ON THE STATIONARY ORBITS OF A HYDROGEN-LIKE ATOM

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Abstract: In this work we discuss the possibility of combining the Coulomb potential with the Yukawa's potential to form a mixed potential and then investigate whether this combination can be used to explain why the electron does not radiate when it manifests in the form of circular motions around the nucleus. We show that the mixed Coulomb-Yukawa potential can yield stationary orbits with zero net force, therefore if the electron moves around the nucleus in these orbits it will not radiate according to classical electrodynamics. We also show that in these stationary orbits, the kinetic energy of the electron is converted into potential energy, therefore the radiation process of a hydrogen-like atom does not related to the transition of the electron as a classical particle between the energy levels. The radial distribution functions of the wave equation determine the energy density rather than the electron density at a distance r along a given direction from the nucleus. It is shown in the appendix that the mixed potential used in this work can be derived from Einstein's general theory of relativity by choosing a suitable energy-momentum tensor. Even though such derivation is not essential in our discussions, it shows that there is a possible connection between general relativity and quantum physics at the quantum level.

In quantum mechanics, especially Bohr's model, the Coulomb force is used to deal with a hydrogen-like atom. This proposition leads to the postulate that the electron does not radiate when it is orbiting the nucleus, because according to classical electrodynamics when the electron of a hydrogen-like atom moves in a circular path it must radiate. In this situation a question arises as to whether the Coulomb force is the correct force to use in the description of the dynamics of the electron. Can we assume a different form of force, such as a combination of known forces, in order to resolve this problem? In classical physics, in order to describe an interaction at a distance between objects we normally use a potential. These are the Newtonian potential of gravity, the Coulomb's potential of electrostatic field, the Hooke's potential and a linear potential. In 1935, Yukawa introduced a short-range potential of an exponential form to describe the force between nucleons [1]. However, in quantum field theories, there are other interactions that need to be described by using a potential that is a combination of those elementary potentials. For example, the interaction between quarks in chromodynamics can be represented by a potential, which is the sum of the Coulomb's potential and a linear potential, of the form $V(r) = -\alpha/r + \beta r$, where the first term dominates at short distances and the second term is used to describe confinement at large distances [2]. Following this type of description, in this work we will discuss the possibility of combining the Coulomb potential with the Yukawa's potential to form a mixed potential and investigate whether this combination can be used to explain why the electron does not radiate when it manifests in the form of circular motions around the nucleus. For references, a summary of the postulates of Bohr's model of a hydrogen-like atom may be summarised as follows [3]:

- The centripetal force required for the electron to orbit the nucleus in a stable circle is the Coulomb force $F = kq^2/r^2$. Using Newton's second law, $ma = F$, we obtain
  $$\frac{mv^2}{r} = \frac{kq^2}{r^2} \qquad (1)$$
- The permissible orbits are those that satisfy the condition that the angular momentum of the electron equals $n\hbar$, that is
  $$mvr = n\hbar \qquad (2)$$
  where the number *n* denotes the *n*th orbit.
- When the electron moves in one of the stable orbits it does not radiate. However, it will radiate when it makes a transition between them.

In order to describe the interaction between the electron and the nucleus of a hydrogen-like atom, instead of Bohr's postulate given in Equation (1), we assume a mixed potential of the form [4]

$$V(r) = -\alpha\frac{e^{-\beta r}}{r} + \frac{Q}{r}, \qquad (3)$$

where Q, α and β are constants that will be determined. This type of potential can be derived from Einstein's general relativity by choosing a suitable energy-momentum tensor, as given in the appendix. From Equation (3), we obtain the following equations by differentiation

$$\frac{dV}{dr} = \alpha e^{-\beta r}\left[\frac{\beta}{r} + \frac{1}{r^2}\right] - \frac{Q}{r^2} \qquad (4)$$

$$\frac{d^2V}{dr^2} = -\alpha e^{-\beta r}\left[\frac{\beta^2}{r} + \frac{2\beta}{r^2} + \frac{2}{r^3}\right] + \frac{2Q}{r^3} \qquad (5)$$

From Equation (4), the corresponding force of interaction $F(r) = -dV/dr$ is obtained as

$$F(r) = -\alpha e^{-\beta r}\left[\frac{\beta}{r} + \frac{1}{r^2}\right] + \frac{Q}{r^2} \qquad (6)$$

In order to comply with classical electrodynamics, the net force acting on the electron must be zero when it is circulating in stable orbits. If we assume the net force acting on the electron to vanish when it moves in a stationary orbit of finite radius $R = 1/\beta$, i.e., $F(r) = 0$ at $r = 1/\beta = R$, then from Equation (6) we obtain the relation

$$\beta^2\left(Q - \frac{2\alpha}{e}\right) = 0 \qquad (7)$$

Since $\beta \neq 0$, we have $Q = 2\alpha/e$. The mixed potential given by Equation (3) now takes the form

$$V(r) = -\frac{eQ}{2}\frac{e^{-\frac{r}{R}}}{r} + \frac{Q}{r} \tag{8}$$

And the corresponding force of interaction $F(r) = -\,dV/dr$ is

$$F(r) = -\frac{eQ}{2}e^{-\frac{r}{R}}\left[\frac{1}{Rr} + \frac{1}{r^2}\right] + \frac{Q}{r^2} \tag{9}$$

The constant $R = 1/\beta$ can be assumed to be the radius of a Bohr's stationary orbit. In order to investigate further we need to know the nature of the stationary point at $r = R$. From Equation (5), the second derivative of $V(r)$ at $r = R$ is found as

$$\frac{d^2V}{dr^2} = -\frac{Q}{2R^3} \tag{10}$$

Depending on the signs of Q we have two different cases.

The case $Q > 0$: In this case $d^2V/dr^2 < 0$, therefore *V(r)* has a local maximum at $r = R$. Since $F(r) = -\,dV/dr$, the force is attractive for $r < R$ and repulsive for $r > R$. This situation can be applied to the process of interaction of two protons. We know that at short distances two protons attract each other according to the strong force, but at long distances they repel each other according to the electrostatic law. Therefore at large distances, $r \gg R$, we expect $V(r) = kq^2/r$, where q is the charge and k is the Coulomb constant, the mixed potential takes the form

$$V(r) = -\frac{ekq^2}{2}\frac{e^{-\frac{r}{R}}}{r} + \frac{kq^2}{r} \tag{11}$$

The corresponding force is

$$F(r) = -\frac{ekq^2}{2}e^{-\frac{r}{R}}\left[\frac{1}{Rr} + \frac{1}{r^2}\right] + \frac{kq^2}{r^2} \tag{12}$$

The value of the potential at $r = R$ is

$$V(R) = \frac{1}{2}\frac{kq^2}{R} \tag{13}$$

The equilibrium of this system at $r = R$ is unstable therefore the system is easily broken into separate states that can be described adequately either by the Yukawa potential or the Coulomb potential alone.

The case $Q < 0$: In this case $d^2V/dr^2 > 0$, therefore *V(r)* has a minimum at $r = R$. Since $F(r) = -\,dV/dr$, the force is repulsive for $r < R$ and attractive for $r > R$. This physical process can be used to describe the interaction of the electron and the nucleus of a hydrogen-like atom. We know that at long distances they attract each other according to the electrostatic force, however, at short distances in this case they repel each other. At large distances, $r \gg R$, we expect $V(r) = -kq^2/r$, where q is the charge and k is the Coulomb constant, the mixed potential now takes the form

$$V(r) = \frac{e}{2}\frac{kq^2 e^{-\frac{r}{R}}}{r} - \frac{kq^2}{r} \tag{14}$$

The corresponding force is

$$F(r) = \frac{ekq^2}{2} e^{-\frac{r}{R}} \left[\frac{1}{Rr} + \frac{1}{r^2}\right] - \frac{kq^2}{r^2} \tag{15}$$

The value of the potential energy at $r = R$ is

$$V(R) = -\frac{1}{2}\frac{kq^2}{R} \tag{16}$$

The values of the potential given by Equation (16) are the levels of the total energy of Bohr's model from which the electron is assumed to absorb or emit electromagnetic radiations when it transits from one level to the other, if R is taken to be the Bohr's radius. In our model, however, since the potential alone is equal to the total energy of the Bohr's model, therefore in order to interpret this result we need to assume that there is no kinetic energy associated with the electron when it is in stationary orbits. It seems as though the whole kinetic energy of the electron has been converted into its potential energy when the electron is in there. So what is the state of the electron when it is in a stationary orbit? Is it still a particle? Or has it turned into a wave? Or is it just floating along with some kind of wave that forms the stationary orbits? It should be mentioned here that, as discussed in our other works [5], the state of the electron in a stationary orbit should be a wavelike state. We will show later that this result is also consistent with wave mechanics when Schrödinger wave equation is applied with the mixed potential given by Equation (14). At distances near the stationary orbits, $r \approx R$, Schrödinger wave equation with the mixed potential reduces to that of a free particle.

In order to obtain Bohr's results, the amount of kinetic energy that has been converted into potential energy should be as follows

$$\frac{mv^2}{2} = \frac{1}{2}\frac{kq^2}{R} \tag{17}$$

As in Bohr's model, we also need to use the quantisation of angular momentum. It has been shown in our work on the principle of least action [6], the quantisation of angular momentum has a topological characteristic that is represented in the form

$$\oint pds = \hbar \oint \kappa ds = \hbar \oint \frac{ds}{r} = \hbar \oint d\theta = nh \tag{18}$$

where κ is the curvature of the path of a particle and n is the winding number of the homotopy group. The relationship between the momentum of a particle and the curvature of its path can be established by applying the Frenet equations in differential geometry and de

Broglie's wavelength of the particle [6]. From Equation (18), for a Bohr's stationary orbit of radius R_n we have

$$mvR_n = n\hbar \tag{19}$$

Using Equations (17) and (19) we then obtain Bohr's results as follows

$$R_n = \frac{\hbar^2 n^2}{mkq^2} \qquad \text{and} \qquad E_n = V(R_n) = -\frac{mk^2q^4}{2\hbar^2}\frac{1}{n^2} \tag{20}$$

In the following, we will discuss further the application of the mixed potential to a hydrogen-like atom in terms of wave mechanics. Since the system of a hydrogen-like atom involves charges, as in the case of Bohr's quantisation of angular momentum that has a topological characteristic, we want to mention here that the quantisation of charge also has a topological feature [6]. This distinctive aspect can be established by identifying Gaussian curvature of a surface with charge density in Gauss's law in classical electrodynamics. Topological methods have been widely used in recent developments of quantum physics. For example, topological concepts and formulations play fundamental roles in topological quantum field theories [7,8] and cobordism relations in algebraic topology are fundamental to theories of quantum gravity [9].

With the potential given in Equation (14), the one-body time-independent Schrödinger equation describing the relative motion is

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\mathbf{r}) + \left(\frac{e}{2}\frac{kq^2e^{-\frac{r}{R}}}{r} - \frac{kq^2}{r}\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{21}$$

where $\mu = mM/(m+M)$ is the reduced mass of the electron and the nucleus. As discussed above, Equation (21) can be interpreted as a wave equation for the description of the dynamics of energy rather than the dynamics of the electron. In this case Equation (21) has the status of Maxwell's equations in classical electrodynamics. Since the mixed Coulomb-Yukawa potential is also spherically symmetric, Equation (21) can be written in the spherical polar coordinates as

$$-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\mathbf{L}^2}{\hbar^2 r^2}\right)\psi(\mathbf{r}) + \left(\frac{e}{2}\frac{kq^2e^{-\frac{r}{R}}}{r} - \frac{kq^2}{r}\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{22}$$

where the orbital angular momentum operator $\mathbf{L}^2$ is given by

$$\mathbf{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right) \tag{23}$$

Solutions of Equation (22) can be found using the separable form

$$\psi_{El}(\mathbf{r}) = R_{El}(r)Y_{lm}(\theta,\phi) \tag{24}$$

where R_{El} is a radial function and Y_{lm} is the spherical harmonic. Applying Equation (24), Equation (22) is reduced to the system of equations

$$\mathbf{L}^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi) \tag{25}$$

$$\left(-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)\hbar^2}{2\mu r^2}+\frac{e}{2}\frac{kq^2 e^{-\frac{r}{R}}}{r}-\frac{kq^2}{r}\right)R_{El}(r) = ER_{El}(r) \tag{26}$$

Even though Equations (26) cannot be solved completely to obtain exact solutions that can be used to describe the physical dynamics of a hydrogen-like atom, it is possible to suggest what type of solutions Equation (26) should admit by considering it for extreme cases with $r \gg R$ and $r \approx R$. First consider the case when $r \gg R$. In this case, the Yukawa term can be ignored and Equation (26) reduces to Schrödinger wave equation with the Coulomb potential

$$\left(-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)\hbar^2}{2\mu r^2}-\frac{kq^2}{r}\right)R_{El}(r) = ER_{El}(r) \tag{27}$$

The normalised radial eigenfunctions of the bound states can be found as [10]

$$R_{nl}(r) = -\left(\left(\frac{2k\mu q^2}{\hbar^2 n}\right)^3 \frac{(n-l-1)!}{2n\big((n+l)!\big)^3}\right)^{1/2} e^{-\rho/2}\rho^l L_{n+l}^{2l+1}(\rho) \tag{28}$$

where $\rho = (-8\mu E/\hbar^2)^{1/2} r$ and $L_{n+l}^{2l+1}(\rho)$ is the associated Laguerre polynomial with the bound state energy eigenvalues given by

$$E_n = -\frac{\mu k^2 q^4}{2\hbar^2}\frac{1}{n^2} \tag{29}$$

Despite the results obtained for the case $r \gg R$ are the same as that of Schrödinger wave equation with the Coulomb potential, the interpretation of what the wavefunctions represent in this case may be different. As we noticed before, in the bound states of the hydrogen-like atom, the kinetic energy of the electron is converted to potential energy, therefore the radiation process of a hydrogen-like atom does not related to the transition of the electron as a classical particle between energy levels when the system absorbs or emits a photon, but is due to the form of the potential. This may be considered as the Aharonov-Bohm effect [11]. The radial distribution function $r^2|R_{nl}(r)|^2$ determines the energy density rather than the electron density at a distance r along a given direction from the nucleus.

Now consider the case when $r \approx R$. This is the case when the electron is in a region when the net force acting on it is negligible. Using first order expansion for the Yukawa potential at $r = R$, Equation (26) becomes

$$\left(-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)\hbar^2}{2\mu r^2}-\frac{kq^2}{2R}\right)R_{El}(r) = ER_{El}(r) \tag{30}$$

As expected, the radial equation given by Equation (30) is that of a free-particle in spherical polar coordinates, except for the constant term $-kq^2/2R$. The radial eigenfunction can be found as [10]

$$R_{El}(r) = Cj_l(\alpha r) \tag{31}$$

where $\alpha = \sqrt{2\mu(E + kq^2/(2R))/\hbar^2}$, C is a constant and $j_l(\alpha r)$ is the spherical Bessel functions. However, unlike the case of $r \gg R$ in which the energy spectrum contains an infinite number of discrete energy levels, the energy spectrum at $r \approx R$ is continuous.

From the above considerations for extreme cases, it is possible to suggest that the exact solutions to Equation (26) should be of such a form, solutions given by Equations (28) and (31) are their asymptotic solutions. At near stationary orbits $r \approx R$ the exact solutions will approach the spherical Bessel solutions $R_{El}(r) \to Cj_l(\alpha r)$ and at large distances $r \gg R$ they will approach solutions $R_{nl}(r) \to -\left(\left(\frac{2k\mu q^2}{\hbar^2 n}\right)^3 \frac{(n-l-1)!}{2n\left((n+l)!\right)^3}\right)^{1/2} e^{-\rho/2}\rho^l L_{n+l}^{2l+1}(\rho)$. Furthermore, the energy of these solutions also needs to change from a continuous spectrum to a discrete spectrum.

Appendix: A line element of the Yukawa potential

The type of potential given in Equation (3) can be derived from Einstein's general theory of relativity by choosing a suitable energy-momentum tensor. The field equations of general relativity are [12]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{32}$$

Assuming a centrally symmetric field, the space-time metric can be written as [13]

$$ds^2 = e^{\psi} c^2 dt^2 - e^{\chi} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{33}$$

Denoting the coordinates (ct, r, θ, ϕ) by $\{x^\mu\}$ $(\mu = 0,1,2,3)$, the metric tensor $g_{\mu\nu}$ of this line element is

$$g_{\mu\nu} = \begin{pmatrix} e^{\psi} & 0 & 0 & 0 \\ 0 & -e^{\chi} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{34}$$

In terms of the metric $g_{\mu\nu}$ given in Equation (34), the non-zero components of the connection

$$\Gamma_{\mu\nu}^{\sigma} = \frac{1}{2} g^{\sigma\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\mu g_{\mu\lambda} - \partial_\mu g_{\mu\nu}\right) \tag{35}$$

are found as

$$\Gamma_{00}^0 = \frac{1}{2}\frac{\partial \psi}{\partial t}, \quad \Gamma_{00}^1 = \frac{e^{\psi-\chi}}{2}\frac{\partial \psi}{\partial r}, \quad \Gamma_{01}^0 = \frac{1}{2}\frac{\partial \psi}{\partial r}, \quad \Gamma_{01}^1 = \frac{1}{2}\frac{\partial \chi}{\partial t}$$

$$\Gamma_{11}^0 = \frac{e^{\psi-\chi}}{2}\frac{\partial \chi}{\partial t}, \quad \Gamma_{11}^1 = \frac{1}{2}\frac{\partial \chi}{\partial r}, \quad \Gamma_{12}^2 = \frac{1}{r}, \quad \Gamma_{13}^3 = \frac{1}{r}, \quad \Gamma_{23}^3 = \cot\theta$$

$$\Gamma_{22}^1 = -re^{-\chi}, \quad \Gamma_{33}^1 = -r\sin^2\theta e^{-\chi}, \quad \Gamma_{33}^2 = -\sin\theta\cos\theta \tag{36}$$

With the line element in Equation (33), and $\Lambda \equiv 0$, the vacuum solutions satisfy the following system of equations

$$\frac{\partial \psi}{\partial r} + \frac{1}{r} - \frac{e^{\chi}}{r} = 0$$

$$\frac{\partial \chi}{\partial r} - \frac{1}{r} + \frac{e^{\chi}}{r} = 0$$

$$\frac{\partial \chi}{\partial t} = 0$$

$$2\frac{\partial^2 \psi}{\partial r^2} + \left(\frac{\partial \psi}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial \psi}{\partial r} - \frac{\partial \chi}{\partial r}\right) - \frac{\partial \psi}{\partial r}\frac{\partial \chi}{\partial r} - e^{\psi-\chi}\left(2\frac{\partial^2 \chi}{\partial t^2} + \left(\frac{\partial \chi}{\partial t}\right)^2 - \frac{\partial \psi}{\partial t}\frac{\partial \chi}{\partial t}\right) = 0 \tag{37}$$

These equations are not independent, since it can be verified that the last equation follows from the first three equations. Furthermore, the first two equations result in $\partial\psi/\partial r + \partial\chi/\partial r = 0$, which leads to $\psi + \chi = 0$, due to the possibility of an arbitrary transformation of the time coordinate. The system of equations (37) admits the Schwarzschild line element which can be used to describe the massless force carriers of the gravitational field. In this work, however, we need a non-vacuum solution. In order to find an appropriate solution to describe the potential required, an energy-momentum tensor must be specified. Obviously, the present state of atomic physics does not allow us to specify a precise form for this kind of energy-momentum tensor. In this situation, it is appropriate to construct an energy-momentum tensor so that it not only gives rise to an exact solution with the desired metric of the Yukawa form, but also satisfies the conservation law $T_{\mu;\nu}^{\nu} = 0$. As an illustration, we consider in the following form of the energy-momentum tensor

$$T_\mu^\nu = \begin{pmatrix} -\frac{\alpha\beta}{\kappa}\frac{e^{-\beta r}}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{\alpha\beta}{\kappa}\frac{e^{-\beta r}}{r^2} & 0 & 0 \\ 0 & 0 & \frac{\alpha\beta^2}{2\kappa}\frac{e^{-\beta r}}{r} & 0 \\ 0 & 0 & 0 & \frac{\alpha\beta^2}{2\kappa}\frac{e^{-\beta r}}{r} \end{pmatrix} \tag{38}$$

where the constants α and β will be needed to be specified. It can be verified that the energy-momentum tensor given in Equation (38) satisfies the conservation law

$$\nabla_\nu T^\nu_\mu = \frac{1}{\sqrt{-g}}\frac{\partial T^\nu_\mu \sqrt{-g}}{\partial x^\nu} - \frac{1}{2}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu} T^{\lambda\sigma} = 0 \tag{39}$$

It can also be shown that this tensor admits a line element of the Yukawa potential as an exact solution to the field equations of general relativity. With the energy-momentum tensor given in Equation (38) together with the metric tensor (34), the field equations of general relativity reduce to the following system of equations

$$e^{-\chi}\left(\frac{\partial \chi}{\partial r} - \frac{1}{r}\right) + \frac{1}{r} = -\alpha\beta\frac{e^{-\beta r}}{r}$$

$$e^{-\chi}\left(\frac{\partial \psi}{\partial r} - \frac{1}{r}\right) + \frac{1}{r} = -\alpha\beta\frac{e^{-\beta r}}{r}$$

$$\frac{\partial \chi}{\partial t} = 0$$

$$-e^{\chi}\left(2\frac{\partial^2 \psi}{\partial r^2} + \left(\frac{\partial \psi}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial \psi}{\partial r} - \frac{\partial \chi}{\partial r}\right) - \frac{\partial \psi}{\partial r}\frac{\partial \chi}{\partial r}\right) + e^{\psi}\left(2\frac{\partial^2 \chi}{\partial t^2} + \left(\frac{\partial \chi}{\partial t}\right)^2 - \frac{\partial \psi}{\partial t}\frac{\partial \chi}{\partial t}\right)$$
$$= \alpha\beta\frac{e^{-\beta r}}{r} \tag{40}$$

The system of equations given in Equation (40) when integrated gives a metric of the form

$$e^{-\chi} = 1 - \alpha\frac{e^{-\beta r}}{r} + \frac{Q}{r} \tag{41}$$

where Q is a constant of integration. The term Q/r can be interpreted as Coulomb potential and the term $-\alpha e^{-\beta r}/r$ can be interpreted as Yukawa potential. This result leads to the conclusion that by specifying an appropriate matter source, it is possible to consider the atomic interaction of a hydrogen-like atom as a manifestation of general relativity at short range.

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