As spinor $\chi = a|\uparrow> +b|\downarrow>$ is physical in $SU(2)_{spin}$ space, then why is isospinor $\psi = a|p> +b|n>$ unphysical in $SU(2)_{isospin}$ space?

Syed Afsar Abbas

Centre for Theoretical Physics, JMI University, New Delhi - 110025, India

and

Jafar Sadiq Research Institute
AzimGreenHome, NewSirSyed Nagar, Aligarh - 202002, India
(e-mail : drafsarabbas@gmail.com)

Abstract

A spin angular momentum state with a polarization orientation in any arbitrary direction can be constructed as a spinor in the SU(2)-spin space as $\chi = a|\uparrow> +b|\downarrow>$. However the corresponding isospinor in the SU(2)-isospin space, $\psi = a|p> +b|n>$ is discarded on empirical grounds. Still, we do not have any sound theoretcal understanding of this phenomenon. Here we provide a consistent explanation of this effect.

Keywords: Nuclear models, group theory, Standard Model, 't Hooft anomaly matching condition, quark model, electric charge

We know that a spin angular momentum state with a polarization orientation in any arbitrary direction can be constructed as a spinor in SU(2)-spin space as $\chi = a|\uparrow> + b|\downarrow>$. However, the same does not hold for the SU(2)-isospin space. The corresponding isospinor $\psi = a|p> + b|n>$ turns out to be unphysical. This is an empirical fact. However one does not have any sound theoretical understanding of this phenomenon. Here we provide a consistent solution of this problem.

The quark model group structure is $SU(6)_{FS} \otimes SU(3)_C \supset SU(3)_F \otimes SU(2)_S \otimes SU(3)_C$. Here in $SU(3)_F$ the quark charges are given as $Q = T_3 + \frac{Y}{2}$ where Y = B + S. As S=0 for proton and neutron $Q_p = \frac{1}{2} + \frac{\frac{1}{3}}{2} = \frac{2}{3}$ and $Q_n = -\frac{1}{2} + \frac{\frac{1}{3}}{2} = -\frac{1}{3}$. These are completely independent of colour. The only way that colour comes into the above picture is by ensuring a colour antisymmetric wave function in the above semi-simple group of $SU(3)_C$. Also note that here the baryon number of 1/3 comes from within as the second diagonal generator λ_8 of $SU(3)_F$. So the baryon number is internally generated in $SU(3)_F$

In contrast, for the first generation of quarks and leptons, in the Standard Model (SM) with group structure $SU(3)_C \otimes SU(2)_L \otimes U(1)_{Y_W}$, the electric charges are defined either as $Q = T_3^W + Y_W$ [1] or as $Q = T_3^W + \frac{Y_W}{2}$ [2]. The hypercharges are put in by hand to provide proper charges for all the matter particles. Again there is no colour present in the electric charge in the SM. However the baryon number 1/3 is colour dependent as arising externally from the group $SU(3)_C$. This is the standard unquantized charge (i.e. arbitrarily put in by hand) which is most commonly used in the SM at presnt[1,2]. The same charges are also used in studies of QCD for arbitrary number of colours [3].

To distinguish the fact that the same group structure and the same matter structure as the above SM has proper charge quantization built into it, we refer to this new structure as the Quantized Charge Standard Model (QCSM). This distinction, as we see below, shall be found to be necessary to avoid undue confusion and also as the QCSM is actually providig physics well ouside the purview of the SM.

In QCSM, it has been shown convingly [4,5], that for the group $SU(N)_C \otimes SU(2)_L \otimes U(1)_{Y_W}$ with $N_C = 3$, the first generation quarks have proper quantized charges,

$$Q(u) = \frac{1}{2}(1 + \frac{1}{N_c}), \quad Q(d) = \frac{1}{2}(-1 + \frac{1}{N_c}) \tag{1}$$

Most significant fact in QCSM is that in spite of the fact that photon does not recognize colour, the electric charge itself has colour sitting inside it! This crucial difference, as to colour in the charges, is the most significant differece with respect the above SM charges. It has been shown [4,5] that this colour dependence is essential to study QCD for arbitrary number of colours. Thus the SM charges fail [3], whereas the QCSM charges succeed [4]. Thus the QCSM is actually an extension of the SM, going beyond its confines and providing new physics beyond the reach of the SM.

However first we are interested in noting the basic differences in how quark charges are represented in the flavour $SU(3)_F$ model and the QCSM. The charges of quarks are completely different as to their intrinsic structure in these two models. One model does not know of any colour (in the charges) while the other one is well-coloured! Also baryon number in the flavour model arises internally from the simple group stucture itself, while in the QCSM it arises due to the colour structure of the semi-simple group for this model.

Given these irreconciliable differences, how can these two models describe the same entities consistently? Let us study this problem now.

Note that the $SU(3)_F$ model successfully describes the baryons as an octet. The nucleon forms the lowest mass isospin doublet. These then provide the proper representation of nucleon as what constitutes the nucleus. Including the isospin in the Generaized Pauli Exclusion Principle along with analysis within the Brueckner-Hartree-Fock view, leads to the successful Independent Particle Model (IPM) of the nucleus.

Next what does this new picture of the nucleon, as viewed within the QCSM analysis, leads to? What information it hides which can help us understand the hadrons better and which may lead to an understanding of the difference between the SU(2)-spin and the SU(2)-isopsin group reperesemtations as pointed out in the title of the paper.

When a theory is strongly coupled, there is often a complete shift in the relevant degrees of freedom; e.g. at short distances strong nuclear force is described by quarks and gluons, while at larger distances the proper degrees of freedom are the hadrons. Imagine a theory is weakly coupled (so perturbation theory works) when we are above a certain energy scale λ. Below this scale let the theory be strongly coupled so that one cannot do perturbation anymore.

$$\textit{weakly coupled theory} \ > \ \lambda \ > \ \textit{strongly coupled theory} \tag{2}$$

Note that we have an advantage if the weakly coupled theory has an anomalous symmetry. 't Hooft showed [6] that regardless of the strength of the interaction, anomaly must be present on both sides of λ.

This allows us to identify the fermion sector of our effective field theory. Canonically, at present the structure of the nucleus at low energies is nucleonic degrees of freedom only; but deep inside, these are made up of quarks which show up at higher energies. However, here we show, that there exists a basic and consistent structure wherein nucleons do appear as fundamental entities. This is indeed made possible due to the 't Hooft anomaly matching condition.

't Hooft anomaly matching condition [6] points out that chirality ensures that the fermions are massless. So composites of fundamental entities in the chiral limit may match each other through the 't Hooft anomaly matching condition. This is possible if the sum of the anomaly coefficients A(r) for the composite fermions (below λ) is equal to that of fundamental fermions (above λ)

$$\sum_r N_r A(r) = \sum_r n_r A(r) \tag{3}$$

(n_r are number of chiral fermions in representation r and N_r are number of massless composite fermions in representation r)

The first generation is unique as the coloured massless u-, d- quarks form an isospin doublet in the SM. Then the only colourless composite spin-half fermions that we can create in the ground state, are proton (uud) and neutron (udd). Now (p,n) do form a massless, chiral, isospin-doublet. Thus the 't Hooft matching condition is indeed satisfied. (However the same logic fails for 3 flavours (u,d,s) to octet baryons $(p, n, \Sigma^{+,-,0}, \Lambda^0, \Xi^{-,0})$).

This leads to a new structure. Due to the above reason, the first generation of quark-lepton goes over to a new and unique single generation of massless chiral nucleon-lepton. Its epresentation in the QCSM group $SU(N)_C \otimes SU(2)_L \otimes U(1)_{Y_W}$ is as follows:

$$N_L = \begin{pmatrix} p \\ n \end{pmatrix}_L, (1, 2, Y_N) \ ; \ \ p_R, (1, 1, Y_p) \ ; \quad n_R, (1, 1, Y_n) \tag{4}$$

$$l_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, (1, 2, Y_l) \ ; \ \ e_R, (1, 1, Y_e) \tag{5}$$

Now let the QCSM symmetry be spontaneously broken (SSB) to $SU(N_C) \otimes U(1)_{em}$ by an Englert-Brout-Higgs (EBH) field - $SU(2)_L$ group doublet in another phase transition. There are five unknown hypercharges plus the above unknown Y_ϕ of the EBH doublet (similar to the case in ref. [4]) Let us define the electric charge operator as

$$Q = T_3 + b\, Y \tag{6}$$

In QCSM we have three massless generators W_1, W_2, W_3 of $SU(2)_L$ and X of $U(1)_Y$. SSB by EBH mechanism provides mass to the $W^\pm$ and Z^0 gauge particles while ensuring zero mass for photons. Let $T_3 = -\frac{1}{2}$ of the EBH field develop a nonzero vacuum expectation value $< \phi >_0$. One of the four generators $(W_1 W_2 W_3, X)$ is thereby left unbroken, (meaning that we ensure a massless photon as a generator of the $U(1)_{em}$ group), we demand:

$$Q < \phi >_0 = 0 \tag{7}$$

This fixes the unknown b and we obtain,

$$Q = T_3 + (\frac{1}{2Y_\phi})Y \tag{8}$$

Anomalies play a very significant role in quantum field theories [1,2] As we require the SM to be renormalisable, we have to ensure that all the anomalies vanish. Thus we have three anomalies [4.5] listed as A, B and C as below

$$Anomaly\ A: \quad TrY[SU(N_C)]^2 = 0 \ ; \ 2Y_N = Y_p + Y_n \tag{9}$$

$$Anomaly\ B: \quad TrY[SU(2)_L]^2 = 0 \ ; \ giving\ \ Y_N = -Y_l \tag{10}$$

$$Anomaly\ C:\ Tr[Y^3] = 0;\ \ 2Y_N^3 - Y_p^3 - Y_n^3 + 2Y_l^3 - Y_e^3 = 0 \tag{11}$$

We need more constraints on the hypercharges. The Yukawa mass terms provide these:

$$Y_p = Y_N + Y_\phi, \quad Y_n = Y_N - Y_\phi, \quad Y_e = Y_l - Y_\phi \tag{12}$$

$$\rightarrow \quad Y_l = -Y_\phi \tag{13}$$

Finally, we get quantized electric charges for this unique nucleon-lepton single generation as,

$$Q(p) = 1,\ \ Q(n) = 0; \quad\ Q(\nu_e) = 0,\ \ Q(e) = -1 \tag{14}$$

The three anomalies, SSB through EBH mechanism, and Yukawa masses, gives consistent charge quantization. Most important to see that these nucleons are taken as fundamental particles and not as composites of quarks. The 't Hooft anomaly matching had made these nucleons massless and point-like chiral fermions as fundamental particles.

Now as in the quark model, we have the isospin doublet $\left(\begin{smallmatrix} p \\ n \end{smallmatrix}\right)$ arising in the flavour group $SU(3)_F \supset SU(2)_F$ above. Now the the same isopin pair arises independemtly in this other model due to the QCSM and conjoined with the 'tHooft anomaly matching condition. These clearly are dual description of the same $\left(\begin{smallmatrix} p \\ n \end{smallmatrix}\right)$ which make up the nucleus. How is this duality justified empirically?

Now in the IPM of the nucleus, the SU(2)-isopin symmetry arises from the quarks in the $SU(3)_F$ model. Note that proton and neutron are indistinguishable particle in this model. Thus the proton-neutron pair wave function is antisymmetric as follows:

$$\Phi = \frac{1}{2}(p(1)n(2) - n(1)p(2)) \tag{15}$$

Note that the position order (12) is fixed by definition while the p and n labels are exchanged. This arises due to the fact that the fundamenatl representation in the isopin group is a single entity called the nucleon N = $\left(\begin{smallmatrix} p \\ n \end{smallmatrix}\right)$

Let a single nucleon be made up of three quarks of $SU(2)_F$ group as

$$q_1(1) = \begin{pmatrix} p(1) \\ n(1) \end{pmatrix}\ ;\ q_2(2) = \begin{pmatrix} p(2) \\ n(2) \end{pmatrix}\ ;\ q_3(3) = \begin{pmatrix} p(3) \\ n(3) \end{pmatrix} \tag{16}$$

where we have put position labels on the nucleons. As colour sits outside in the group $SU(3)_C \otimes SU(2)_F$, so for a particular doublet

$$q_1(1) = \begin{pmatrix} p_R(1) \\ n_R(1) \end{pmatrix}\ ;\ \begin{pmatrix} p_B(2) \\ n_B(2) \end{pmatrix}\ ;\ \begin{pmatrix} p_G(3) \\ n_G(3) \end{pmatrix} \tag{17}$$

Now when three quarks make up proton and neutron, the quark content may be given as follows:

$$N_1(1) = \begin{pmatrix} p(1') \\ n(1') \end{pmatrix}; \; p(1') = u_R(1)u_B(2)d_G(3); \; n(1') = d_R(1)d_B(2)u_G(3) \quad (18)$$

where we have put $1'$ as some common centre of the positions 1, 2, and 3 above. As one builds proper symmetry into the flavour space, the colour antisymmetry would ensure that proton and neutron have the same base on the position labels (123) above.

Now the antisymmetric wave function of two nucleons N_1 *and* N_2 is

$$\Phi = \frac{1}{2}(N_1(1)N_2(2) - N_2(1)N_1(2)) \quad (19)$$

which then leads to the n-p pair antisymmetric wave function above in eqn. (15). Note the significance of the labels 1, 2, 3, and $1'$ in the above wave functions!

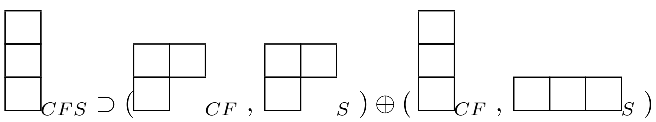
Now let us study the group structure relevant for the new doublet $\binom{p}{n}$ of the QCSM conjoined with the anomaly matching case.

In the canonical quark model the group structure is $SU(6)_{FS} \otimes SU(3)_C \supset SU(3)_F \otimes SU(2)_S \otimes SU(3)_C$. Antisymmetry arises from the colour part and the SU(6) part gives symmetric states for baryons. What is the meaning of $SU(6)_{FS} \supset SU(3)_F \otimes SU(2)_S$? We know that $SU(3)_F$ is pretty badly broken. It works at low non-relativistic energies. From this we work up to the bigger group $SU(6)_{FS}$ by including the purely "static" SU(2)-spin group. It is broken atleast as badly as its flavour subgroup. However it works pretty well in resolving some basic puzzles of the $SU(3)_F$ model (such as lack of flavour singlet repreesentation of spin 1/2 baryons etc.).

However in the new model due to chiralty, the quark masses are exactly zero and thus $SU(2)_F$ is an exact symmetry. As $SU(3)_C$ is an exact symmetry anyway, thus the larger group $SU(6)_{CF} \supset SU(3)_C \otimes SU(2)_F$ is a very good symmetry. Note that this is true at relativistic energies. This though is slightly broken due to SSB by EBH mechanism by fixing the slightly different masses of neutron and proton by Yukawa coupling. However it still remains a good symmetry to classify the states even at relativistic energies. Given the fact that $SU(2)_S$ should be a good symmetry, the new group structure would be

$$SU(12)_{CFS} \supset SU(6)_{CF} \otimes SU(2)_S \quad (20)$$

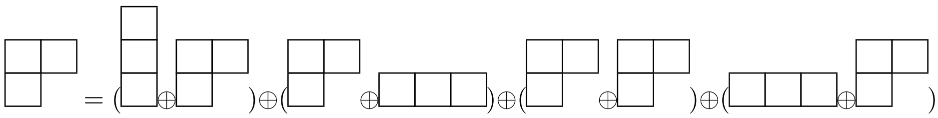
Now the three quark antisymmetric state in this bigger group, decomposed as above [7],



$$220_{CFS} \rightarrow (70_{CF}, 2_S) \oplus (20_{CF}, 4_S) \quad (21)$$

Clearly the doublet spin state above should be the representation which would provide our new (p,n) doublet.

The colour and flavour content of the above $SU(6)_{CF}$ representation is



$$70_{CF} = (1_C, 2_F) \oplus (8_C, 4_F) \oplus (8_C, 2_F) \oplus (10_C, 2_F) \tag{22}$$

We see from the first set on the right hand side that this state indeed has the proper colour singlet and flavour doublet of the new anomaly matched (p,n).

Thus we see that the fundamental representation of our $SU(6)_{CF}$ group is,

$$Q = \begin{array}{|c|} \hline u_R \\ \hline u_B \\ \hline u_G \\ \hline d_R \\ \hline d_B \\ \hline d_G \\ \hline \end{array} \tag{23}$$

We can put position labels, just like in eqn. (16), and construct proton and neutron from these. But now there are more than two states (actually six) at each position and thus when we construct wave functions for proton and neutron, differences shall arise. What it means is that for the proton = [u (1) u(2) d(3)] including colour for the group $SU(6)_{CF}$, given a state say $u_R(1)$, then the corresponding d-quark for the corresponding neutron may exist in any of the states $d_R(1)$, $d_B(1)$, $d_G(1)$. Thus for this group structure it can not be guaranteed that both p and n exist at the same position. This is a major difference with respect to the result in eqn. (18). And thus $\binom{p}{n}$ pair is not a nucleon (i.e. existing at one specific point as in eqn.(18)).

This means that the proton and neutron of a pair are located at different points in this new model. Therefor proton and neutron are not identical and indistiguishable particles here. Thus a nucleus made up of these should be treated as made up of distinguishable and different proton and neutron Fermi seas.

What do the empirical results say? *Answer:* **success** *! phenomenologically this model finds full support in the nucleus.*

In fact, right up to the $\sim 1960's$ most of the nuclear physics models treated protons and neutrons as distinguishable fermions, for e.g. see Blatt and Weisskopf [8].

But as we have discussed above, the $SU(2)_I$ models is today the best and the most successful model of the nucleus. But the earlier results [8], were equally good too. Thus there is actually a duality of models here. Therefor a nucleus can be described well in an $SU(2)_I$ model (where (p-n) are indistinguishable) and in another independent picture where the pair (p-n) is treated as made up of distinguishable fermions. Lawson [9] has shown, in a complete section entitled "Isospin and non-isospin methods of calculation", that these two independent methods yield essentially identical results in the nucleus.

The relationship between the two formalisms here is discussed at many places [8,9,10]. These demonstrate that it is merely a formal requirement to move from one formalism to another. So taking the Pauli Exclusion Principle for the proton and neutron separately in a conventional manner or by requiring antisymmetry under the exchange of two nucleons in isospin formalism, i.e. no matter whether we had $(p-p)$ or $(p-p)$ or $(n-p)$ pairs, we are able to build an antisymmetric wave function from the conventional wave function [8,10]

Thus in our model here, the $SU(2)_I$ forlamism and the anomaly matching (p,n) doublet, give identical results and so represent dual model structures of the nucleus.

There is a subtle difference though. As discussed above, every conventional wave (of distinguishable proton and neutron) function can be generalized to be written in the proper isospin formalism. But the converse does not always hold. Take the case of a simple isospin wave function of the single nucleon,

$$\psi(\vec{r},\chi,\eta) = \phi\,(\vec{r},\chi)\,\frac{1}{\sqrt{2}}\,\{\nu(\eta)+\pi(\eta)\} \tag{24}$$

This function corresponds to a nucleon in the ordinary states $\phi(\vec{r},\chi)$ (notation as in [8])]. However, this nucleon has equal probability of being a proton or a neutron at any particular time. This is the same isospinor described in the title of this paper. This state does not correspond to any physically known states of a proton or a neutron. Thus the isospin formalism provides us with spurious states which do not correspond to any physical reality whatsoever. Hence the need to use proper states in the isospin formalism, which means that we ensure that states with a definite number of protons and neutrons only are constructed.

Thus as both these independent pictures, describe the same reality. Then because of the existence of each other, both these model strutures should provide the same representations. It seems that the anomaly matching (p,n) doublet structure is more basic and thus it enforces its dictat on the $SU(2)_I$ model and thus forbids the above isospinor.. Thus the spinors that may exist in $SU(2)_S$ space, do not have a counterpart as isospinors of $SU(2)_I$ space. This is made possible due to the duality of model structures of the nucleus as shown here. This answers the questions raised in the title of the paper.

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