Levi-Civita Rhymes with Lolita

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February 2, 2017

Ladies and gentlemen of the jury, exhibit number one is what the seraphs, the misinformed, simple, noble-winged seraphs, envied. Look at this tangle of thorns. — Humbert Humbert in *Lolita* by Vladimir Nabokov (1955)

The Levi-Civita symbol is arguably the simplest mathematical quantity of importance that one can imagine. In n dimensions, it carries n indices whose sole purpose is to keep track of the signs of various indexed mathematical quantities that it operates on. Unlike matrices, vectors and tensors, the Levi-Civita symbol (also called the permutation symbol) has no individual components of its own to speak of—the indices all act in unison, returning either 0 or ± 1, depending only on how they're lined up. Nevertheless, the Levi-Civita symbol is ubiquitous in elementary vector and matrix algebra, though it also appears routinely in general relativity, quantum mechanics and even topology.

Despite its apparent simplicity, textbook derivations of some of the Levi-Civita symbol's properties are notoriously hard to come by. Indeed, to the consternation of the student the symbol's properties are often just written down as if they were obvious. Here we provide a simple overview of the symbol's properties along with several of its applications, with particular attention given to basic derivations.

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1. Introduction and Notation

The Italian mathematician Tullio Levi-Civita (1873-1941) was the only doctoral student of the great mathematician Gregorio Ricci-Curbastro, the Italian inventor of tensor calculus. Prolific in mathematics and several other fields, Levi-Civita is acknowledged as being the inventor (or discoverer) of the handy symbol that now carries his name. In n dimensions, the symbol is generally given as $\varepsilon_{a_1 a_2 a_3 \ldots a_n}$ or $\varepsilon^{a_1 a_2 a_3 \cdots a_n}$, where the a's are integers comprising some permutation of the integer set $\{1, 2, 3 \ldots n\}$. The symbol expresses the permutation property

$$\varepsilon_{123\ldots n} = \varepsilon^{123\ldots n} = \begin{cases} +1 & \text{if the integers are an even permutation of } 1, 2, 3 \ldots n \\ -1 & \text{if the integers are an odd permutation of } 1, 2, 3 \ldots n \\ 0 & \text{if any two integers are the same} \end{cases}$$

Consequently, the symbol's sign depends upon an overall even or odd number of exchanges of the indices, going to zero only when two or more indices are identical (older texts often refer to the LC symbol as the *permutation symbol* or the *antisymmetry symbol*). Here are some examples:

$$\begin{aligned} \varepsilon_{12} &= +1 \\ \varepsilon_{21} &= -1 \\ \varepsilon_{1324} &= -1 \\ \varepsilon_{7125634} &= +1 \\ \varepsilon_{7163145} &= \ \ 0 \end{aligned}$$

By convention, consecutive ordering of the integers results in $+1$, so that $\varepsilon_{123\ldots} = \varepsilon^{123\cdots} = 1$. As we will see, the *exchange antisymmetry* of the symbols neatly parallels the *symmetry* of the ordinary two-index Kronecker symbol $\delta_{\mu\nu}$. Indeed, we will see that any LC symbol consists entirely of a string of Kronecker symbols. This relationship is often called the ε-δ formalism.

There are n^n possible number strings in an LC symbol of dimension n, but most of the symbols are zero due to repeated indices. It is easily shown that there are always only $n!$ non-zero terms in the permutation of the $123\ldots n$ string, with half of them being $+1$ and half -1. This behavior parallels the Laplace expansion of an $n \times n$

matrix A in the calculation of the matrix's determinant $|A|$. This turns out to be no coincidence, and we will see later that the LC symbol provides a compact way of representing the determinant of any square matrix.

2. Preliminaries

We will be dealing with a Minkowski (flat) space throughout. Consequently, the Levi-Civita symbol can be treated as a tensor of rank n whose indices are raised and lowered with the flat-space metric tensors $\eta^{\mu\nu}$ (and $\eta_{\mu\nu}$) respectively, with the important distinction that the signature of the metric be given by $(+++\ldots+)$; that is, in any dimension the metric tensor is just the Kronecker delta $\delta^{\mu\nu}$, $\delta_{\mu\nu}$, where

$$\delta_{ab} = \delta^{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

While LC symbols can also have mixed indices ($\varepsilon^{\lambda}_{\ \mu\nu}$, etc.), we will not be using them. Hopefully, the student will realize that the derivation of many of the properties of the LC symbol is difficult without the use of the special metric $\eta_{\mu\nu}$ we will be using.

Lastly, while we will use Einstein notation throughout (in which any repeated index in a term is to be summed over from 1 to n), we won't pay much attention to conventional niceties—for example, the sum $\varepsilon_{ab} T_{ac}$ will be treated the same as the more proper notation $\varepsilon_{ab} T^{ac}$.

3. The Levi-Civita Symbols as Determinants

Consider the simple two-dimensional LC symbol ε_{ab}, which obviously has only two non-vanishing terms, $\varepsilon_{12} = 1$ and $\varepsilon_{21} = -1$. We can express the symbol by lowering the indices of its upper-indexed form via

$$\varepsilon_{ab} = \eta_{ac} \eta_{bd} \varepsilon^{cd}$$

Expanding, and replacing the metric terms with their equivalent Kronecker symbols, we have

$$\begin{aligned} \varepsilon_{ab} &= \delta_{a1} \delta_{bd} \varepsilon^{1d} + \delta_{a2} \delta_{bd} \varepsilon^{2d} \\ &= \delta_{a1} \delta_{b2} \varepsilon^{12} + \delta_{a2} \delta_{b1} \varepsilon^{21} \\ &= \varepsilon^{12} (\delta_{a1} \delta_{b2} - \delta_{a2} \delta_{b1}) \end{aligned}$$

or

$$\varepsilon_{ab} = \begin{vmatrix} \delta_{a1} & \delta_{a2} \\ \delta_{b1} & \delta_{b2} \end{vmatrix} \tag{3.1}$$

since $\varepsilon^{12} = 1$. Thus, the 2D LC symbol is a simple 2×2 determinant of Kronecker deltas. We can repeat this exact same argument for any LC symbol to show that in general

$$\varepsilon_{abcd\ldots n} = \begin{vmatrix} \delta_{a1} & \delta_{a2} & \delta_{a3} & \ldots & \delta_{an} \\ \delta_{b1} & \delta_{b2} & \delta_{b3} & \ldots & \delta_{bn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_{n1} & \delta_{n2} & \delta_{n3} & \ldots & \delta_{nn} \end{vmatrix} \tag{3.2}$$

It is often necessary to evaluate products of same-dimensional LC symbols, such as $\varepsilon_{ab} \varepsilon_{cd}$ and $\varepsilon^{abc} \varepsilon_{def}$. Even for simple products, the number of terms quickly becomes unwieldy upon direct multiplication:

$$\varepsilon_{ab} \varepsilon_{cd} = \begin{vmatrix} \delta_{a1} & \delta_{a2} \\ \delta_{b1} & \delta_{b2} \end{vmatrix} \cdot \begin{vmatrix} \delta_{c1} & \delta_{c2} \\ \delta_{d1} & \delta_{d2} \end{vmatrix},$$

which expands out to

$$\varepsilon_{ab} \varepsilon_{cd} = \delta_{a1} \delta_{b2} \delta_{c1} \delta_{d2} - \delta_{a1} \delta_{b2} \delta_{c2} \delta_{d1} + \delta_{a2} \delta_{b1} \delta_{c2} \delta_{d1} - \delta_{a2} \delta_{b1} \delta_{c1} \delta_{d2}$$

Well, this is a real mess. In an effort to simplify matters, let's consider the identity

$$\begin{aligned}\delta_{ab} &= \eta_{ac}\eta_{bd}\,\delta^{cd} \\ &= \delta_{a1}\delta_{bd}\,\delta^{1d} + \delta_{a2}\delta_{bd}\,\delta^{2d} \\ &= \delta_{a1}\delta_{b1} + \delta_{a2}\delta_{b2} \end{aligned} \tag{3.3}$$

(where, of course, $\delta^{11} = \delta^{22} = 1$ and $\delta^{12} = \delta^{21} = 0$). Using the identity in (1) to calculate $\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$, we get precisely the same expression in (2). But this shows that

$$\varepsilon_{ab}\,\varepsilon_{cd} = \begin{vmatrix} \delta_{ac} & \delta_{ad} \\ \delta_{bc} & \delta_{bd} \end{vmatrix} \tag{3.4}$$

A similar argument can be used for larger LC symbols (although you really wouldn't want to), in which case we have

$$\varepsilon_{ABC\ldots n}\,\varepsilon_{abc\ldots n} = \begin{vmatrix} \delta_{Aa} & \delta_{Ab} & \delta_{Ac} & \ldots & \delta_{An} \\ \delta_{Ba} & \delta_{Bb} & \delta_{Bc} & \ldots & \delta_{Bn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_{na} & \delta_{nb} & \delta_{nc} & \ldots & \delta_{nn} \end{vmatrix} \tag{3.5}$$

Upper-index and mixed products can be shown to work the same way. For example,

$$\varepsilon^{ABC\ldots n}\,\varepsilon_{abc\ldots n} = \begin{vmatrix} \delta^A_a & \delta^A_b & \delta^A_c & \ldots & \delta^A_n \\ \delta^B_a & \delta^B_b & \delta^B_c & \ldots & \delta^B_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta^n_a & \delta^n_b & \delta^n_c & \ldots & \delta^n_n \end{vmatrix} \tag{3.6}$$

4. The Levi-Civita Symbols in Determinant Computation

The alternating signs associated with the terms of the LC symbol provide a convenient means of connecting them to the determinants of square matrices. There's no magic formula for easily computing determinants, but we can use the fact that for any $n \times n$ matrix, its determinant—obtained by Laplace expansion about any row or column—always has $n!$ product terms, with an equal number of positive and negative elements. The same applies to any LC symbol of the same dimension. So, if we can associate the positive and negative LC terms with those of the determinant of a matrix, we might be able to identify a relationship between the two.

Consider the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

whose determinant is the $3! = 6$-term quantity

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

The six non-zero LC symbol terms are

$$\varepsilon_{123} = +1,\ \varepsilon_{231} = +1,\ \varepsilon_{312} = +1,\ \varepsilon_{132} = -1,\ \varepsilon_{213} = -1,\ \varepsilon_{321} = -1$$

By matching up the LC terms with the determinant products, it should be clear that the expression

$$|A| = \varepsilon_{ijk}\,a_{i1}a_{j2}a_{k3}$$

is valid for the 3×3 case. It is in fact true for all square matrices of any dimension:

$$|A| = \varepsilon_{ijk\ldots}\,a_{i1}a_{j2}a_{k3}\ldots \tag{4.1}$$

We can “derive” one more expression relating LC symbols and determinants. If we multiply the left side of (1) by $\varepsilon_{123} = 1$, it is not difficult to see that the general expression

$$\varepsilon_{\mu\nu\lambda\ldots}\,|A| = \varepsilon_{abc\ldots} a_{a\mu} a_{b\nu} a_{c\lambda} \ldots \tag{4.2}$$

is also valid.

5. Densities

Although the LC symbol can be used to generate scalar-like quantities (such as $\varepsilon^{ab} V_a V_b$, where V_i is some vector), such quantities do not transform like scalars under coordinate transformations. Instead, use of the LC symbol results in the creation of scalar, vector and tensor *densities*. This behavior is of signficant importance in topological field theory and the related Chern-Simons theory of quantum mechanics.

Let us see how the index-less vector quantity $T = \varepsilon^{abc} V_a V_b V_c$ behaves under the coordinate tranformation $x \rightarrow \bar{x}$. While the LC symbol remains unchanged, we have

$$\bar{V}_a = \frac{\partial x^\mu}{\partial \bar{x}^a} V_\mu \tag{5.1}$$

so that

$$\bar{T} = \varepsilon^{abc}\, \bar{V}_a \bar{V}_b \bar{V}_c \rightarrow \varepsilon^{abc} \frac{\partial x^\mu}{\partial \bar{x}^a} \frac{\partial x^\nu}{\partial \bar{x}^b} \frac{\partial x^\lambda}{\partial \bar{x}^c} V_\mu V_\nu V_\lambda$$

Treating the differential terms as matrices, we can use (4.1) to express this as

$$\bar{T} = \varepsilon^{\mu\nu\lambda}\, V_\mu V_\nu V_\lambda \left| \frac{\partial x}{\partial \bar{x}} \right|$$

where the end term is the Jacobian determinant of the coordinate transformation,

$$\left| \frac{\partial x}{\partial \bar{x}} \right| = \begin{vmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^1}{\partial \bar{x}^3} \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^3} \\ \frac{\partial x^3}{\partial \bar{x}^1} & \frac{\partial x^3}{\partial \bar{x}^2} & \frac{\partial x^3}{\partial \bar{x}^3} \end{vmatrix} \tag{5.2}$$

But this is just

$$\bar{T} = \left| \frac{\partial x}{\partial \bar{x}} \right| T,$$

so the quantity

$$\frac{T}{\left| \frac{\partial x}{\partial \bar{x}} \right|} \tag{5.3}$$

is now a true scalar under coordinate transformations. Thus, the LC symbol in this case creates what is known as a scalar density or *pseudoscalar*. In view of this, the symbol itself is often referred to as a *pseudotensor*.

Along with vectors, the LC symbol also acts on tensors in a similar manner. Just for the hell of it, let’s take the metric tensor $g_{\mu\nu}(x)$ in four dimensions and see where the expression

$$\varepsilon_{\mu\nu\alpha\beta}\, |g| = \varepsilon^{abcd}\, g_{\mu a}\, g_{\nu b}\, g_{\alpha c}\, g_{\beta d}$$

leads under the change of coordinates $x \rightarrow \bar{x}$. Let us therefore break down

$$\varepsilon_{\mu\nu\alpha\beta}\, |\bar{g}| = \varepsilon^{abcd}\, \bar{g}_{\mu a}\, \bar{g}_{\nu b}\, \bar{g}_{\alpha c}\, \bar{g}_{\beta d}$$

where

$$\bar{g}_{\mu a} = \frac{\partial x^e}{\partial \bar{x}^\mu} \frac{\partial x^f}{\partial \bar{x}^a}\, g_{ef} \tag{5.4}$$

We then have

$$\begin{aligned}\varepsilon_{\mu\nu\alpha\beta}\,|\bar{g}| &= \varepsilon^{abcd}\,\frac{\partial x^e}{\partial\bar{x}^\mu}\frac{\partial x^f}{\partial\bar{x}^a}\frac{\partial x^h}{\partial\bar{x}^\nu}\frac{\partial x^i}{\partial\bar{x}^b}\frac{\partial x^j}{\partial\bar{x}^\alpha}\frac{\partial x^k}{\partial\bar{x}^c}\frac{\partial x^l}{\partial\bar{x}^\beta}\frac{\partial x^m}{\partial\bar{x}^d}\,g_{ef}\,g_{hi}\,g_{jk}\,g_{lm} \\ &= \varepsilon^{abcd}\left(\frac{\partial x^f}{\partial\bar{x}^a}\frac{\partial x^i}{\partial\bar{x}^b}\frac{\partial x^k}{\partial\bar{x}^c}\frac{\partial x^m}{\partial\bar{x}^d}\right)\frac{\partial x^e}{\partial\bar{x}^\mu}\frac{\partial x^h}{\partial\bar{x}^\nu}\frac{\partial x^j}{\partial\bar{x}^\alpha}\frac{\partial x^l}{\partial\bar{x}^\beta}\,g_{ef}\,g_{hi}\,g_{jk}\,g_{lm}\end{aligned}$$

Using (4.2), this reduces to

$$\begin{aligned}\varepsilon_{\mu\nu\alpha\beta}\,|\bar{g}| &= \varepsilon^{fikm}\left|\frac{\partial x}{\partial\bar{x}}\right|\frac{\partial x^e}{\partial\bar{x}^\mu}\frac{\partial x^h}{\partial\bar{x}^\nu}\frac{\partial x^j}{\partial\bar{x}^\alpha}\frac{\partial x^l}{\partial\bar{x}^\beta}\,g_{ef}\,g_{hi}\,g_{jk}\,g_{lm} \\ &= \varepsilon_{\mu\nu\alpha\beta}\,|g|\left|\frac{\partial x}{\partial\bar{x}}\right|^2\end{aligned}$$

Of course, all of this is just a long-winded way of saying that

$$\bar{g} = \left|\frac{\partial x}{\partial\bar{x}}\right|^2 g$$

or

$$\sqrt{|\bar{g}|} = \left|\frac{\partial x}{\partial\bar{x}}\right|\sqrt{|g|}\ ,$$

which could have been deduced immediately from (5.4). Nevertheless, it is heartening to see that the LC formalism is consistent with other (and sometimes more obvious) methods.

6. Advanced Applications

In general relativity it is conventional to use $\sqrt{|g|}$ in the scalar densities associated with the Lagrangians of gravitation. For example, the integral

$$I_{EH} = \int\sqrt{|g|}\,R\,d^4x$$

(where R is the Ricci scalar) defines the Einstein-Hilbert action for free-space general relativity. Note that the determinant of *any* second-rank tensor $A_{\mu\nu}$ could be used in this regard, as the overall goal is to simply make $\sqrt{|A|}\,d^4x$ a coordinate-invariant quantity. (It is amusing to note that the integral $\int\sqrt{|g|}\,d^4x$ is a perfectly good action Lagrangian. Its only problem is that it doesn't lead to anything meaningful.)

Lastly, in view of the foregoing it should be noted that quantities such as

$$\int R\,\varepsilon^{\mu\nu\alpha\beta}\,R_{\mu\nu\alpha\beta}\,d^4x, \quad \int\varepsilon^{\mu\nu\alpha\beta}\,R_{\mu\nu}R_{\alpha\beta}\,d^4x$$

are completely valid candidates for gravity theories since the presence of the LC symbol guarantees coordinate invariance of the integrands. What is particularly fascinating about actions such as these is that they don't necessarily involve the metric tensor $g_{\mu\nu}$ or its determinant $\sqrt{|g|}$. (One might argue that, since Ricci curvature terms are composed of the metric tensor, this claim in not true. However, all curvature tensors can be expressed purely in terms of *affine connections*, negating this argument.) As Zee has rightly noted, these actions are completely *unaware* of any metric quantity, so things like time (clocks) and distance (rulers) don't enter into the formalism at all. They are purely *topological* in nature—one may stretch and squeeze spacetime in any arbitrary way, and the actions will remain unscathed.

The above observation has led to the search for topologically invariant quantities in gravity theory. Of particular interest is the formalism known as *Regge calculus*, in which spacetime is *discretized* into a mesh of interconnecting, purely flat shapes (usually triangles). The primary advantage of this formalism involves the development of efficient *numerical* methods in general relativity which can be run on high-speed computers. Exact analogies of this approach are the finite-difference and finite-element methods commonly used in structural engineering.

Regge calculus is fascinating, and the student is encouraged to look into it. It begins with the Euler characteristic (or symbol), which relates the connectedness of vertices, edges and loops (or faces) in a simple closed graph, and progresses to the notion of *angular deficits* associated with a topological construction (such as a geodesic dome). Simplicity is the rule in this calculus—the basic shapes (triangles, tetrahedrons, dodecahedrons, etc.) used to describe a spacetime can be mangled in almost any possible way without altering the underlying invariants of the topology. The notion of spacetime curvature itself in Regge calculus is topological in nature, providing a hint that the natural world is entirely grounded in geometry.

I am thinking of aurochs and angels, the secret of durable pigments, prophetic sonnets, the refuge of art. And this is the only immortality you and I may share, my Lolita.

References

1. V. Nabokov, *Lolita*, 1955. In spite of its controversial subject matter, Nabokov's book is frequently rated as one of the top five novels in all English literature, and deservedly so. It says nothing about the LC symbol of course, but, like Shakespeare's *Hamlet* and Joseph Conrad's *Heart of Darkness*, it's great reading.

2. P. Renteln, *Lectures on vector calculus*, 2011. A fine summary of vector calculus and its applications, with considerable details provided on the LC symbol. It can be downloaded at

   `http://physics.csusb.edu/~prenteln/notes/vc_notes.pdf`

3. A. Zee, *Einstein gravity in a nutshell*, Chapter X.5. Princeton University Press, 2013. Already a classic, Zee's text is the best introductory/intermediate book on general relativity available. Stranded on a desert island? You should have this book with you.