sigma_{\alpha\beta}] = -\frac{4i}{3}(\eta_{\mu\alpha}\Sigma_{\nu\beta} - \eta_{\mu\beta}\Sigma_{\nu\alpha} + \eta_{\nu\beta}\Sigma_{\mu\alpha} - \eta_{\nu\alpha}\Sigma_{\mu\beta}) \; (\forall \mu, \nu, \alpha, \beta = 1\,\text{to}\,7)\\
&[\Xi_{\mu\nu}, \Xi_{\alpha\beta}] = -i\frac{62}{3}(\eta_{\mu\alpha}\Xi_{\nu\beta} - \eta_{\mu\beta}\Xi_{\nu\alpha} + \eta_{\nu\beta}\Xi_{\mu\alpha} - \eta_{\nu\alpha}\Xi_{\mu\beta}) \; (\forall \mu, \nu, \alpha, \beta = 1\,\text{to}\,7)\\
&[\Xi_{\mu\nu}, \Sigma_{8\alpha}] = -\frac{2i}{3}(\eta_{\nu\alpha}\Sigma_{8\mu} - \eta_{\mu\alpha}\Sigma_{8\nu}) \qquad (\forall \mu, \nu, \alpha = 1 \text{ to } 7).
\end{aligned} \tag{29}$$

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3 The Poincaré group of $SO(1,9)$:

A proper Lorentz transformation in $D = 10$ space is defined as

$$x'^\mu =\Lambda^\mu_\nu x^\nu \qquad (\forall \Lambda \in SO(1,9), \forall \mu, \nu = 0\,to\,9) \tag{30}$$

The elements of rotation group $SO(1,9)$ in ten dimensional space-time satisfy the metric preserving condition [20]

$$\Lambda^T \eta \mathit{\Lambda} =\eta. \qquad (\forall \Lambda \in SO(1,9)) \tag{31}$$

Here the metric is defined as $\eta_{\mu\nu} = \{1, -1, -1, -1, -1, -1, -1, -1\}$ and T for the transpose of matrix. The determinant of $\mathit{\Lambda}$ comes out to be unity. From the Lie algebra theory it is well known that for each $\Lambda \in SO(1,9)$ may be define as $\Lambda(a) = exp(aR)$. Where a is a real parameter and R is an element of the Lie algebra $so(1,9)$. By putting this on equation (31) and differentiating w.r.t.$'a'$. We get the following condition for R (the element of Lie algebra $so(8)$ or the generator of Lie group $SO(8)$) as

$$R^T = -\eta R \eta \qquad (\forall R \in so(8)) \tag{32}$$

where R is to be taken as

$$R = [r_{ab}]_{a,b=0}^{9} \,. \tag{33}$$

Substituting this to (32) we get $r_{mm} = 0$, $r_{mn} = -r_{nm}$, $r_{om} = r_{m0}$ $(\forall m, n = 1\, to\, 9)$, and corresponding generators for $SO(1,9)$ group may be written accordingly. The determinant of $\varLambda$ turns out to be unity. Thus we may easily define the rotation and Lorentz boost generators of $SO(1,9)$ group respectively denoted by L_{mn} and N_{0m}. For $SO(1,9)$ group there exists 35 generators of rotation associated with L_{mn} matrices followed by 10 generators corresponding to the Lorentz boost matrices N_{0m}. Both L_{mn} and N_{0m} are traceless matrices. The five Lorentz boost generators N_{0m} are symmetric, while the other ten rotation generators L_{mn} are antisymmetric. Matrices L_{mn} and N_{0m} are discussed in the appendix$-I$. In physics generally it is prefered to write the group elements as $\varLambda(a) = exp(-i\omega^{mn} M_{mn})\, (\forall m, n = 0\, to\, 9)\, \omega^{mn}\, is\, antisymmetric\, parameter)$. where $M'_{mn}s$ are the generators of group transformations with the extra $-i$ in the exponent, because the group transformation must act as a unitary operator and so the generators to be act as Hermitian operators. Therefore for Hermitian generators we may define matrices $M_{\mu\nu}$ $(\forall \mu, \nu = 0\, to\, 9)$ corresponding to the L_{mn} and N_{0m}, whose matrix elements are defined by the mappings $(M)_{\mu\nu} = i\eta_{\mu\rho}(L)_{\nu}^{\rho}$ and $(M)_{\sigma\gamma} = i\eta_{\sigma\delta}(N)_{\gamma}^{\delta}$ $(\forall \mu, \nu, \rho, \sigma, \delta, \gamma = 0\, to\, 9)$. So the algebra of quaternion Lorentz group $SO(1,9)$ describes the following structure

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma}) \tag{34}$$

where the metric for $SO(1,9)$ group is defined as $\eta_{\mu\nu} = (1, -1, -1, -1, -1, -1, -1, -1, -1, -1)$. The Poincaré group transformation induces the changes in vector x^{μ}as

$$x'^{\mu} = \varLambda_{\nu}^{\mu} x^{\nu} + a^{\mu} \tag{35}$$

leaving the space-time interval $(\triangle x)^2$ constant. The group elements follow the composition rule

$$(\Lambda_1, a_1) \,.\, (\Lambda_2, a_2) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2) \,. \tag{36}$$

We may associate with the transformation (Λ, a) a matrix as [21]

$$(\Lambda, a) \rightarrow \begin{bmatrix} \Lambda & a \\ 0 & 1 \end{bmatrix} \tag{37}$$

the Λ is an element of $SO(1,9)$ corresponding to rotation and Lorentz boost in the space $D = 10$ space time. The a corresponding to the translation vector in $D = 10$ space time. For the case of homogeneous Lorentz transformation we have the 11×11 transformation matrices as

$$G = \begin{bmatrix} \Lambda & 0 \\ 0 & 1 \end{bmatrix} \tag{38}$$

The generators of Lorentz transformation are already evaluated and described in Appendix$-I$. While the transformation matrices for the translation is defined as

$$W = \begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix} \tag{39}$$

where $'a'$ is a 10 dimensional vector defined in 10×1 matrix representation. The generators of translation defined as

$$(P_\mu)_{\rho\sigma} = -\, i\eta_{\mu\rho}\delta_{\sigma 6} \qquad (\forall \mu = 0\, to\, 5, \quad \sigma, \rho = 0\, to\, 10). \tag{40}$$

Where the generators of Lorentz transformations and of linear transformations are described as

$$S_{\mu\nu} = \begin{pmatrix} M_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_\rho = \begin{pmatrix} 0 & T_\rho \\ 0 & 0 \end{pmatrix} \tag{41}$$

Where the T_ρ are matrices in 10×1 representation as

$$T_0 = \begin{pmatrix} -i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, T_1 = \begin{pmatrix} 0 \\ i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 \\ 0 \\ i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, T_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, T_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$T_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, T_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i \\ 0 \\ 0 \\ 0 \end{pmatrix}, T_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i \\ 0 \\ 0 \end{pmatrix}, T_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i \\ 0 \end{pmatrix}, T_9 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i \end{pmatrix} \tag{42}$$

It may be easily find that $S_{\mu\nu}$ and P_ρ satisfy the relations

$$\begin{aligned} [S_{\mu\nu}, S_{\rho\sigma}] &= -\, i(\eta_{\mu\rho}S_{\nu\sigma} - \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\nu\rho}S_{\mu\sigma} + \eta_{\nu\sigma}S_{\mu\rho}) \\ [S_{\mu\nu}, P_\rho] &= -\, i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\ [P_\mu, P_\nu] &= 0 \end{aligned} \tag{43}$$

4 Octonion realization of $SO(1,9)$ and Poincaré algebra in $D = 10$:

The Space-time vector x^μ in $D = 10$ may be written as

$$x^\mu = (x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9) = (x^0, \vec{x}) \tag{44}$$

It is well known that there exist homomorphism [8] between the proper Lorentz group in critical dimensions $3, 4, 6, 10$ and group involving matrices having elements of division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathcal{O}$ such as

$$\begin{aligned} \overline{SO}(1,2) &\cong Sl(2;\mathbb{R}) \\ \overline{SO}(1,3) &\cong Sl(2;\mathbb{C}) \\ \overline{SO}(1,5) &\cong Sl(2;\mathbb{H}) \\ \overline{SO}(1,9) &\cong Sl(2;\mathcal{O}) \end{aligned} \tag{45}$$

Hence a ten dimensional vector x^{μ}can be made corresponding to a octonionic 2×2 matrices as follows

$$X = x^{\mu}\Gamma_{\mu} = \left(\begin{array}{cc} x^0 + x^9 & x^8 + e_1x^1 + e_2x^2 + e_3x^3 + e_4x^4 + e_5x^5 + e_6x^6 + e_7x^7 \\ x^8 - e_1x^1 - e_2x^2 - e_3x^3 - e_4x^4 - e_5x^5 - e_6x^6 - e_7x^7 & x^0 - x^9 \end{array}\right) \tag{46}$$

where may take octonion 2×2 Γ_{μ} matrices such as

$$\Gamma_0 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right),\ \Gamma_i = \left(\begin{array}{cc} 0 & e_i \\ -e_i & 0 \end{array}\right),\ \Gamma_8 = \left(\begin{array}{cc} 0 & e_0 \\ e_0 & 0 \end{array}\right),\ \Gamma_9 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right),\quad (\forall i = 1\, to\, 7). \tag{47}$$

These Γ_{μ} matrices satisfy the following relation

$$\Gamma_{\mu}\overline{\Gamma}_{\nu} + \Gamma_{\nu}\overline{\Gamma}_{\mu} = 2\eta_{\mu\nu} \tag{48}$$

where the metric structure taken as $\eta_{\mu\nu} = (1, -1, -1, -1, -1, -1, -1, -1, -1, -1)$. And $\overline{\Gamma}_{\mu} = (\Gamma_0, -\Gamma_i)$ $(\forall i = 1\, to\, 9)$. The determinent of the (34) which is already unambiguously defined for hermitian 2×2 matrices leads to

$$det(X) = x_{\mu}x^{\mu} = \eta_{\mu\nu}x^{\nu}x^{\mu} \tag{49}$$

For finite Lorentz transformation the

$$X' = YXY^{\dagger} \qquad with\ Y \in Sl(2;\mathcal{O}) \tag{50}$$

is ambiguous for octonions due to the non-associativity of octonions but it's infinitesimal version is still valid [18] for $Y = I + \in M$ with $trM = 0$. The 4×4 representation of Dirac represenatation of Υ_{μ}matrices are defined as

$$\Upsilon_0 = \left(\begin{array}{cc} I_2 & 0 \\ 0 & -I_2 \end{array}\right),\ \Upsilon_j = \left(\begin{array}{cc} 0 & \Gamma_j \\ -\Gamma_j & 0 \end{array}\right) \quad (\forall j = 1\, to\, 9). \tag{51}$$

The 4×4 representation of Weyl represenatation of γ_{μ}matrices in $D = 10$ space defined as

$$\gamma_0 = \left(\begin{array}{cc} 0 & I_2 \\ I_2 & 0 \end{array}\right),\ \gamma_l = \Gamma_l \otimes \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \quad (\forall l = 1\, to\, 9). \tag{52}$$

Lorentz generators are defined in $D = 10$ dimensional space as

$$\Sigma^{(4)}_{\mu\nu} = \frac{i}{4}[\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}] \qquad (\forall \mu, \nu = 0\, to\, 9). \tag{53}$$

The number of generators in $SO(1,9)$ is 45. There are 31 independent generators come from $\Sigma^{(4)}_{\mu\nu}$. Since there are three degenerate plains of rotation which generate rotation about the same axis. The other 14 come again from the derivative algebra G_2 of octonion. Here we define again the generators $\Xi^{(4)}_{\mu\nu}$ defined as

$$\Xi^{(4)}_{\mu\nu} =\frac{i}{2}D_{\mu\nu}\begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \qquad (\forall \mu, \nu = 0\,to\,7). \tag{54}$$

We have thus the 4×4 matrices representation for the Lie algebra of $SO(1,9)$ as

$$\begin{aligned}
[\Sigma^{(4)}_{\alpha\beta}, \Sigma^{(4)}_{lm}] =& -i(\eta_{\alpha l}\Sigma^{(4)}_{\beta m} - \eta_{\alpha m}\Sigma^{(4)}_{\beta l} + \eta_{\beta m}\Sigma^{(4)}_{\alpha l} - \eta_{\beta l}\Sigma^{(4)}_{\alpha m}) + K^{(4)}_{\alpha\beta lm} \qquad (\forall \alpha, \beta, m, l = 1\,to\,7)\\
[\Sigma^{(4)}_{8\alpha}, \Sigma^{(4)}_{\beta\gamma}] =& -i(\eta_{\alpha\gamma}\Sigma^{(4)}_{8\beta} - \eta_{\alpha\beta}\Sigma^{(4)}_{8\gamma}) + X^{(4)}_{\beta\gamma\alpha\delta} \qquad (\forall \alpha, \beta, \gamma, \delta = 1\,to\,7)\\
[\Sigma^{(4)}_{8\alpha}, \Sigma^{(4)}_{8\beta}] =& -i\eta_{88}\Sigma^{(4)}_{\alpha\beta} \qquad (\forall \alpha, \beta = 1\,to\,7)\\
[\Xi^{(4)}_{\mu\nu}, \Sigma^{(4)}_{\alpha\beta}] =& -\frac{2i}{3}[\eta_{\mu\beta}\Sigma^{(4)}_{\nu\alpha} - \eta_{\nu\beta}\Sigma^{(4)}_{\mu\alpha} - \eta_{\mu\alpha}\Sigma^{(4)}_{\nu\beta} + \eta_{\nu\alpha}\Sigma^{(4)}_{\mu\beta}] + Y^{(4)}_{\alpha\mu\nu\beta} \; (\forall \alpha, \mu, \nu, \beta = 1\,to\,7)\\
[\Xi^{(4)}_{\mu\nu}, \Sigma^{(4)}_{8\alpha}] =& -\frac{i}{3}[\eta_{8\nu}\Sigma^{(4)}_{\mu\alpha} - \eta_{8\mu}\Sigma^{(4)}_{\nu\alpha}] + Z^{(4)}_{\mu\nu\alpha r} \qquad (\forall \mu, \nu, \alpha, r = 1\,to\,7)\\
[\Sigma^{(4)}_{9\alpha}, \Sigma^{(4)}_{\beta\gamma}] =& -i[\eta_{\gamma\alpha}\Sigma^{(4)}_{9\beta} - \eta_{\beta\alpha}\Sigma^{(4)}_{9\gamma}] - T^{(4)}_{\beta\gamma\alpha\sigma} \qquad (\forall \alpha\beta\gamma\sigma = 0\,to\,7)\\
[\Sigma^{(4)}_{0\alpha}, \Sigma^{(4)}_{\beta\gamma}] =& -i[\eta_{\gamma\alpha}\Sigma^{(4)}_{0\beta} - \eta_{\beta\alpha}\Sigma^{(4)}_{0\gamma}] - S^{(4)}_{\beta\gamma\alpha\sigma} \qquad (\forall \alpha\beta\gamma\sigma = 0\,to\,7)\\
[\Xi^{(4)}_{\mu\nu}, \Sigma^{(4)}_{0\alpha}] =& -i[\eta_{0\mu}\Sigma^{(4)}_{\nu\alpha} - \eta_{0\nu}\Sigma^{(4)}_{\mu\alpha}] + R^{(4)}_{\mu\nu\alpha\rho} \qquad (\forall \alpha = 0\,to\,9)\\
[\Sigma^{(4)}_{9\alpha}, \Sigma^{(4)}_{9\beta}] =& -i\eta_{99}\Sigma^{(4)}_{\alpha\beta} \qquad (\forall \alpha, \beta = 1\,to\,7)
\end{aligned} \tag{55}$$

For the extension of octonion realization for $SO(1,9)$ to the Poincaré algebra in $D = 10$ space time by introducing the generators of translation and Lorentz group as

$$\begin{aligned}
P_\mu =& \frac{i}{2}\begin{pmatrix} 0 & 0 \\ \gamma_\mu & 0 \end{pmatrix} \qquad (\forall \mu = 0\,to\,9)\\
\Sigma^{(8)}_{\mu\nu} =& \begin{pmatrix} \Sigma^{(4)}_{\mu\nu} & 0 \\ 0 & \Sigma^{(4)}_{\mu\nu} \end{pmatrix} \qquad (\forall \mu, \nu = 0\,to\,9)\\
\Xi^{(8)}_{\mu\nu} =& \begin{pmatrix} \Xi^{(4)}_{\mu\nu} & 0 \\ 0 & \Xi^{(8)}_{\mu\nu} \end{pmatrix} \qquad (\forall \mu, \nu = 1\,to\,7)
\end{aligned} \tag{56}$$

Now the octonion realization of Poincaré algebra in $D = 10$ may be defined as (including the Lie algebra of equation (55)

$$\begin{aligned}
[P_\mu, P_\nu] =& 0\\
[\Sigma^{(8)}_{0n}, P_j] =& -i(\eta_{0j}P_n - \eta_{nj}P_0)\\
[\Sigma^{(8)}_{8k}, P_j] =& -i(\eta_{8j}P_k - \eta_{kj}P_8)\\
[\Sigma^{(8)}_{9k}, P_j] =& -i(\eta_{9j}P_k - \eta_{kj}P_9)\\
[\Sigma^{(8)}_{\alpha\beta}, P_\sigma] =& -i(\eta_{\alpha\sigma}P_\beta - \eta_{\beta\sigma}P_\alpha) + K''_{\alpha\beta\sigma\gamma}\\
[\Xi^{(8)}_{\mu\nu}, P_\sigma] =& -\frac{i}{6}(\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu) - K_{\mu\nu\sigma\rho}
\end{aligned} \tag{57}$$

5 Conformal algebra of $SO(1,9)$:

A conformal transformation of the coordinates is a mapping which leaves invariant the metric $\eta_{\mu\nu}$ up to a scale[22]

$$\eta'_{\mu\nu}(x') = \Lambda(x)\eta_{\mu\nu}. \tag{58}$$

The set of all conformal transformations form a group which has Poincaré group as subgroup corresponds to $\Lambda(x) = 1$. The conformal group of $SO(1,9)$ have the following generators: 28 Generators of rotation $SO(8)$as

$$\begin{aligned}
\Sigma_{\alpha\beta} =& -\frac{i}{2} f_{\alpha\beta\gamma} \begin{pmatrix} e_\gamma & 0 \\ 0 & e_\gamma \end{pmatrix} \quad (\forall \alpha, \beta, \gamma = 1\, to\, 7) \\
\Sigma_{0\alpha} =& \frac{i}{2} \begin{pmatrix} -e_\alpha & 0 \\ 0 & e_\alpha \end{pmatrix} \quad (\forall \alpha = 1\, to\, 7) \\
\Xi_{\mu\nu} =& -\frac{i}{2} \begin{pmatrix} D_{\mu\nu} & 0 \\ 0 & D_{\mu\nu} \end{pmatrix} \quad (\forall \mu, \nu \in 1\, to\, 7).
\end{aligned} \tag{59}$$

Eight generators of translation may be defined as

$$P_\mu = \frac{i}{2} \begin{pmatrix} 0 & e_\mu \\ 0 & 0 \end{pmatrix} \quad (\forall \mu = 0\, to\, 7). \tag{60}$$

Eight generators of conformal accelerations may be defined as

$$K_0 = \frac{i}{2} \begin{pmatrix} 0 & 0 \\ -e_0 & 0 \end{pmatrix}, K_\mu = \begin{pmatrix} 0 & 0 \\ e_\mu & 0 \end{pmatrix} \quad (\forall \mu = 1\, to\, 7). \tag{61}$$

One generator of dilations may be defined as

$$D = \frac{i}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{62}$$

So the total number of generators of conformal algebra in $D = 10$ space is equal to $28+8+8+1 = 45$. The commutation relations in conformal group in $D = 10$ space have the following form:

The commutator relations of P_μ with the seven generators $\Sigma_{8\alpha}$ $(\forall \alpha = 1\, to\, 7)$ evaluated as

$$\begin{aligned}
[\Sigma_{8\alpha}, P_\mu] =& \left[\frac{i}{2} \begin{pmatrix} -e_\alpha & 0 \\ 0 & e_\alpha \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 0 & e_\mu \\ 0 & 0 \end{pmatrix} \right] \\
=& -\frac{1}{4} \begin{pmatrix} 0 & -e_\alpha e_\mu - e_\mu e_\alpha \\ 0 & 0 \end{pmatrix} \\
=& -\frac{1}{2} \begin{pmatrix} 0 & \delta_{\alpha\mu} e_0 \\ 0 & 0 \end{pmatrix} = i\delta_{\alpha\mu} P_8.
\end{aligned} \tag{63}$$

Similarly the commutator relations of P_μ with the generators $\Sigma_{\alpha\beta}$ $(\forall \alpha, \beta = 1\, to\, 7)$ evaluated as

$$\begin{aligned}
[\Sigma_{\alpha\beta}, P_\mu] =& \left[-\frac{i}{2} f_{\alpha\beta\gamma} \begin{pmatrix} e_\gamma & 0 \\ 0 & e_\gamma \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 0 & e_\mu \\ 0 & 0 \end{pmatrix} \right] \\
=& i\delta_{\beta\mu} P_\alpha - i\delta_{\alpha\mu} P_\beta + K'_{\alpha\beta\mu p}
\end{aligned} \tag{64}$$

with $K'_{\alpha\beta\mu p}$has the value

$$K'_{\alpha\beta\mu p} = \frac{1}{2} g_{\alpha\beta\mu p} \begin{pmatrix} 0 & e_p \\ 0 & 0 \end{pmatrix}. \tag{65}$$

Also the commutator relations of P_μ with the generators of Lorentz group comes from the derivative algebra of octonion (G_2) evaluated as

$$\begin{aligned}[\Xi_{\mu\nu}, P_\sigma] &= \left[-\frac{i}{2} \begin{pmatrix} D_{\mu\nu} & 0 \\ 0 & D_{\mu\nu} \end{pmatrix}, \frac{i}{2} \begin{pmatrix} 0 & e_\sigma \\ 0 & 0 \end{pmatrix} \right] \\ &= \frac{1}{4} \begin{pmatrix} 0 & 0 \\ [D_{\mu\nu}, e_\sigma] & 0 \end{pmatrix} \\ &= \frac{i}{3}\delta_{\nu\sigma} P_\mu - \frac{i}{3}\delta_{\mu\sigma} P_\nu + \frac{8}{3} K'_{\mu\nu\sigma r}. \end{aligned} \tag{66}$$

The octonionic conformal algebra in $SO(1,9)$ is defined a

$$\begin{aligned}
[\Sigma_{\alpha\beta}, \Sigma_{lm}] &= -i(\eta_{\alpha l}\Sigma_{\beta m} - \eta_{\alpha m}\Sigma_{\beta l} + \eta_{\beta m}\Sigma_{\alpha l} - \eta_{\beta l}\Sigma_{\alpha m}) && (\forall \alpha, \beta, m, l = 1 \text{ to } 7) \\
[\Sigma_{8\alpha}, \Sigma_{\beta\gamma}] &= -i(\eta_{\alpha\gamma}\Sigma_{8\beta} - \eta_{\alpha\beta}\Sigma_{8\gamma}) && (\forall \alpha, \beta, \gamma, \delta = 1 \text{ to } 7) \\
[\Sigma_{8\alpha}, \Sigma_{0\beta}] &= -i\eta_{88}\Sigma_{\alpha\beta} && (\forall \alpha, \beta = 1 \text{ to } 7) \\
[\Xi_{\mu\nu}, \Sigma_{\alpha\beta}] &= -\frac{4i}{3}[\eta_{\nu\beta}\Sigma_{\mu\alpha} - \eta_{\mu\beta}\Sigma_{\nu\alpha} + \eta_{\mu\alpha}\Sigma_{\nu\beta} - \eta_{\nu\alpha}\Sigma_{\mu\beta}] && (\forall \alpha, \mu, \nu, \beta = 1 \text{ to } 7) \\
[\Xi_{\mu\nu}, \Xi_{\alpha\beta}] &= -\frac{4i}{3}(\eta_{\mu\alpha}\Sigma_{\nu\beta} - \eta_{\mu\beta}\Sigma_{\nu\alpha} + \eta_{\nu\beta}\Sigma_{\mu\alpha} - \eta_{\nu\alpha}\Sigma_{\mu\beta}) && (\forall \mu, \nu, \alpha, \beta = 1 \text{ to } 7)
\end{aligned}$$

$$\begin{aligned}
[\Xi_{\mu\nu}, \Sigma_{8\alpha}] &= -\frac{2i}{3}[\eta_{\mu 8}\Sigma_{\nu\alpha} - \eta_{\nu 8}\Sigma_{\mu\alpha}] && (\forall \mu, \nu, \alpha, r = 1 \text{ to } 7) \\
[\Sigma_{8\alpha}, P_\mu] &= i\eta_{\alpha\mu}P_8 && (\forall \mu, \alpha = 0 \text{ to } 7) \\
[\Sigma_{\alpha\beta}, P_\mu] &= i(\eta_{\beta\mu}P_\alpha - \eta_{\mu\alpha}P_\beta) && (\forall \alpha, \beta, \mu, p = 0 \text{ to } 7) \\
[\Xi_{\mu\nu}, P_\sigma] &= \frac{i}{3}(\eta_{\mu\sigma}P_\nu - \eta_{\sigma\nu}P_\mu) && (\forall \sigma, \mu, \nu, r = 0 \text{ to } 7) \\
[\Sigma_{8\alpha}, K_\mu] &= i\eta_{\alpha\mu}K_8 && (\forall \mu, \alpha = 0 \text{ to } 7) \\
[\Sigma_{\alpha\beta}, K_\mu] &= i(\eta_{\beta\mu}K_\alpha - \delta_{\alpha\mu}K_\beta) && (\forall \alpha, \beta, \mu, p = 0 \text{ to } 7) \\
[\Xi_{\mu\nu}, K_\sigma] &= \frac{i}{3}(\eta_{\nu\sigma}K_\mu - \eta_{\mu\sigma}K_\nu) && (\forall \sigma, \mu, \nu, r = 0 \text{ to } 7) \\
[P_\mu, P_\nu] &= 0 && (\forall \mu, \nu = 0 \text{ to } 7) \\
[K_\mu, K_\nu] &= 0 && (\forall \mu, \nu = 0 \text{ to } 7) \\
[K_\mu, P_\nu] &= i\eta_{\mu\nu}D - \frac{i}{2}\Sigma_{\mu\nu} && (\forall \mu, \nu = 0 \text{ to } 7) \\
[D, P_\mu] &= \frac{i}{2}P_\mu && (\forall \mu = 0 \text{ to } 7) \\
[D, K_\mu] &= -\frac{i}{2}K_\mu && (\forall \mu = 0 \text{ to } 7) \\
[\Sigma_{\mu\nu}, D] &= 0 && (\forall \mu, \nu = 0 \text{ to } 7) \\
[\Xi_{\mu\nu}, D] &= 0 && (\forall \mu, \nu = 0 \text{ to } 7)
\end{aligned} \tag{67}$$

6 Supersymmetrization of Conformal Group in $SO(1,9)$:

The $N=1$ supersymmetrization of conformal group in $D=10$ dimensional space (conformal algebra of $SO(1,9)$) may be done by the introduction of the following representation of conformal group of superchages as

$$Q_1(e_\mu)=\begin{pmatrix} 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & e_\mu \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & 0 \end{pmatrix}, Q_2(e_\mu)=\begin{pmatrix} 0 & 0 & \vdots & e_\mu \\ 0 & 0 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & 0 \end{pmatrix}$$

$$S_1(e_\nu)=\begin{pmatrix} 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & e_\nu & \vdots & 0 \end{pmatrix}, S_2(e_\nu)=\begin{pmatrix} 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ e_\nu & 0 & \vdots & 0 \end{pmatrix} \tag{68}$$

The anti commutation relation for the octonionic supercharges may be obtained as

$$\begin{aligned}
\{Q_1(e_\mu),S_2(e_\nu)\} =& \frac{i}{2}(-\delta_{\mu\nu}K_0+f_{\mu\nu\sigma}K_\sigma)\\
\{Q_2(e_\mu),S_1(e_\nu)\} =& \frac{i}{2}(-\delta_{\mu\nu}P_0+f_{\mu\nu\sigma}P_\sigma)\\
\{Q_1(e_\mu),Q_2(e_\nu)\} =& \{S_1(e_\mu),S_2(e_\nu)\}=0\\
\{Q_1(e_\mu),S_1(e_\nu)\} =& \frac{i}{4}T(e_\nu e_\mu)-\frac{i}{4}(e_\nu e_\mu)A+\frac{i}{2}\Sigma_{\mu\nu}+(e_\mu e_\nu)D\\
\{Q_2(e_\mu),S_2(e_\nu)\} =& \frac{i}{4}T(e_\nu e_\mu)-\frac{i}{4}(e_\nu e_\mu)A+\frac{i}{2}\Sigma_{\mu\nu}-(e_\mu e_\nu)D
\end{aligned} \tag{69}$$

where T is the generator of internal symmetry and A is the non-compact chiral generator such as

$$T(e_\mu)=\begin{pmatrix} e_\mu & 0 & 0 \\ 0 & e_\mu & 0 \\ 0 & 0 & e_\mu \end{pmatrix}, A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{70}$$

The covariance relations of the superalgebra are

$$\begin{aligned}
[\Sigma_{0\alpha}, Q_\beta(e_\mu)] =& iQ_\beta(-e_\alpha e_\mu) && (\forall \mu, \alpha = 0\, to\, 7)\\
[\Sigma_{\alpha\beta}, Q_\sigma(e_\mu),] =& iQ_\sigma(f_{\alpha\beta\gamma} e_\gamma e_\mu) && (\forall \alpha, \beta, \mu, \gamma = 0\, to\, 7)\\
[\Xi_{\mu\nu}, Q_\sigma(e_\mu)] =& iQ_\sigma(D_{\mu\nu}(e_\mu)) && (\forall \sigma, \mu, \nu = 0\, to\, 7)\\
[P_\mu, Q_1(e_\nu)] =& iQ_2(e_\mu e_\nu) && (\forall \mu, \nu = 0\, to\, 7)\\
[P_\mu, Q_2(e_\nu)] =& 0 && (\forall \mu, \nu = 0\, to\, 7)\\
[P_\mu, S_1(e_\nu)] =& 0 && (\forall \mu, \nu = 0\, to\, 7)\\
[P_\mu, S_2(e_\nu)] =& -iS_1(e_\nu e_\mu) && (\forall \mu, \nu = 0\, to\, 7)\\
[\Sigma_{0\alpha}, S_\beta(e_\mu)] =& iS_\beta(-e_\alpha e_\mu) && (\forall \mu, \alpha = 0\, to\, 7)\\
[\Sigma_{\alpha\beta}, S_\sigma(e_\mu),] =& iS_\sigma(f_{\alpha\beta\gamma} e_\gamma e_\mu) && (\forall \alpha, \beta, \mu, \gamma = 0\, to\, 7)\\
[\Xi_{\mu\nu}, S_\sigma(e_\mu)] =& iS_\sigma(D_{\mu\nu}(e_\mu)) && (\forall \sigma, \mu, \nu = 0\, to\, 7)\\
[K_\mu, Q_1(e_\nu)] =& 0 && (\forall \mu, \nu = 0\, to\, 7)\\
[K_\mu, Q_2(e_\nu)] =& iQ_1(e_\mu e_\nu) && (\forall \mu, \nu = 0\, to\, 7)\\
[K_\mu, S_1(e_\nu)] =& -iS_2(e_\nu e_\mu) && (\forall \mu, \nu = 0\, to\, 7)\\
[K_\mu, S_2(e_\nu)] =& 0 && (\forall \mu, \nu = 0\, to\, 7)
\end{aligned} \tag{71}$$

7 Conformal group in D=10 space-time:

A conformal transformation of the coordinates is a mapping which leaves invariant the metric $\eta_{\mu\nu}$ up to a scale

$$\eta'_{\mu\nu}(x') = \Lambda(x)\eta_{\mu\nu}. \tag{72}$$

The set of all conformal transformations form a group which has Poincaré group (discuss in previuos section) as subgroup corresponds to $\Lambda(x) = 1$. The conformal group of $D = 10$ space is isomorphic to the group $SO(1, 11)$ have the following set of generators:

45 Generators of rotation $SO(1, 9)$ as

$$\begin{aligned}
\Sigma^{(8)}_{\mu\nu} =& \begin{pmatrix} \Sigma^{(4)}_{\mu\nu} & 0 \\ 0 & \Sigma^{(4)}_{\mu\nu} \end{pmatrix} && (\forall \mu, \nu = 0\, to\, 9)\\
\Xi^{(8)}_{\mu\nu} =& \begin{pmatrix} \Xi^{(4)}_{\mu\nu} & 0 \\ 0 & \Xi^{(8)}_{\mu\nu} \end{pmatrix} && (\forall \mu, \nu = 1\, to\, 7)
\end{aligned} \tag{73}$$

10 generators of translation dfeined as

$$P_\mu = \frac{i}{2} \begin{pmatrix} 0 & 0 \\ \gamma_\mu & 0 \end{pmatrix} \qquad (\forall \mu = 0\, to\, 9). \tag{74}$$

10 generators of conformal accelerations may be defined as

$$K_\mu = \frac{i}{2} \begin{pmatrix} 0 & \gamma_\mu \\ 0 & 0 \end{pmatrix} \qquad (\forall \mu = 0\, to\, 9). \tag{75}$$

One generator of dilations may be defined as

$$D = \frac{1}{4}\begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}. \tag{76}$$

The conformal algebra in $D = 10$ space-time has the following commutation relations

$$\begin{aligned}
[K_\mu, K_\nu] =& 0 \\
[\Sigma^{(8)}_{0n}, K_j] =& -i(\eta_{0j}K_n - \eta_{nj}K_0) \\
[\Sigma^{(8)}_{8k}, K_j] =& -i(\eta_{8j}K_k - \eta_{kj}K_8) \\
[\Sigma^{(8)}_{9k}, K_j] =& -i(\eta_{9j}K_k - \eta_{kj}K_9) \\
[\Sigma^{(8)}_{\alpha\beta}, K_\sigma] =& -i(\eta_{\alpha\sigma}K_\beta - \eta_{\beta\sigma}K_\alpha) + W''_{\alpha\beta\sigma\gamma} \\
[\Xi^{(8)}_{\mu\nu}, K_\sigma] =& -\frac{i}{6}(\eta_{\mu\sigma}K_\nu - \eta_{\nu\sigma}K_\mu) - W_{\mu\nu\sigma\rho} \\
[P^{(8)}_\mu, K^{(8)}_\nu] =& -2\eta_{\mu\nu}D + \Sigma^{(8)}_{\mu\nu} \\
[P^{(8)}_\mu, D] =& \frac{1}{4}P_\mu \\
[K^{(8)}_\mu, D] =& -\frac{1}{4}K_\mu \\
[\Sigma^{(8)}_{\alpha\beta}, D] =& 0 \\
[\Xi^{(8)}_{\mu\nu}, D] =& 0
\end{aligned} \tag{77}$$

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