Solving Inconsistent Systems of Linear Equations (Part I)

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Abstract

Using Geometric Algebra consistent solutions of inconsistent systems of linear equations can be found.

Keywords

Geometric Algebra, Clifford Algebra, Linear Algebra, Systems of Linear Equations

Contents

1 Basic philosophy of this paper

Usually the mathematics of linear algebra is used to model the physics of quantum mechanics. In this paper all this will be reversed: Some ideas of quantum mechanics will be used to model linear algebra and the mathematics of consistent solutions of inconsistent linear equations.

To describe these mathematical structures of inconsistent linear equations, Geometric Algebra will be applied. More than 150 years ago Hermann Grassmann showed in his miraculous and tremendously important book about the theory of extensions [1] how to solve systems of linear equations with Geometric Algebra. He introduced his ideas with the words, that: "… the applicability of outer multiplication emerges with such a striking determination and firmness, that … algebra will gain a substantial different shape" [1, § 45, p. 71][1]. This modern algebra of different shape even allows to consistently solve systems of inconsistent linear equations.

The foundations of Geometric Algebra (or Clifford Algebra if you like) are explained in many well written introductory books [2], [3], [4], [5] or papers [6], [7], [8], [9]. All this is really "simple. The only rules to remember is that different orthogonal generating units (vectors) anticommute and that their square is +1,−1 (or 0)" [10, p. 823]. Thus vectors are represented by Pauli matrices in three-dimensional Euclidean space or by generalized Pauli or Dirac matrices in higher-dimensional spaces or spacetimes.

2 Simple examples of systems of consistent linear equations

Let us have a short look on some very simple examples first: Systems of two linear equations with one unknown variable only.

The first system of two linear equations will be the following consistent system:

$$\begin{matrix} x = 1 \\ x = 1 \end{matrix} \qquad \text{or} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{or} \qquad (\sigma_x + \sigma_y)\, x = \sigma_x + \sigma_y$$

The version on the left simply states two given linear equations. The first equation $x = 1$ describes a point in a one dimensional world (thus sitting on an axis) at the position "1". The second equation $x = 1$ describes another point in this one dimensional world at the same position "1" of the x-axis.

The version in the middle shows the conventional column vector representation, which is a restricted and limited way of writing a system of linear equations. Only vectors but no higher-dimensional bivectors, trivectors, etc. can be represented when column vectors are used.

The version on the right shows the modern Geometric Algebra version. The two linear equations are fused together into one equation. The σ_x-components describe the first linear equation $\sigma_x\, x = \sigma_x$ and the σ_y-components describe the second linear equation $\sigma_y\, x = \sigma_y$.

How can we find now the solution of this system of linear equations? Following the ideas of Grassmann and Clifford, we simply multiply by the coefficient vector $\mathbf{a} = \sigma_x + \sigma_y$ and divide by the square of this coefficient vector $\mathbf{a}^2 = 2$:

$$\begin{aligned} (\sigma_x + \sigma_y)\,(\sigma_x + \sigma_y)\, x &= (\sigma_x + \sigma_y)\,(\sigma_x + \sigma_y) \\ 2\, x &= 2 \\ x &= 1 \end{aligned}$$

[1] Original German quotation: "…dass die Anwendbarkeit der äusseren Multiplikation mit einer so schlagenden Entschiedenheit heraustritt, dass … durch diese Anwendung auch die Algebra eine wesentlich veränderte Gestalt gewinnen (werde)" [1, § 45, p. 71].

The quantum mechanical interpretation of this result is interesting: The first equation x = 1 tells us that there is a ½ = 50 % chance of finding the position x = 1, if we identify the probability of this first linear equation to be measured proportional to the square of the σ_x-component:

$$p_{(1)} = \frac{\sigma_x{}^2}{\mathbf{a}^2} = \frac{\sigma_x{}^2}{(\sigma_x + \sigma_y)^2} = \frac{1}{2} = 50\ \%$$

And the second equation x = 1 tells us again that there is a ½ = 50 % chance of finding the position x = 1, if we identify the probability of this second linear equation to be measured with the square of the σ_y-component:

$$p'_{(1)} = \frac{\sigma_y{}^2}{\mathbf{a}^2} = \frac{\sigma_y{}^2}{(\sigma_x + \sigma_y)^2} = \frac{1}{2} = 50\ \%$$

Therefore we will find an average position of

$$\bar{x} = 0.5 \cdot 1 + 0.5 \cdot 1 = 1$$

which is identical to the positions of the points of first and second linear equations.

The second system of two linear equations will be the following consistent system:

$$\begin{aligned} x &= 1 \\ 2\,x &= 2 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{or} \qquad (\sigma_x + 2\,\sigma_y)\,x = \sigma_x + 2\,\sigma_y$$

The first equation x = 1 describes a point in a one dimensional world (thus sitting on an axis) at the position 1/1 = 1. The second equation 2 x = 2 describes another point in this one dimensional world at the same position 2/2 = 1 of the x-axis.

To find the solution of this second system of linear equations, we follow again the ideas of Grassmann and Clifford and simply multiply by the coefficient vector $\mathbf{a} = \sigma_x + 2\,\sigma_y$ and divide by the square of this coefficient vector $\mathbf{a}^2 = 5$:

$$\begin{gathered} (\sigma_x + 2\,\sigma_y)\,(\sigma_x + 2\,\sigma_y)\,x = (\sigma_x + 2\,\sigma_y)\,(\sigma_x + 2\,\sigma_y) \\ 5\,x = 5 \\ x = 1 \end{gathered}$$

Again a quantum mechanical interpretation of the result can be found: The first equation x = 1 tells us that there is a $^1/_5$ = 20 % chance of finding the position x = 1, if we identify the probability of this first linear equation to be measured again proportional to the square of the σ_x-component:

$$p_{(1)} = \frac{\sigma_x{}^2}{\mathbf{a}^2} = \frac{\sigma_x{}^2}{(\sigma_x + 2\,\sigma_y)^2} = \frac{1}{5} = 20\ \%$$

And the second equation 2 x = 2 tells us that there also is a $^4/_5$ = 80 % chance of finding the position x = 1, if we identify the probability of this second linear equation to be measured proportional to the square of the σ_y-component:

$$p'_{(1)} = \frac{(2\,\sigma_y)^2}{\mathbf{a}^2} = \frac{(2\,\sigma_y)^2}{(\sigma_x + 2\,\sigma_y)^2} = \frac{4}{5} = 80\ \%$$

Therefore we will find an average position of

$$\bar{x} = 0.2 \cdot 1 + 0.8 \cdot 1 = 1$$

which is again identical to the positions of the points of first and second linear equations.

The third example system of two linear equations will be the following consistent system:

$$3\,x = 3 \quad\quad 4\,x = 4 \qquad \text{or} \qquad \begin{bmatrix} 3 \\ 4 \end{bmatrix} x = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \text{or} \qquad (3\,\sigma_x + 4\,\sigma_y)\,x = 3\,\sigma_x + 4\,\sigma_y$$

The first equation 3 x = 3 describes a point in a one dimensional world (thus sitting on an axis) at the position 3/3 = 1. The second equation 4 x = 4 describes another point in this one dimensional world at the same position 4/4 = 1 of the x-axis.

To find the solution of this third system of linear equations, we follow again the ideas of Grassmann and Clifford and simply multiply by the coefficient vector $\mathbf{a} = 3\,\sigma_x + 4\,\sigma_y$ and divide by the square of this coefficient vector $\mathbf{a}^2 = 25$:

$$(3\,\sigma_x + 4\,\sigma_y)\,(3\,\sigma_x + 4\,\sigma_y)\,x = (3\,\sigma_x + 4\,\sigma_y)\,(3\,\sigma_x + 4\,\sigma_y)$$
$$25\,x = 25$$
$$x = 1$$

From a quantum mechanical perspective, the first equation 3 x = 3 tells us that there is a $^{9}/_{25}$ = 36 % chance of finding the position x = 1. And the second equation 4 x = 4 tells us that there is a $^{16}/_{25}$ = 64 % chance of finding the position x = 1, if we identify the probabilities of first and second linear equations to be proportional to the squares of their coefficient vector components:

$$p_{(1)} = \frac{(3\sigma_x)^2}{\mathbf{a}^2} = \frac{(3\sigma_x)^2}{(3\sigma_x + 4\sigma_y)^2} = \frac{9}{25} = 36\ \%$$

$$p'_{(1)} = \frac{(4\sigma_y)^2}{\mathbf{a}^2} = \frac{(4\sigma_y)^2}{(3\sigma_x + 4\sigma_y)^2} = \frac{16}{25} = 64\ \%$$

Therefore we will find an average position of

$$\bar{x} = 0.36 \cdot 1 + 0.64 \cdot 1 = 1$$

which is again identical to the positions of the points of first and second linear equations. And that is all that we can do in standard quantum mechanics: We are only able to measure average values.

3 Simple examples of systems of inconsistent linear equations

Now we modify the systems of linear equations of section (2) to get some inconsistent systems of two linear equations with one unknown variable only.

The first inconsistent system of two linear equations will be the following system:

$$x = 1 \quad\quad x = 2 \qquad \text{or} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{or} \qquad (\sigma_x + \sigma_y)\,x = \sigma_x + 2\,\sigma_y$$

Obviously the first equation x = 1 describes a point in a one dimensional world (thus sitting on an axis) at the position 1/1 = 1. But the second equation x = 2 describes another point in the same one dimensional world at a different position 2/1 = 2 of the x-axis. We now have two different points at two different positions. But these points are interconnected and correlate. They should not be considered as two different objects. On contrary, these two points form one and only one object: they are a pair of points mathematically bound together.

The conceptual difference between having two different and isolated points and having one interconnected pair of points is important. In Conformal Geometric Algebra (CGA) for instance, a pair of points is described mathematically as an intersection of a circle and a plane or as an intersection of three spheres. Thus points and point pairs are of different grade, e.g. see the discussions of inner product null space (IPNS) and outer product null space (OPNS) of CGA in [11], [12], [13].

To find the solution of this first inconsistent system of linear equations, we follow again the ideas of Grassmann and Clifford and simply multiply by the coefficient vector $\mathbf{a} = \sigma_x + \sigma_y$ and divide by the square of this coefficient vector $\mathbf{a}^2 = 2$:

$$(\sigma_x + \sigma_y)(\sigma_x + \sigma_y)\, x = (\sigma_x + \sigma_y)(\sigma_x + 2\,\sigma_y)$$
$$2\,x = 3 + \sigma_x\sigma_y$$
$$x = \frac{3}{2} + \frac{1}{2}\sigma_x\sigma_y = 1.5 + 0.5\,\sigma_x\sigma_y$$

And $x = 1.5 + 0.5\,\sigma_x\sigma_y$ should be indeed called a consistent solution of the inconsistent system of linear equations $(\sigma_x + \sigma_y)\,x = \sigma_x + 2\,\sigma_y$ because this solution solves this system in a correct way, as a short check clearly shows:

$$(\sigma_x + \sigma_y)\,x = (\sigma_x + \sigma_y)(1.5 + 0.5\,\sigma_x\sigma_y) = (1.5 - 0.5)\,\sigma_x + (1.5 + 0.5)\,\sigma_y = \sigma_x + 2\,\sigma_y$$

We are faced with the hard fact, that in Geometric Algebra $x = 1.5 + 0.5\,\sigma_x\sigma_y$ is the **correct solution of the inconsistent system of linear equations** $(\sigma_x + \sigma_y)\,x = \sigma_x + 2\,\sigma_y$.

From a quantum mechanical perspective, the first equation x = 1 now tells us that there is a ½ = 50 % chance of finding the position x = 1. And the second equation x = 2 tells us that there now is a ½ = 50 % chance of finding the position x = 2, if we identify the probabilities of first and second linear equations to be proportional to the squares of their coefficient vector components:

$$p_{(1)} = \frac{\sigma_x^{\;2}}{\mathbf{a}^2} = \frac{\sigma_x^{\;2}}{(\sigma_x + \sigma_y)^2} = \frac{1}{2} = 50\ \%$$

$$p'_{(2)} = \frac{\sigma_y^{\;2}}{\mathbf{a}^2} = \frac{\sigma_y^{\;2}}{(\sigma_x + \sigma_y)^2} = \frac{1}{2} = 50\ \%$$

Therefore we will find an average position of

$$\overline{x} = 0.5 \cdot 1 + 0.5 \cdot 2 = 1.5$$

which is identical to the scalar part of the solution $x = 1.5 + 0.5\,\sigma_x\sigma_y$. And that is all that we can do in standard quantum mechanics: We are only able to measure average values.

In a similar way we can now solve the second inconsistent system of two linear equations, which will have the following structure:

$$\begin{aligned} x &= 1 \\ 2\,x &= 4 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \text{or} \qquad (\sigma_x + 2\,\sigma_y)\,x = \sigma_x + 4\,\sigma_y$$

Again the first equation x = 1 describes a point in a one dimensional world (thus sitting on an axis) at the position 1/1 = 1 and the second equation 2 x = 4 describes another point in the same one dimensional world at a different position 4/2 = 2 of the x-axis. Thus we have again two different points at two different positions, which are interconnected and correlate and should be considered as a pair of points mathematically bound together.

To find the solution of the second inconsistent system of linear equations, we follow again the ideas of Grassmann and Clifford and simply multiply by the coefficient vector $\mathbf{a} = \sigma_x + 2\,\sigma_y$ and divide by the square of this coefficient vector $\mathbf{a}^2 = 5$:

$$(\sigma_x + 2\,\sigma_y)\,(\sigma_x + 2\,\sigma_y)\,x = (\sigma_x + 2\,\sigma_y)\,(\sigma_x + 4\,\sigma_y)$$

$$5\,x = 9 + 2\,\sigma_x\sigma_y$$

$$x = \frac{9}{5} + \frac{2}{5}\,\sigma_x\sigma_y = 1.8 + 0.4\,\sigma_x\sigma_y$$

$x = 1.8 + 0.4\,\sigma_x\sigma_y$ should be indeed called a consistent solution of the inconsistent system of linear equations $(\sigma_x + 2\,\sigma_y)\,x = \sigma_x + 4\,\sigma_y$ because this solution solves this system in a correct way, as a short check clearly shows:

$$(\sigma_x + 2\,\sigma_y)\,x = (\sigma_x + 2\,\sigma_y)\,(1.8 + 0.4\,\sigma_x\sigma_y) = (1.8 - 0.8)\,\sigma_x + (3.6 + 0.4)\,\sigma_y = \sigma_x + 4\,\sigma_y$$

In Geometric Algebra $x = 1.8 + 0.4\,\sigma_x\sigma_y$ is the **correct solution of the inconsistent system of linear equations** $(\sigma_x + 2\,\sigma_y)\,x = \sigma_x + 4\,\sigma_y$.

From a quantum mechanical perspective, the first equation $x = 1$ tells us that there is a $^1/_5 = 20\,\%$ chance of finding the position $x = 1$, while the second equation $2\,x = 4$ tells us that there is a $^4/_5 = 80\,\%$ chance of finding the position $x = 2$, as we identify the probabilities of first and second linear equations to be proportional to the squares of their components again:

$$p_{(1)} = \frac{\sigma_x^{\,2}}{\mathbf{a}^2} = \frac{\sigma_x^{\,2}}{(\sigma_x + 2\,\sigma_y)^2} = \frac{1}{5} = 20\,\%$$

$$p'_{(2)} = \frac{(2\,\sigma_y)^2}{\mathbf{a}^2} = \frac{(2\,\sigma_y)^2}{(\sigma_x + 2\,\sigma_y)^2} = \frac{4}{5} = 80\,\%$$

Therefore we will find an average position of

$$\bar{x} = 0.2 \cdot 1 + 0.8 \cdot 2 = 1.8$$

which is identical to the scalar part of the solution $x = 1.8 + 0.4\,\sigma_x\sigma_y$. That is all we can do in standard quantum mechanics: We are only able to measure average values. We do not know the complete history of a physical situation. And therefore it is possible, that the average position of $\bar{x} = 1.8$ is the result of a totally different story. Instead of having had a 20 % chance of finding the position $x = 1$ and an 80 % chance of finding the position $x = 2$, we might have measured $\bar{x} = 1.8$ because there has been a 50 % chance of finding the position $x = 1.4$ and a 50 % chance of finding the position $x = 2.2$, as the following calculation shows:

$$(\sigma_x + \sigma_y)\,(\sigma_x + 2\,\sigma_y)\,(\sigma_x + 2\,\sigma_y)\,x = (\sigma_x + \sigma_y)\,(\sigma_x + 2\,\sigma_y)\,(\sigma_x + 4\,\sigma_y)$$

$$(\sigma_x + \sigma_y)\,x = (\sigma_x + \sigma_y)\,(1.8 + 0.4\,\sigma_x\sigma_y)$$

$$(\sigma_x + \sigma_y)\,x = 1.4\,\sigma_x + 2.2\,\sigma_y$$

Having now probabilities of

$$p_{(1.4)} = \frac{\sigma_x^{\,2}}{\mathbf{a}^2} = \frac{\sigma_x^{\,2}}{(\sigma_x + \sigma_y)^2} = p'_{(2.2)} = \frac{\sigma_y^{\,2}}{\mathbf{a}^2} = \frac{\sigma_y^{\,2}}{(\sigma_x + \sigma_y)^2} = \frac{1}{2} = 50\,\%$$

the average position, which can be measured, does not change:

$$\bar{x} = 0.5 \cdot 1.4 + 0.5 \cdot 2.2 = 1.8$$

Thus from a quantum mechanical perspective the two systems of linear equations

$$\begin{matrix} x = 1 \\ 2\,x = 4 \end{matrix} \qquad \text{or} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \text{or} \qquad (\sigma_x + 2\,\sigma_y)\, x = \sigma_x + 4\,\sigma_y$$

and

$$\begin{matrix} x = 1.4 \\ x = 2.2 \end{matrix} \qquad \text{or} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} 1.4 \\ 2.2 \end{bmatrix} \qquad \text{or} \qquad (\sigma_x + \sigma_y)\, x = 1.4\,\sigma_x + 2.2\,\sigma_y$$

are equivalent.

4 Making the simple examples a little bit more complicated

Do we really know that there are only two different values of x? Let us suppose that there is a third hidden position which we might measure with a disappearing chance of 0 %. The two equivalent systems of inconsistent linear equations given above might then be written as

$$\begin{matrix} x = 1 \\ 2\,x = 4 \\ 0\,x = 0 \end{matrix} \qquad \text{or} \qquad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \qquad \text{or} \qquad (\sigma_x + 2\,\sigma_y)\, x = \sigma_x + 4\,\sigma_y$$

and

$$\begin{matrix} x = 1.4 \\ x = 2.2 \\ 0\,x = 0 \end{matrix} \qquad \text{or} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x = \begin{bmatrix} 1.4 \\ 2.2 \\ 0 \end{bmatrix} \qquad \text{or} \qquad (\sigma_x + \sigma_y)\, x = 1.4\,\sigma_x + 2.2\,\sigma_y$$

Again the solution of x will be

$$x = (\sigma_x + 2\,\sigma_y)^{-1}\,(\sigma_x + 4\,\sigma_y) = (\sigma_x + \sigma_y)^{-1}\,(1.4\,\sigma_x + 2.2\,\sigma_y) = 1.8 + 0.4\,\sigma_x\sigma_y$$

Now we try to recover this hidden third position by finding a system of three inconsistent linear equations with a $^{25}/_{50}$ = 50 % chance of measuring a first value of x, a a $^{16}/_{50}$ = 32 % chance of measuring a second value of x, and a a $^{9}/_{50}$ = 18 % chance of measuring a third value of x. This inconsistent system of linear equations should look like:

$$\begin{matrix} 5\,x = \,? \\ 4\,x = \,? \\ 3\,x = \,? \end{matrix} \qquad \text{or} \qquad \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} x = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \qquad \text{or} \qquad (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\, x = \,???$$

A short calculation shows an unexpected result:

$$\begin{aligned} (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\, x &= (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,(1.8 + 0.4\,\sigma_x\sigma_y) \\ &= 7.4\,\sigma_x + 9.2\,\sigma_y + 5.4\,\sigma_z + 1.2\,\sigma_x\sigma_y\sigma_z \end{aligned}$$

Check of result:

$$\left.\begin{matrix} 5^2 + 4^2 + 3^2 = 25 + 16 + 9 = 50 \\ 1.8^2 - 0.4^2 = 3.24 + 0.16 = 3.4 \end{matrix}\right\} \qquad 50 \cdot 3.4 = 170$$

$$7.4^2 + 9.2^2 + 5.4^2 + 1.2^2 = 54.76 + 84.64 + 29.16 + 1.44 = 170$$

Thus there now is a 50 % chance of measuring the first value x = 7.4/5 = 1.48, a 32 % chance of measuring the second value x = 9.2/4 = 2.3, and an 18 % chance of measuring the third value x = 5.4/3 = 1.8. And there is a 0 % chance of measuring an unexpected forth, elsewhere value $\mathbf{x}_{tri}$ = 1.2

of grade 3. This additional elsewhere value might be considered as a hidden trivectorial something, and we might call it a field, which distorts our results. Thus by disclosing the hidden third value, we have created another hidden entity.

Nevertheless a measurement of the average position will give the old result:

$$\overline{x} = 0.5 \cdot 1.48 + 0.32 \cdot 2.3 + 0.18 \cdot 1.8 = 1.8$$

Therefore by measurements of the average positions $\overline{x}$ only, we are not able to distinguish between the three equivalent inconsistent systems of linear equations

$x = 1$	$x = 1.4$	$5\,x = 7.4$	$(\sigma_x + 2\,\sigma_y)\,x = 1\,\sigma_x + 4\,\sigma_y$
$2\,x = 4$	$x = 2.2$	$4\,x = 9.2$	or $(\sigma_x + \sigma_y)\,x = 1.4\,\sigma_x + 2.2\,\sigma_y$
$0\,x = 0$	$0\,x = 0$	$3\,x = 5.4$	$(5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,x$
without field	without field	with field	$= 7.4\,\sigma_x + 9.2\,\sigma_y + 5.4\,\sigma_z + 1.2\,\sigma_x\sigma_y\sigma_z$
$\mathbf{x}_{tri} = 0$	$\mathbf{x}_{tri} = 0$	$\mathbf{x}_{tri} = 1.2$	Solution: $x = 1.8 + 0.4\,\sigma_x\sigma_y$

and the different inconsistent system of linear equations

$$\begin{aligned} 5\,x &= 7.4 \\ 4\,x &= 9.2 \\ 3\,x &= 5.4 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} x = \begin{bmatrix} 7.4 \\ 9.2 \\ 5.4 \end{bmatrix} \quad \text{or} \quad (5\,\sigma_x + 4\,\sigma_y + 3\,\sigma_z)\,x = 7.4\,\sigma_x + 9.2\,\sigma_y + 5.4\,\sigma_z$$

without field

$\mathbf{x}_{tri} = 0$

Solution: $x = 1.8 + 0.328\,\sigma_x\sigma_y - 0.12\,\sigma_y\sigma_z - 0.096\,\sigma_z\sigma_x$

which is not equivalent to the first three inconsistent systems, because the last system has a different solution.

And we are able now to create or to annihilate fields by simple multiplications. To create a field, simply pre-multiply the solution by a coefficient vector, which is not parallel to the trivectorial field part of the solution. And to annihilate a field, simply pre-multiply the solution by a coefficient vector, which is parallel to the trivectorial field part of the solution.

And we are even able to scratch at the riddles of infinity. To transform a quantum mechanical solution into a system of linear equations which seems to describe a classical situation, just pre-multiply the solution by a coefficient vector with one component only, e.g. $\mathbf{a}_{new} = \sigma_x$:

$$\begin{aligned} \mathbf{a}_{new}\,x &= \mathbf{a}_{new}\,(1.8 + 0.4\,\sigma_x\sigma_y) \\ \sigma_x\,x &= \sigma_x\,(1.8 + 0.4\,\sigma_x\sigma_y) \\ \sigma_x\,x &= 1.8\,\sigma_x + 0.4\,\sigma_y \end{aligned} \quad \text{or} \quad \begin{aligned} x &= 1.8 \\ 0\,x &= 0.4 \end{aligned} \quad (\text{without field: } \mathbf{x}_{tri} = 0)$$

Now we have a 100 % chance of measuring $x = \overline{x} = 1.8$. But there exists a second position, which is indefinitely far away at the position $x = \lim_{h\to 0} \frac{0.4}{h} = \infty$ and which will be measured with the non-existing chance of 0 %.

4 Generalizing the examples

A system of n linear equations with one unknown variable only (which might be consistent or inconsistent) is given by the following Geometric Algebra equation:

$$\mathbf{a}\, x = \mathbf{r} \qquad \text{with} \quad \mathbf{a} = \sum_{i=1}^{n} a_i \sigma_i = a_1\, \sigma_1 + a_2\, \sigma_2 + \ldots + a_n\, \sigma_n$$

$$\mathbf{r} = \sum_{i=1}^{n} r_i \sigma_i = r_1\, \sigma_1 + r_2\, \sigma_2 + \ldots + r_n\, \sigma_n$$

a is called coefficient vector and **r** is called resulting vector of constant terms of the system of linear equations **a** x = **r**. The solution of this system then is:

$$x = \mathbf{a}^{-1}\, \mathbf{r} = \frac{1}{\mathbf{a}^2}\, \mathbf{a}\, \mathbf{r}$$

If $\mathbf{a} \wedge \mathbf{r} = 0$ and therefore $\mathbf{a}\,\mathbf{r} = \mathbf{a} \bullet \mathbf{r}$, then x will be a scalar and the system of linear equations will be consistent. This solution might be geometrically represented by a point.

If $\mathbf{a} \wedge \mathbf{r} \neq 0$ and therefore $\mathbf{a}\,\mathbf{r} = \mathbf{a} \bullet \mathbf{r} + \mathbf{a} \wedge \mathbf{r}$, then x will be a linear combination of a scalar and a bivector and the system of linear equations will be inconsistent. The solution might be geometrically represented by a pair of points if n = 2 or by a set of points if n > 2.

The probabilities of measuring the different values of x are proportional to the squares of their coefficient vector components:

First value of x:	$x_1 = \frac{r_1}{a_1}$	Probability of measuring the first value of x:	$p_1 = \frac{a_1^{\,2}}{\mathbf{a}^2}$
Second value of x:	$x_2 = \frac{r_2}{a_2}$	Probability of measuring the second value of x:	$p_2 = \frac{a_2^{\,2}}{\mathbf{a}^2}$
	$\vdots$		$\vdots$
ith value of x:	$x_i = \frac{r_i}{a_i}$	Probability of measuring the ith value of x:	$p_i = \frac{a_i^{\,2}}{\mathbf{a}^2}$

The average value of x will then be identical to the scalar part $\langle x \rangle_0$ of the solution:

$$\bar{x} = \sum_{i=1}^{n} p_i\, x_i = \sum_{i=1}^{n} \frac{a_i^{\,2}}{\mathbf{a}^2}\, \frac{r_i}{a_i} = \frac{1}{\mathbf{a}^2} \sum_{i=1}^{n} a_i\, r_i = \frac{1}{\mathbf{a}^2}\, \mathbf{a} \bullet \mathbf{r} = \langle x \rangle_0$$

Different systems of linear equations are equivalent, if they have the same solution. Different systems of linear equations are not equivalent, if they have different solutions. Thus different systems of linear equations which show the same average value at measurements, are only equivalent, if they possess an identical bivector part of the solution.

Different equivalent systems of linear equations can be constructed by pre-multiplication of different coefficient vectors $\mathbf{a}_{\mathbf{new}}$:

$$\mathbf{a}_{new}\, x = \frac{1}{\mathbf{a}^2}\, \mathbf{a}_{\text{new}}\, \mathbf{a}\, \mathbf{r} = \mathbf{r}_{\text{new}} \qquad \text{with} \quad \mathbf{a}_{\text{new}} = \sum_{i=1}^{n} a_{\text{new}\, i} \sigma_i = a_{\text{new}\,1}\, \sigma_1 + a_{\text{new}\,2}\, \sigma_2 + \ldots + a_{\text{new}\,n}\, \sigma_n$$

As the coefficient vectors describe the state of the system of linear equations, they can be called "state vectors". And of course the solution remains unchanged:

$$x = \mathbf{a}^{-1}\,\mathbf{r} = \frac{1}{\mathbf{a}^2}\,\mathbf{a}\,\mathbf{r} = \mathbf{a}_{new}{}^{-1}\,\mathbf{r}_{new} = \frac{1}{\mathbf{a}_{new}{}^2}\,\mathbf{a}_{new}\,\mathbf{r}_{new}$$

If the new state vector $\mathbf{a}_{new}$ is parallel to the bivector part of the solution

$$\mathbf{a}_{new} \wedge \langle x\rangle_2 = \mathbf{a}_{new} \wedge \mathbf{a} \wedge \mathbf{r} = 0 \qquad \Leftrightarrow \qquad \mathbf{a}_{new} \text{ and } \langle x\rangle_2 \text{ are parallel,}$$

then the new resulting vector of constant terms $\mathbf{r}_{new}$ will be a pure vector

$$\mathbf{r}_{new} = \sum_{i=1}^{n} r_{new\,i}\,\sigma_i = r_{new\,1}\,\sigma_1 + r_{new\,2}\,\sigma_2 + \ldots + r_{new\,n}\,\sigma_n = \mathbf{a}_{new}\langle x\rangle_0 + \mathbf{a}_{new} \bullet \langle x\rangle_2$$

If the new state vector $\mathbf{a}_{new}$ is not parallel to the bivector part of the solution

$$\mathbf{a}_{new} \wedge \langle x\rangle_2 = \frac{\mathbf{a}_{new} \wedge \mathbf{a} \wedge \mathbf{r}}{\mathbf{a}^2} \neq 0 \qquad \Leftrightarrow \qquad \mathbf{a}_{new} \text{ and } \langle x\rangle_2 \text{ are not parallel,}$$

then the new resulting vector of constant terms $\mathbf{r}_{new}$ will be a linear combination of a vector and a trivector

$$\mathbf{r}_{new} = \underbrace{\mathbf{a}_{new}\langle x\rangle_0 + \mathbf{a}_{new} \bullet \langle x\rangle_2}_{\text{vector}} + \underbrace{\mathbf{a}_{new} \wedge \langle x\rangle_2}_{\text{trivector}} = \langle \mathbf{r}_{new}\rangle_1 + \langle \mathbf{r}_{new}\rangle_3$$

magnitude of trivector: $\mathbf{x}_{tri} = |\langle \mathbf{r}_{new}\rangle_3|$

and

$$\mathbf{r}_{new} = r_{new\,1}\,\sigma_1 + r_{new\,2}\,\sigma_2 + \ldots + r_{new\,n}\,\sigma_n + r_{new\,123}\,\sigma_1\sigma_2\sigma_3 + r_{new\,124}\,\sigma_1\sigma_2\sigma_4 + \ldots + r_{new\,234}\,\sigma_2\sigma_3\sigma_4 + \ldots + r_{new\,n-2,n-1,n}\,\sigma_{n-2}\sigma_{n-1}\sigma_n$$

The trivector part can be considered as an additional field.

The probabilities of measuring the new values of x are proportional to the squares of their new coefficient vector components:

First value of x: $\quad x_1 = \dfrac{\langle \mathbf{r}_{new}\sigma_1\rangle_0}{a_{new\,1}} = \dfrac{\langle \mathbf{r}_{new}\rangle_1 \bullet \sigma_1}{a_{new\,1}} = \dfrac{r_{new\,1}}{a_{new\,1}}$

Probability of measuring the first value of x: $\quad p_1 = \dfrac{a_{new\,1}{}^2}{\mathbf{a}_{new}{}^2}$

Second value of x: $\quad x_2 = \dfrac{\langle \mathbf{r}_{new}\sigma_2\rangle_0}{a_{new\,2}} = \dfrac{\langle \mathbf{r}_{new}\rangle_1 \bullet \sigma_2}{a_{new\,2}} = \dfrac{r_{new\,2}}{a_{new\,2}}$

Probability of measuring the second value of x: $\quad p_2 = \dfrac{a_{new\,2}{}^2}{\mathbf{a}_{new}{}^2}$

$\vdots \qquad\qquad \vdots$

ith value of x: $\quad x_i = \dfrac{\langle \mathbf{r}_{new}\sigma_i\rangle_0}{a_{new\,i}} = \dfrac{\langle \mathbf{r}_{new}\rangle_1 \bullet \sigma_i}{a_{new\,i}} = \dfrac{r_{new\,i}}{a_{new\,i}}$

Probability of measuring the ith value of x: $\quad p_i = \dfrac{a_{new\,i}{}^2}{\mathbf{a}_{new}{}^2}$

And this is one of the main findings of this paper: The hidden field does not influence the measurement, as the average value of x, which can be measured, will be again identical to the scalar part of the solution. Thus the result the measurement of $\bar{x}$ is unchanged:

$$\overline{x}_{new} = \sum_{i=1}^{n} p_i \, x_i = \sum_{i=1}^{n} \frac{a_{new\,i}^{\;2}}{\mathbf{a}^2} \frac{r_{new\,i}}{a_{new\,i}} = \frac{1}{\mathbf{a}_{new}^{\;2}} \sum_{i=1}^{n} a_{new\,i} \, r_{new\,i} = \frac{1}{\mathbf{a}_{new}^{\;2}} \mathbf{a}_{new} \bullet \langle \mathbf{r}_{new} \rangle_1$$

As

$$x = \frac{1}{\mathbf{a}_{new}^{\;2}} \mathbf{a}_{new} \, \mathbf{r}_{new} = \frac{1}{\mathbf{a}_{new}^{\;2}} \mathbf{a}_{new} \, (\underbrace{\mathbf{a}_{new} \langle x \rangle_0 + \mathbf{a}_{new} \bullet \langle x \rangle_2}_{\text{vector}} + \underbrace{\mathbf{a}_{new} \wedge \langle x \rangle_2}_{\text{trivector}})$$

the scalar part $\overline{x}$ must be

$$\overline{x} = \frac{1}{\mathbf{a}_{new}^{\;2}} \mathbf{a}_{new} \bullet (\underbrace{\mathbf{a}_{new} \langle x \rangle_0 + \mathbf{a}_{new} \bullet \langle x \rangle_2}_{\text{vector}}) = \frac{1}{\mathbf{a}_{new}^{\;2}} \mathbf{a}_{new} \bullet \langle \mathbf{r}_{new} \rangle_1$$

with the clear conclusion:

$$\overline{x}_{new} = \overline{x}$$

Even if the new state produces an additional field, the average value does not chance and a measurement will give an identical result.

11 Outlook

At BSEL and MSB I discussed Geometric Algebra solution strategies of solving consistent systems of linear equations with first-year students. So please have a look at the material of these business mathematics courses [14], [15], [16], [17], [18], [19] to understand how Grassmann solved systems of linear equations with more than one variable. The material can be downloaded at PhyDid (www.phydid.de).

The solution strategy of Grassmann can also be used to solve higher-dimensional inconsistent systems of linear equations by calculating inverses of non-square matrices (e.g. see [20], [21], [22]). The interpretation and quantum mechanical meaning of these consistent solutions of inconsistent systems of linear equations with more than one unknown variable will be discussed in more detail in the second part of this paper.

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