Double Conformal Space-Time Algebra for General Quadric Surfaces in Space-Time

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Abstract

The $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) is a high-dimensional 12D Geometric Algebra that extends the concepts introduced with the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) with entities for Darboux cyclides (incl. parabolic and Dupin cyclides, general quadrics, and ring torus) in spacetime with a new boost operator. The base algebra in which spacetime geometry is modeled is the $\mathcal{G}_{1,3}$ Space-Time Algebra (STA). Two $\mathcal{G}_{2,4}$ Conformal Space-Time subalgebras (CSTA) provide spacetime entities for points, hypercones, hyperplanes, hyperpseudospheres (and their intersections) and a complete set of versors for their spacetime transformations that includes rotation, translation, isotropic dilation, hyperbolic rotation (boost), planar reflection, and (pseudo)spherical inversion. $\mathcal{G}_{4,8}$ DCSTA is a doubling product of two orthogonal $\mathcal{G}_{2,4}$ CSTA subalgebras that inherits doubled CSTA entities and versors from CSTA and adds new 2-vector entities for general (pseudo)quadrics and Darboux (pseudo)cyclides in spacetime that are also transformed by the doubled versors. The "pseudo" surface entities are spacetime surface entities that use the time axis as a pseudospatial dimension. The (pseudo)cyclides are the inversions of (pseudo)quadrics in hyperpseudospheres. An operation for the directed non-uniform scaling (anisotropic dilation) of the 2-vector general quadric entities is defined using the boost operator and a spatial projection. Quadric surface entities can be boosted into moving surfaces with constant velocities that display the Thomas-Wigner rotation and length contraction of special relativity. DCSTA is an algebra for computing with general quadrics and their inversive geometry in spacetime. For applications or testing, $\mathcal{G}_{4,8}$ DCSTA can be computed using various software packages, such as the symbolic computer algebra system *SymPy* with the $\mathcal{G}$*Algebra* module.

Keywords: Clifford algebra; conformal geometric algebra; space-time algebra

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1 Introduction

This is an extended paper[1] on the Double Conformal Space-Time Algebra [11] (DCSTA) $\mathcal{G}_{4,8}$ [7]. DCSTA $\mathcal{G}_{4,8}$ is a high-dimensional 12D Geometric Algebra [3][15][17] over the twelve-dimensional (12D) vector space $\mathbb{R}^{4,8}$ that extends the concepts introduced with the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) $\mathcal{G}_{8,2}$ [4][5][6][8][9][10] with entities for Darboux cyclides (incl. parabolic and Dupin cyclides, general quadrics, and ring torus) in spacetime with a new boost operator.

The base algebra in which spacetime geometry is modeled is the Space-Time Algebra (STA) $\mathcal{G}_{1,3}$ [14]. Two orthogonal, and isomorphic, Conformal Space-Time subalgebras (CSTA) $\mathcal{G}_{(1+1),(3+1)}$ [2] provide spacetime entities for points, hypercones, hyperplanes, and hyperpseudospheres (and their intersections) and a complete set of versors for their spacetime transformations that includes rotation, translation, isotropic dilation, hyperbolic rotation (boost), planar reflection, and (pseudo)spherical inversion.

The double CSTA (DCSTA) $\mathcal{G}_{4,8}$ is a doubling product of two orthogonal CSTA subalgebras $\mathcal{G}_{2,4}$ that inherits doubled CSTA entities and versors from CSTA and adds new bivector entities for general (pseudo)quadrics and Darboux (pseudo)cyclides in space-time that are also transformed by the doubled versors. The "pseudo" surface entities are spacetime surface entities that use the time axis as a pseudospatial dimension. The (pseudo)cyclides are the inversions of (pseudo)quadrics in hyperpseudospheres.

1. Revised version v4 (viXra), *26 Jan 2019* (original v1 is dated *22 Aug 2017*). This paper was written as an invited extended paper, extending on the published 10-page conference proceedings paper *Double Conformal Geometric Algebra* (DOI:10.1063/1.4972658), for possible later submission to the journal *Mathematics in Engineering, Science and Aerospace (MESA)*. This is a working paper that has not been peer-reviewed, may have errors, and may have later revisions.

DCSTA allows general quadric surfaces to be transformed in spacetime by a complete set of doubled CSTA versor (i.e., DCSTA versor) operations. General quadric surface entities can be boosted into moving surfaces with constant velocities that display the Thomas-Wigner rotation and length contraction of special relativity. DCSTA also defines an operation for the directed non-uniform scaling (anisotropic dilation) of the bivector general quadric entities using the boost operator followed by a spatial projection.

As will be explained further in more detail, the new DCSTA bivector entities for quadrics and Darboux cyclides are formed by extracting polynomial terms from the coefficients on the basis 2-blade terms of the DCSTA 2-blade point entity using reciprocal (or pseudoinverse) basis 2-blades as extraction operators. The reciprocal basis 2-blades that extract the same polynomial term s from the DCSTA point entity are added and averaged as the DCSTA 2-vector extraction operator for value s.

The DCSTA $\mathcal{G}_{4,8}$ $\mathcal{D}$ has a basis of twelve orthonormal vector elements $\mathbf{e}_i$, $1 \leq i \leq 12$, with metric (squares or signatures) $m_{\mathcal{D}}$:

$$\begin{aligned} m = m_{\mathcal{D}} &= \operatorname{diag}(1,-1,-1,-1,1,-1,1,-1,-1,-1,1,-1) = [m_{ij}] && (1)\\ &= \operatorname{diag}(m_{\mathcal{C}^1}, m_{\mathcal{C}^2}) = \operatorname{diag}(1, m_{\mathcal{CS}^1}, 1, m_{\mathcal{CS}^2}) && (2)\\ &= \operatorname{diag}(m_{\mathcal{M}^1}, 1, -1, m_{\mathcal{M}^2}, 1, -1) = \operatorname{diag}(1, m_{\mathcal{S}^1}, 1, -1, 1, m_{\mathcal{S}^2}, 1, -1) && (3)\\ m_{\mathcal{DS}} &= \operatorname{diag}(m_{\mathcal{CS}^1}, m_{\mathcal{CS}^2}), \qquad m_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. && (4) \end{aligned}$$

The above metric also includes the metrics of the subalgebras:

- $\mathcal{G}_{2,4}$ CSTA1 $\mathcal{C}^1$: $m_{\mathcal{C}^1}$
- $\mathcal{G}_{1,4}$ Conformal SA1 (CSA1) $\mathcal{CS}^1$: $m_{\mathcal{CS}^1}$
- $\mathcal{G}_{1,3}$ STA1 $\mathcal{M}^1$: $m_{\mathcal{M}^1}$
- $\mathcal{G}_{0,3}$ Space Algebra 1 (SA1) $\mathcal{S}^1$: $m_{\mathcal{S}^1}$
- $\mathcal{G}_{2,8}$ Double Conformal SA (DCSA) $\mathcal{DS}$: $m_{\mathcal{DS}}$
- $\mathcal{G}_{2,4}$ CSTA2 $\mathcal{C}^2$: $m_{\mathcal{C}^2}$
- $\mathcal{G}_{1,4}$ CSA2 $\mathcal{CS}^2$: $m_{\mathcal{CS}^2}$
- $\mathcal{G}_{1,3}$ STA2 $\mathcal{M}^2$: $m_{\mathcal{M}^2}$
- $\mathcal{G}_{0,3}$ SA2 $\mathcal{S}^2$: $m_{\mathcal{S}^2}$

2 Notation of Space-Time Algebra (STA)

The basis of the space-time algebra $\mathcal{G}_{1,3}$ STA $\mathcal{M} \cong \mathcal{G}_{1,3}$ STA1 $\mathcal{M}^1$ is $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \cong \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, and for the second copy of the space-time algebra $\mathcal{G}_{1,3}$ STA2 $\mathcal{M}^2$ we have the basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \cong \{\mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_9, \mathbf{e}_{10}\}$. The space algebra $\mathcal{G}_{0,3}$ SA $\mathcal{S}$ basis, included in the space-time algebra, is $\{\gamma_1, \gamma_2, \gamma_3\}$. The STA unit four-dimensional pseudoscalar is $\mathbf{I}_{\mathcal{M}} = \gamma_0\gamma_1\gamma_2\gamma_3$, and for SA the unit three-dimensional pseudoscalar is $\mathbf{I}_{\mathcal{S}} = \gamma_1\gamma_2\gamma_3$. Moreover, STA defines a symbolic space-time "test" position with symbolic coordinates $(w = ct, x, y, z)$ by the four-dimensional (4D) vector

$$\boldsymbol{t} = \boldsymbol{t}_{\mathcal{M}} = w\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 = w\gamma_0 + \mathbf{t}_{\mathcal{S}}, \tag{5}$$

a specific space-time position with specific coordinates (p_w, p_x, p_y, p_z) by the 4D vector

$$\boldsymbol{p} = \boldsymbol{p}_{\mathcal{M}} = p_w\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 = p_w\gamma_0 + \mathbf{p}_{\mathcal{S}}, \tag{6}$$

and a 4D space-time velocity

$$\boldsymbol{v} = \boldsymbol{v}_{\mathcal{M}} = c\gamma_0 + v_x\gamma_1 + v_y\gamma_2 + v_z\gamma_3 = c\gamma_0 + \mathbf{v}_{\mathcal{S}}, \tag{7}$$

with 4D STA vectors in ***bold italic***, and 3D SA spatial $v_x\gamma_1 + v_y\gamma_2 + v_z\gamma_3$ vectors $\mathbf{v} = \mathbf{v}_{\mathcal{S}}$ in **bold**. An algebra symbol $\mathcal{S}$ as a subscript of an element $A_{\mathcal{S}}$ indicates that $A_{\mathcal{S}} \in \mathcal{S}$, and similarly for the other algebra symbols.

The *geometric product* of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\boldsymbol{u}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}, \tag{8}$$

where the *inner product* is the symmetric product

$$\boldsymbol{u} \cdot \boldsymbol{v} = (\boldsymbol{u}\boldsymbol{v} + \boldsymbol{v}\boldsymbol{u})/2 \tag{9}$$

and the *outer product* is the anti-symmetric product

$$\boldsymbol{u}\wedge\boldsymbol{v}=(\boldsymbol{u}\boldsymbol{v}-\boldsymbol{v}\boldsymbol{u})/2. \tag{10}$$

We will make frequent use of the square of $\boldsymbol{t}_{\mathcal{M}}$, known as the space-time *interval*,

$$\boldsymbol{t}_{\mathcal{M}}^2=w^2-x^2-y^2-z^2. \tag{11}$$

A vector $\boldsymbol{t}_{\mathcal{M}}$ is *time-like* for $\boldsymbol{t}_{\mathcal{M}}^2>0$, *space-like* for $\boldsymbol{t}_{\mathcal{M}}^2<0$, and *null* (or *light-like*) for $\boldsymbol{t}_{\mathcal{M}}^2=0$. The *interval* $\boldsymbol{a}^2$ is often used to avoid imaginary numbers. In (138), the interval $\boldsymbol{a}^2$ is the square of a hyperbolic radius $r^2=\boldsymbol{a}^2$. A null radius $r=0$ is associated with a null hypercone. A real radius $r>0$, for $\boldsymbol{a}^2>0$, is associated with a (hyper)hyperboloid of one sheet. An imaginary $r=|r|\sqrt{-1}$, or $\boldsymbol{a}^2<0$, is associated with a (hyper)hyperboloid of two sheets. The (hyper)hyperboloids are circular and are also called real or imaginary hyperpseudospheres, respectively.

A *null vector* $\boldsymbol{a}^2=(\boldsymbol{s}+\boldsymbol{t})^2=0$ is the sum of space-like $\boldsymbol{s}^2<0$ and time-like $\boldsymbol{t}^2>0$ vectors that are orthogonal $2\boldsymbol{s}\cdot\boldsymbol{t}=\boldsymbol{s}\boldsymbol{t}+\boldsymbol{t}\boldsymbol{s}=0$ and of equal magnitude $|\boldsymbol{t}|=|\boldsymbol{s}|$ (31), where $\boldsymbol{a}^2=\boldsymbol{s}^2+\boldsymbol{t}^2=|\boldsymbol{t}|^2-|\boldsymbol{s}|^2=0$. A *non-null* vector $\boldsymbol{a}$ has a non-zero interval $\boldsymbol{a}^2\neq 0$ and has an inverse $\boldsymbol{a}^{-1}=\boldsymbol{a}/\boldsymbol{a}^2$.

The vector projections [15] of any vector $\boldsymbol{u}=\boldsymbol{u}^{\|\boldsymbol{v}}+\boldsymbol{u}^{\perp\boldsymbol{v}}$ parallel $\boldsymbol{u}^{\|\boldsymbol{v}}$ and perpendicular $\boldsymbol{u}^{\perp\boldsymbol{v}}$ to any *non-null* vector $\boldsymbol{v}$ is defined by

$$\boldsymbol{u}=(\boldsymbol{u}\boldsymbol{v})\boldsymbol{v}^{-1}=(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}^{-1}+(\boldsymbol{u}\wedge\boldsymbol{v})\boldsymbol{v}^{-1}=\boldsymbol{u}^{\|\boldsymbol{v}}+\boldsymbol{u}^{\perp\boldsymbol{v}}=\mathcal{P}_{\boldsymbol{v}}(\boldsymbol{u})+\mathcal{P}_{\boldsymbol{v}}^{\perp}(\boldsymbol{u}). \tag{12}$$

The untranslated (at origin) *observer* worldline, in the rest frame of the observer, is

$$\boldsymbol{o}t=ct\boldsymbol{\gamma}_0 \tag{13}$$

with *proper time* (coordinate time) t and *light speed* c. See also, the CSTA line entity $\boldsymbol{L}_{\mathcal{C}}$ of (151).

The SA *spatial dualization* of SA spatial vector $\mathbf{n}$ is

$$\mathbf{n}_{\mathcal{S}}^*=-\mathbf{n}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1}. \tag{14}$$

For an SA spatial *unit vector* rotation axis $\hat{\mathbf{n}}$, the unit directional 2-blade $\hat{\mathbf{n}}^*$ of the rotation plane is isomorphic to a pure unit quaternion, where $(\hat{\mathbf{n}}^*)^2=-1$. Using (14), the correspondence with unit quaternions is $\{\mathbf{i},\mathbf{j},\mathbf{k}\}\cong\{\boldsymbol{\gamma}_1^*,\boldsymbol{\gamma}_2^*,\boldsymbol{\gamma}_3^*\}$.

The STA *space-time dualization* of STA space-time vector $\boldsymbol{v}_{\mathcal{M}}$ is

$$\boldsymbol{v}_{\mathcal{M}}^*=\boldsymbol{v}_{\mathcal{M}}\mathbf{I}_{\mathcal{M}}^{-1}. \tag{15}$$

The vector (1-blade) *conjugate* $\boldsymbol{a}^\dagger$ [18] of any vector $\boldsymbol{a}$, written using Einstein notation as

$$\boldsymbol{a}=a^i\mathbf{e}_i=\left(\sum_{i=1}^{p}a^i\mathbf{e}_i+\sum_{i=p+1}^{p+q}a^i\mathbf{e}_i\right)\in\mathcal{G}_{p,q}^1 \tag{16}$$

on the standard orthonormal basis of vectors $\{\ \mathbf{e}_i\ :\ 1\leq i\leq p+q\ \}$ having pseudo-Euclidean signature $(p,\ q)$ with Euclidean signatures $\left\{\ \mathbf{e}_i\ :\ \mathbf{e}_i^2=1, 1\leq i\leq p\ \right\}$ and anti-Euclidean signatures $\left\{\ \mathbf{e}_i\ :\ \mathbf{e}_i^2=-1, p+1\leq i\leq p+q\ \right\}$, is

$$\boldsymbol{a}^\dagger=\sum_{i=1}^{p}a^i\mathbf{e}_i-\sum_{i=p+1}^{p+q}a^i\mathbf{e}_i, \tag{17}$$

such that all of the anti-Euclidean basis vector terms are multiplied by -1. For any STA vector (1-blade) $\boldsymbol{a}=a_w\boldsymbol{\gamma}_0+\mathbf{a}=a_w\boldsymbol{\gamma}_0+a_x\boldsymbol{\gamma}_1+a_y\boldsymbol{\gamma}_2+a_z\boldsymbol{\gamma}_3\in\mathcal{G}_{1,3}^1$, its conjugate is (changing the sign of the spatial component)

$$\boldsymbol{a}^\dagger=\boldsymbol{\gamma}_0\boldsymbol{a}\boldsymbol{\gamma}_0=a_w\boldsymbol{\gamma}_0-\mathbf{a}=a_w\boldsymbol{\gamma}_0-a_x\boldsymbol{\gamma}_1-a_y\boldsymbol{\gamma}_2-a_z\boldsymbol{\gamma}_3. \tag{18}$$

A *k-blade* $\boldsymbol{A}_{\langle k\rangle}$, of grade k denoted by subscript $\langle k\rangle$ [18], is the outer product of k vectors $\boldsymbol{a}_i$,

$$\boldsymbol{A}_{\langle k\rangle}=\bigwedge\boldsymbol{a}_i=\bigwedge_{i=1}^{k}\boldsymbol{a}_i=\boldsymbol{a}_1\wedge\boldsymbol{a}_2\wedge\ldots\wedge\boldsymbol{a}_k. \tag{19}$$

A scalar a is also called a 0-blade. A k-vector $A_{\langle k \rangle}$, often denoted A_k, is a sum of k-blades. A multivector A is a sum of k-vectors of various grades k. The *reverse* $A^\sim$ of any multivector A reverses the products of all vectors in A (e.g., $\mathbf{I}_{\mathcal{M}}^\sim = \gamma_3\gamma_2\gamma_1\gamma_0$). The reverse of a k-blade $\boldsymbol{A}_{\langle k \rangle}$ is

$$\boldsymbol{A}_{\langle k \rangle}^\sim = (-1)^{k(k-1)/2} \boldsymbol{A}_{\langle k \rangle} = \boldsymbol{a}_k \wedge \boldsymbol{a}_{k-1} \wedge \ldots \wedge \boldsymbol{a}_1. \tag{20}$$

The k-blade *conjugate* $\boldsymbol{A}_{\langle k \rangle}^\dagger$ [18] of any k-blade $\boldsymbol{A}_{\langle k \rangle}$ is (n.b. the reverse order)

$$\boldsymbol{A}_{\langle k \rangle}^\dagger = \bigwedge_{i=1}^{k} \boldsymbol{a}_{k+1-i}^\dagger = \boldsymbol{a}_k^\dagger \wedge \boldsymbol{a}_{k-1}^\dagger \wedge \ldots \wedge \boldsymbol{a}_1^\dagger. \tag{21}$$

For any STA k-blade $\boldsymbol{A}_{\langle k \rangle} = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \ldots \wedge \boldsymbol{a}_k \in \mathcal{G}_{1,3}^k$, $1 \leq k \leq 4$, its conjugate is (a composition of reversion $\boldsymbol{A}_{\langle k \rangle}^\sim$ with sandwiching between $\boldsymbol{\gamma}_0$ factors)

$$\boldsymbol{A}_{\langle k \rangle}^\dagger = \boldsymbol{\gamma}_0 \boldsymbol{A}_{\langle k \rangle}^\sim \boldsymbol{\gamma}_0 = \boldsymbol{a}_k^\dagger \wedge \boldsymbol{a}_{k-1}^\dagger \wedge \ldots \wedge \boldsymbol{a}_1^\dagger. \tag{22}$$

The *Euclidean norm* (l_2-norm [16]) $\|\boldsymbol{a}\|_2$ of any STA vector $\boldsymbol{a}$ is

$$\|\boldsymbol{a}\|_2 = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}^\dagger} \geq 0, \tag{23}$$

which is positive or zero. Similarly, the Euclidean norm $\|\boldsymbol{A}_{\langle k \rangle}\|_2$ of a k-blade $\boldsymbol{A}_{\langle k \rangle}$ is

$$\|\boldsymbol{A}_{\langle k \rangle}\|_2 = \sqrt{\boldsymbol{A}_{\langle k \rangle} \cdot \boldsymbol{A}_{\langle k \rangle}^\dagger} \geq 0. \tag{24}$$

The *Euclidean normalization* of any (but restricting to *null*) STA vector $\boldsymbol{a}$ is the Euclidean *normalized vector*

$$\hat{\boldsymbol{a}} = \boldsymbol{a} / \|\boldsymbol{a}\|_2, \tag{25}$$

where $\|\hat{\boldsymbol{a}}\|_2 = 1$ is unit Euclidean norm. Although the Euclidean normalization (25) is defined for *any* STA vector $\boldsymbol{a}$, in this paper we restrict the Euclidean normalization to STA *null* vectors $\boldsymbol{a}$, where $\boldsymbol{a}^2 = 0$. For any STA *non-null* vector $\boldsymbol{a}$, we define $\hat{\boldsymbol{a}}$ by the pseudo-Euclidean normalization (30) as a *unit vector*, where $\hat{\boldsymbol{a}}^2 = \pm 1$.

A *null* vector (1-blade) $\boldsymbol{a}$ has a zero interval $\boldsymbol{a}^2 = 0$ and no inverse, but has a *pseudoinverse* [18]

$$\boldsymbol{a}^+ = \boldsymbol{a}^\dagger / \|\boldsymbol{a}\|_2^2, \tag{26}$$

where $\boldsymbol{a} \cdot \boldsymbol{a}^+ = 1$.

For any k-blade $\boldsymbol{A}_{\langle k \rangle}$, its *pseudoinverse* [18] is

$$\boldsymbol{A}_{\langle k \rangle}^+ = \boldsymbol{A}_{\langle k \rangle}^\dagger / \|\boldsymbol{A}_{\langle k \rangle}\|_2^2 = \boldsymbol{A}_{\langle k \rangle}^\dagger / (\boldsymbol{A}_{\langle k \rangle} \cdot \boldsymbol{A}_{\langle k \rangle}^\dagger), \tag{27}$$

where $\boldsymbol{A}_{\langle k \rangle} \cdot \boldsymbol{A}_{\langle k \rangle}^+ = 1$.

For any *non-null* k-blade $\boldsymbol{A}_{\langle k \rangle}$, its *inverse* is

$$\boldsymbol{A}_{\langle k \rangle}^{-1} = \boldsymbol{A}_{\langle k \rangle}^\sim / (\boldsymbol{A}_{\langle k \rangle} \boldsymbol{A}_{\langle k \rangle}^\sim) = \boldsymbol{A}_{\langle k \rangle} / \boldsymbol{A}_{\langle k \rangle}^2, \tag{28}$$

where $\boldsymbol{A}_{\langle k \rangle} \boldsymbol{A}_{\langle k \rangle}^{-1} = 1$. A null k-blade has no inverse, but has a pseudoinverse.

The *pseudo-Euclidean norm* (or *seminorm*) $\|\boldsymbol{a}\|$ of any STA vector $\boldsymbol{a}$ is

$$\|\boldsymbol{a}\| = \sqrt{|\boldsymbol{a}^2|} = \sqrt{|\boldsymbol{a} \cdot \boldsymbol{a}|} \geq 0, \tag{29}$$

and the *pseudo-Euclidean normalization* of any non-null vector $\boldsymbol{a}$ is the *unit vector*

$$\hat{\boldsymbol{a}} = \boldsymbol{a} / \|\boldsymbol{a}\|, \tag{30}$$

such that if $\boldsymbol{a}$ is time-like ($\boldsymbol{a}^2 > 0$) then $\hat{\boldsymbol{a}}^2 = 1$, and if $\boldsymbol{a}$ is space-like ($\boldsymbol{a}^2 < 0$) then $\hat{\boldsymbol{a}}^2 = -1$. For a *null* (light-like) vector $\boldsymbol{a}$, the notation $\hat{\boldsymbol{a}}$ is the Euclidean (l_2) normalization $\hat{\boldsymbol{a}} = \boldsymbol{a} / \|\boldsymbol{a}\|_2$ (25). The pseudo-Euclidean norm $\|\boldsymbol{a}\|$ is a seminorm since $\|\boldsymbol{a}\| = 0$ for a null vector $\boldsymbol{a} \neq 0$ [16].

The pseudo-Euclidean norm $\|\boldsymbol{a}\|$ (29) is equivalent to hyperbolic *modulus* [21] (or magnitude)

$$|\boldsymbol{a}| = \|\boldsymbol{a}\| = \sqrt{|\boldsymbol{a}\cdot\boldsymbol{a}|} = \sqrt{|a_w^2 - (a_x^2 + a_y^2 + a_z^2)|}, \tag{31}$$

and the non-null unit vector $\hat{\boldsymbol{a}}$ (30) is also called a *unimodular vector*. For scalar a, $|a|$ is the *absolute value* of a. For SA spatial vector $\mathbf{a}$, the pseudo-Euclidean norm and Euclidean norm are equal,

$$\|\mathbf{a}\| = |\mathbf{a}| = \|\mathbf{a}\|_2 = \sqrt{\mathbf{a}\cdot\mathbf{a}^\dagger} = \sqrt{-\mathbf{a}\cdot\mathbf{a}} \geq 0. \tag{32}$$

Similarly, the pseudo-Euclidean norm $\|\boldsymbol{A}_{\langle k\rangle}\|$ of any STA k-blade $\boldsymbol{A}_{\langle k\rangle}$ is

$$|\boldsymbol{A}_{\langle k\rangle}| = \|\boldsymbol{A}_{\langle k\rangle}\| = \sqrt{|\boldsymbol{A}_{\langle k\rangle}^2|} = \sqrt{|\boldsymbol{A}_{\langle k\rangle}\cdot\boldsymbol{A}_{\langle k\rangle}|} \geq 0. \tag{33}$$

For any STA vector (1-blade) $\boldsymbol{a}$, the *unit vector* $\hat{\boldsymbol{a}}$ is defined by

$$\hat{\boldsymbol{a}} = \begin{cases} \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/\sqrt{|\boldsymbol{a}^2|} & : \ \boldsymbol{a}^2 \neq 0 \\ \boldsymbol{a}/\|\boldsymbol{a}\|_2 = \boldsymbol{a}/\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}^\dagger} & : \ \boldsymbol{a}^2 = 0. \end{cases} \tag{34}$$

For any STA k-blade $\boldsymbol{A}_{\langle k\rangle}$, the *unit k-blade* $\hat{\boldsymbol{A}}_{\langle k\rangle}$ is defined by

$$\hat{\boldsymbol{A}}_{\langle k\rangle} = \begin{cases} \boldsymbol{A}_{\langle k\rangle}/\|\boldsymbol{A}_{\langle k\rangle}\| = \boldsymbol{A}_{\langle k\rangle}/\sqrt{|\boldsymbol{A}_{\langle k\rangle}^2|} & : \ \boldsymbol{A}_{\langle k\rangle}^2 \neq 0 \\ \boldsymbol{A}_{\langle k\rangle}/\|\boldsymbol{A}_{\langle k\rangle}\|_2 = \boldsymbol{A}_{\langle k\rangle}/\sqrt{\boldsymbol{A}_{\langle k\rangle}\cdot\mathbf{A}_{\langle k\rangle}^\dagger} & : \ \boldsymbol{A}_{\langle k\rangle}^2 = 0. \end{cases} \tag{35}$$

In STA, null k-blades exist, such as the null 2-blade $(\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1)\boldsymbol{\gamma}_2$, where $((\boldsymbol{\gamma}_0+\boldsymbol{\gamma}_1)\boldsymbol{\gamma}_2)^2 = 0$. Note that, the conjugate $\boldsymbol{A}_{\langle k\rangle}^\dagger$, Euclidean norm $\|\boldsymbol{A}_{\langle k\rangle}\|_2$, and pseudoinverse $\boldsymbol{A}_{\langle k\rangle}^+$ exist for null k-blades $\boldsymbol{A}_{\langle k\rangle}^2 = 0$ (including null 1-blade vectors) and are mainly used for the algebra of null k-blades.

The *canonical basis* of $\mathcal{G}_{p,q}$ has $n = p+q$ orthonormal basis vector elements $\boldsymbol{a}_i = \mathbf{e}_i$ and a total of 2^n basis unit k-blade elements

$$\boldsymbol{A}_b = \bigwedge \mathbf{e}_i^{b_i} = \mathbf{e}_1^{b_1}\mathbf{e}_2^{b_2}\ldots\mathbf{e}_n^{b_n}, \tag{36}$$

where the exponents b_i are n binary digits of the binary number $b = b_1 b_2 \ldots b_n$, essentially acting as presence bits (generally $A^0 = 1$, $A^1 = A$). A k-blade has a number b with k ones. The basis unit 0-blade (unit scalar) is $\boldsymbol{A}_0 = 1$, and the n-blade unit pseudoscalar is $\mathbf{I} = \boldsymbol{A}_{2^n-1} = \mathbf{e}_1\mathbf{e}_2\ldots\mathbf{e}_n$. However, for the decimal number $\operatorname{dec}(b)$ of b, in general $\operatorname{dec}(b) \neq i$ of $\mathbf{e}_i$ etc., and the binary number b is the subscript on $\boldsymbol{A}_b$ since it intuitively relates to the construction of $\boldsymbol{A}_b$ as the product of canonical basis unit vectors $\mathbf{e}_i$ in *ascending order* of subscripts i. The basis k-blade $\boldsymbol{A}_b$ on an *arbitrary basis* $\boldsymbol{a}_i$, where the $\boldsymbol{a}_i$ are not necessarily orthonormal vectors, is

$$\boldsymbol{A}_b = \bigwedge \boldsymbol{a}_i^{b_i} = \boldsymbol{a}_1^{b_1} \wedge \boldsymbol{a}_2^{b_2} \wedge \ldots \wedge \boldsymbol{a}_n^{b_n}, \tag{37}$$

which is not in general equal to the geometric product of the $\boldsymbol{a}_i^{b_i}$, as for orthogonal vectors.

On an arbitrary vector basis $\boldsymbol{a}_i$, $1 \leq i \leq n$, of an n-dimensional algebra $\mathcal{G}_{p,q}$, $n = p+q$, where $\boldsymbol{a} = \boldsymbol{a}_j$ is the jth linearly independent basis vector, the pseudoscalar is $\mathbf{I} = \boldsymbol{A}_{\langle n\rangle} = \bigwedge \boldsymbol{a}_i$ (19) (not necessarily a unit n-blade) and the *reciprocal basis vector* $\boldsymbol{a}^j$, to $\boldsymbol{a} = \boldsymbol{a}_j$, is [15, page 28]

$$\boldsymbol{a}^j = (-1)^{j-1}(\mathbf{I}\backslash\boldsymbol{a}_j)\mathbf{I}^{-1}, \tag{38}$$

where $\mathbf{I}\backslash\boldsymbol{a}_j = \bigwedge_{i\neq j}\boldsymbol{a}_i$ (i.e., $\mathbf{I}$ without $\boldsymbol{a}_j$) and j is still the index (not exponent), such that

$$\boldsymbol{a}_j\cdot\boldsymbol{a}^j = (-1)^{j-1}\boldsymbol{a}_j\cdot((\mathbf{I}\backslash\boldsymbol{a}_j)\cdot\mathbf{I}^{-1}) = (-1)^{j-1}(\boldsymbol{a}_j\wedge(\mathbf{I}\backslash\boldsymbol{a}_j))\mathbf{I}^{-1} = \mathbf{I}\mathbf{I}^{-1} = 1, \tag{39}$$

and $\boldsymbol{a}_i\cdot\boldsymbol{a}^j = 0$ for $i \neq j$. Using the Kronecker delta

$$\delta_i^j = \begin{cases} 0 & : \ i \neq j \\ 1 & : \ i = j, \end{cases} \tag{40}$$

the reciprocal basis vectors are often *defined* by the expression $\boldsymbol{a}_i\cdot\boldsymbol{a}^j = \delta_i^j$.

The reciprocal basis vector $\boldsymbol{a}^j$ is a coefficient *extraction operator* that extracts the (contravariant) scalar coefficient v^j on (covariant) basis vector $\boldsymbol{a}_j$ in any vector $\boldsymbol{v} = v^i\boldsymbol{a}_i = v_i\boldsymbol{a}^i$ as $v^j = \boldsymbol{v}\cdot\boldsymbol{a}^j$, and $v_j = \boldsymbol{v}\cdot\boldsymbol{a}_j$. For basis vector $\boldsymbol{a} = \boldsymbol{a}_j$, its reciprocal basis vector $\boldsymbol{a}^j$ and its inverse $\boldsymbol{a}^{-1}$ are *not necessarily* equal, especially since a null basis vector has no inverse but does have a reciprocal.

Similar to the case of vectors (1-blades), it is possible to compute the reciprocal basis k-blade of a canonical (non-null) basis unit k-blade $\boldsymbol{A}_b$ (36) as

$$\boldsymbol{A}^b = s(\mathbf{I}\backslash\boldsymbol{A}_b)\mathbf{I}^{-1}, \tag{41}$$

where b is still the index (not exponent), and

$$s = (\boldsymbol{A}_b \wedge \boldsymbol{A}_{\mathrm{NOT}b})\cdot\mathbf{I}^{-1} \text{ and } \boldsymbol{A}_{\mathrm{NOT}b} = \mathbf{I}\backslash\boldsymbol{A}_b \text{ where } (\mathbf{I} = \mathbf{I}\wedge 1)\backslash\mathbf{I} = 1, \tag{42}$$

such that $\boldsymbol{A}_b\cdot\boldsymbol{A}^b = 1$ and $\boldsymbol{A}^b = \boldsymbol{A}_b^+ = \boldsymbol{A}_b^{-1}$ on the canonical basis. The notation $\mathrm{NOT}\, b = b\,\mathrm{XOR}\,(2^n-1)$ is the bitwise complement. The formula (41) can also be used to compute the reciprocal basis k-blade $\boldsymbol{A}^b$ on an arbitrary basis $\boldsymbol{a}_i$ by replacing $\mathbf{I}$ with the pseudoscalar $\boldsymbol{A}_{\langle n\rangle}$ (19) of the arbitrary basis $\boldsymbol{a}_i$, but then $\boldsymbol{A}^b = \boldsymbol{A}_b^+ = \boldsymbol{A}_b^{-1}$ does not hold in general on an arbitrary basis.

There are distinctions between *pseudoinverse* $\boldsymbol{a}^+$, *reciprocal* $\boldsymbol{a}^j$, and *inverse* $\boldsymbol{a}^{-1}$ vectors (and of k-blades $\boldsymbol{A}_{\langle k\rangle}$) that can be clarified in the context of STA by some further explanation, which follows (until (46)).

In STA $\mathcal{G}_{1,3}$, an arbitrary basis can include null basis vectors. There are three Minkowski planes, $\gamma_0\gamma_1$, $\gamma_0\gamma_2$, and $\gamma_0\gamma_3$, that each span γ_0, so we have to choose one Minkowski plane for any STA basis. A 2D Minkowski (pseudo-Euclidean) plane can be spanned by two null vectors, e.g., $\boldsymbol{n}_{0-1} = \gamma_0 - \gamma_1$ and $\boldsymbol{n}_{0+1} = \gamma_0 + \gamma_1 = \boldsymbol{n}_{0-1}^\dagger$, using (25) for a null unit vector $\hat{\boldsymbol{n}}_{0\pm1}$ (25), and then the rest of 4D spacetime is spanned by the other two vectors, e.g. γ_2 and γ_3. Such a basis is comprised of orthonormal vectors $\boldsymbol{a}_i \in \{\hat{\boldsymbol{n}}_{0-1}, \hat{\boldsymbol{n}}_{0+1}, \gamma_2, \gamma_3\}$, as are the canonical basis vectors $\boldsymbol{a}_i \in \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, so it is not an entirely arbitrary basis, but a special basis. The three different unit pseudoscalars, for a different orthonormal basis with null vectors, are $\mathbf{I}_{\mathcal{M}} = \hat{\boldsymbol{n}}_{0-1}\wedge\hat{\boldsymbol{n}}_{0+1}\wedge\gamma_2\wedge\gamma_3$ or $\mathbf{I}_{\mathcal{M}} = \hat{\boldsymbol{n}}_{0-2}\wedge\gamma_1\wedge\hat{\boldsymbol{n}}_{0+2}\wedge\gamma_3$ or $\mathbf{I}_{\mathcal{M}} = \hat{\boldsymbol{n}}_{0-3}\wedge\gamma_1\wedge\gamma_2\wedge\hat{\boldsymbol{n}}_{0+3}$ (19), but they are not actually different since all equal $\mathbf{I}_{\mathcal{M}}$, which is the unit pseudoscalar of the canonical basis of STA. For the canonical basis unit vectors $\boldsymbol{a}_i \in \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ and the null basis unit vectors $\hat{\boldsymbol{n}}_{0\pm i}$ (25), it can be shown that their pseudoinverse vector (26) and reciprocal basis vector (38) are equal, and we find it more convenient to use the pseudoinverse to obtain a reciprocal basis vector, since we are then not concerned with determining the sign $(-1)^{j-1}$ in (38). More generally, for any pseudo-Euclidean $\gamma_0\gamma_i$-plane, with $\gamma_0^2 = 1$ and $\gamma_i^2 = -1$, the canonical orthonormal basis $\{\gamma_0, \gamma_i\}$ and any basis of pseudoinverses $\{\boldsymbol{a} = |\boldsymbol{a}|\hat{\boldsymbol{a}} = \boldsymbol{a}_j, \boldsymbol{a}^+ = \hat{\boldsymbol{a}}^\dagger/|\boldsymbol{a}| = \boldsymbol{a}^j\}$ in the plane, have the property that their pseudoinverses (26) equal their reciprocals (38). The basis $\{\boldsymbol{a}, \boldsymbol{a}^+\}$ includes the null basis $\{\alpha\hat{\boldsymbol{n}}_{0-i}, \hat{\boldsymbol{n}}_{0+i}/\alpha\}$ or any non-null skew basis $\{\ \boldsymbol{a} = \alpha\gamma_0 - \beta\gamma_i, \boldsymbol{a}^+ \ :\ \alpha\neq\beta\ \}$ around γ_0 in the $\gamma_0\gamma_i$-plane. The choice of basis for STA, between the *canonical basis* (of vectors) $\boldsymbol{a}_i \in \{\gamma_j\}$ or a *special basis*

$$\boldsymbol{a}_i \in \{\gamma_{j\notin\{0,i\}}, \boldsymbol{a}, \boldsymbol{a}^+\}, \tag{43}$$

having a particular skew basis $\{\boldsymbol{a}, \boldsymbol{a}^+\}$ for just one of the three particular Minkowski $\gamma_0\gamma_i$-planes, is arbitrary since the pseudoinverse basis k-blade $\boldsymbol{A}_b^+$ (27) provides a uniform expression of the reciprocal basis k-blade $\boldsymbol{A}^b = \boldsymbol{A}_b^+$ (41) on these special choices of basis. For the general choice of an arbitrary STA vector basis $\boldsymbol{a}_i$, the pseudoinverse $\boldsymbol{a}_i^+$ (26) is *not in general* equal to the reciprocal $\boldsymbol{a}^i \neq \boldsymbol{a}_i^+$ (38), and then the general reciprocal vector $\boldsymbol{a}^i$ by (38) can always be used instead of the pseudoinverse to obtain the correct reciprocal.

If we arbitrarily choose one of the three Minkowski planes $\gamma_0\gamma_i$, then the *special null basis* for STA is an orthonormal basis, $\hat{\boldsymbol{a}}_i \in \{\hat{\boldsymbol{n}}_{0-1}, \hat{\boldsymbol{n}}_{0+1}, \gamma_2, \gamma_3\}$ or $\hat{\boldsymbol{a}}_i \in \{\hat{\boldsymbol{n}}_{0-2}, \gamma_1, \hat{\boldsymbol{n}}_{0+2}, \gamma_3\}$ or $\hat{\boldsymbol{a}}_i \in \{\hat{\boldsymbol{n}}_{0-3}, \gamma_1, \gamma_2, \hat{\boldsymbol{n}}_{0+3}\}$ with unit pseudoscalar $\mathbf{I}_{\mathcal{M}}$, that includes the two null unit vectors $\hat{\boldsymbol{n}}_{0\pm i}$ of the $\gamma_0\gamma_i$-plane and the two other canonical basis vectors $\gamma_{j\notin\{0,i\}}$. Then, a basis unit k-blade $\boldsymbol{A}_b$ (37) that includes *one* of the null basis unit vectors $\hat{\boldsymbol{n}}_{0\pm i}$ is a null basis unit k-blade $\boldsymbol{A}_b^2 = 0$, where its reciprocal basis k-blade (41) $\boldsymbol{A}^b$ and pseudoinverse k-blade (27) $\boldsymbol{A}_b^+$ are equal $\boldsymbol{A}^b = \boldsymbol{A}_b^+$, but no inverse exists. For a null basis unit k-blade, we find it again convenient to use its pseudoinverse (27) as its reciprocal (41) and avoid the determination of the sign s in the more general formula for a reciprocal (41).

For a canonical (non-null) basis unit k-blade (36) or special null basis unit k-blade $\boldsymbol{A}_b$, we have

$$\boldsymbol{A}_b = \bigwedge \hat{\boldsymbol{a}}_i^{b_i} = \hat{\boldsymbol{a}}_1^{b_1} \wedge \hat{\boldsymbol{a}}_2^{b_2} \wedge \ldots \wedge \hat{\boldsymbol{a}}_n^{b_n} = \hat{\boldsymbol{a}}_1^{b_1} \hat{\boldsymbol{a}}_2^{b_2} \ldots \hat{\boldsymbol{a}}_n^{b_n}, \tag{44}$$

where the basis unit vectors $\hat{\boldsymbol{a}}_j$ (34) are *orthonormal*, and where just *one* of $\hat{\boldsymbol{a}}_i^{b_i} = \hat{\boldsymbol{a}}_j$ may be present ($b_j = 1$) that is a special null basis unit vector $\hat{\boldsymbol{a}}_j = \hat{\boldsymbol{n}}_{0\pm i}$ for a special null basis unit k-blade $\boldsymbol{A}_b$. Its reciprocal basis unit k-blade $\boldsymbol{A}^b$ (41) equals its pseudoinverse (27) $\boldsymbol{A}_b^+$ as (n.b. the reverse order)

$$\boldsymbol{A}^b = \boldsymbol{A}_b^+ = \bigwedge_{i=1}^{n} (\hat{\boldsymbol{a}}_{n+1-i}^{b_{n+1-i}})^+ = (\hat{\boldsymbol{a}}_n^{b_n})^+ \wedge (\hat{\boldsymbol{a}}_{n-1}^{b_{n-1}})^+ \wedge \ldots \wedge (\hat{\boldsymbol{a}}_1^{b_1})^+ = (\hat{\boldsymbol{a}}_n^{b_n})^+ (\hat{\boldsymbol{a}}_{n-1}^{b_{n-1}})^+ \ldots (\hat{\boldsymbol{a}}_1^{b_1})^+. \tag{45}$$

This result $\boldsymbol{A}_b^+$, on the canonical or any special null basis, is easy to use in the sequel on DCSTA, where the DCSTA bivector (2-vector) extraction operators T_s are sums of 2-blade extraction operators $\boldsymbol{A}_b^+$. We simply multiply pseudoinverse (reciprocal) CSTA basis vectors $\mathbf{a}_j^+$, as CSTA extraction operators $\boldsymbol{a}_j^+ = T_{\mathcal{C}^i}^{s_i}$ (129), in reverse order to form a DCSTA 2-blade extraction operator $\boldsymbol{A}_b^+ = T_{\mathcal{D}}^{s} = T_{\mathcal{C}^2}^{s_2} T_{\mathcal{C}^1}^{s_1}$ (Table 1) for polynomial term $s = s_2 s_1$. In further extensions of CGA beyond doubling, called Extended CGA (k-CGA), this result $\boldsymbol{A}_b^+$ is used for defining the k-vector extraction operators T_s of k-CGA, which are sums of k-blade extraction operators $\boldsymbol{A}_b^+$.

A k-versor V_k is the product of k non-null vectors $\boldsymbol{a}_i$ with inverses $\boldsymbol{a}_i^{-1} = \boldsymbol{a}_i / \boldsymbol{a}_i^2$ as

$$V_k = \boldsymbol{a}_k \boldsymbol{a}_{k-1} \ldots \boldsymbol{a}_1. \tag{46}$$

A unimodular k-versor $\hat{V}_k$ is the product of k unimodular (non-null) vectors $\hat{\boldsymbol{a}}_i^2 = \pm 1$ as

$$\hat{V}_k = \hat{\boldsymbol{a}}_k \hat{\boldsymbol{a}}_{k-1} \ldots \hat{\boldsymbol{a}}_1. \tag{47}$$

The modulus $|V_k|$ of k-versor V_k is

$$|V_k| = \sqrt{|V_k V_k^\sim|} = \sqrt{|V_k^\sim V_k|} = \sqrt{|\boldsymbol{a}_k \boldsymbol{a}_{k-1} \ldots \boldsymbol{a}_1 \boldsymbol{a}_1 \ldots \boldsymbol{a}_{k-1} \boldsymbol{a}_k|} = |\boldsymbol{a}_k||\boldsymbol{a}_{k-1}| \ldots |\boldsymbol{a}_1|, \tag{48}$$

and the unimodular k-versor $\hat{V}_k$ can be expressed as

$$\hat{V}_k = |V_k|^{-1} V_k = V_k / \sqrt{|V_k V_k^\sim|}. \tag{49}$$

The inverse V_k^{-1} of k-versor V_k is

$$V_k^{-1} \;=\; V_k^\sim / (V_k V_k^\sim) = \boldsymbol{a}_1^{-1} \boldsymbol{a}_2^{-1} \ldots \boldsymbol{a}_k^{-1}. \tag{50}$$

The 1-versor $V_1 = \boldsymbol{a}$ is an operator for vector reflection $\boldsymbol{v}' = V_1 \boldsymbol{v} V_1^{-1} = \boldsymbol{a}\boldsymbol{v}\boldsymbol{a}^{-1} = \hat{\boldsymbol{a}}^2 \hat{\boldsymbol{a}}\boldsymbol{v}\hat{\boldsymbol{a}} = \boldsymbol{v}^{||\boldsymbol{a}} - \boldsymbol{v}^{\perp \boldsymbol{a}}$. The negative, $-\boldsymbol{a}\boldsymbol{v}\boldsymbol{a}^{-1} = -\hat{\boldsymbol{a}}^2 \hat{\boldsymbol{a}}\boldsymbol{v}\hat{\boldsymbol{a}} = \boldsymbol{v}^{\perp \boldsymbol{a}} - \boldsymbol{v}^{||\boldsymbol{a}}$, reflects $\boldsymbol{v}$ in the hyperplane orthogonal to $\boldsymbol{a}$. The unimodular 1-versor $\hat{V}_1 = \hat{\boldsymbol{a}}$ is an operator $\boldsymbol{v}' = \hat{V}_1 \boldsymbol{v} \hat{V}_1^\sim = \hat{\boldsymbol{a}}\boldsymbol{v}\hat{\boldsymbol{a}}$ for reflection in the vector $\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}^{-1}$ when $\hat{\boldsymbol{a}}^2 = 1$, and for reflection in the hyperplane orthogonal to $\hat{\boldsymbol{a}} = -\hat{\boldsymbol{a}}^{-1}$ when $\hat{\boldsymbol{a}}^2 = -1$.

A unimodular k-versor $\hat{V}_k$ operates on a vector $\boldsymbol{v}$ using the *versor "sandwich" operation*

$$\boldsymbol{v}' = \hat{V}_k \boldsymbol{v} \hat{V}_k^\sim, \tag{51}$$

where $\hat{V}_k^\sim = \pm \hat{V}_k^{-1}$ (use $\hat{V}_k^{-1}$ when $\pm$ orientation is significant). The general k-versor V_k operation is

$$\boldsymbol{v}' = V_k \boldsymbol{v} V_k^{-1}. \tag{52}$$

We primarily use (51) since the $\pm$ orientation is usually not significant for CGA geometric entities.

The unimodular k-versor $\hat{V}_k$ is an operator for successive reflections in k vectors or hyperplanes, depending on the particular signatures of the unit vectors $\hat{\boldsymbol{a}}_i^2 = \pm 1$. Reflection in two unimodular spatial vectors $\hat{\mathbf{a}}_i$ is spatial rotation in the plane of the two vectors by *twice* the angle θ between them. Reflection in two unimodular time-like space-time vectors $\hat{\boldsymbol{a}}_i^2 = 1$ in the same space-time $\hat{\mathbf{v}}\boldsymbol{\gamma}_0$-plane is hyperbolic rotation (boost in direction $\hat{\mathbf{v}}$) by *twice* the hyperbolic angle φ between them.

By outermorphism [18], the unimodular k-versor $\hat{V}_k$ (47) transforms the k-blade $\boldsymbol{A}_{\langle k \rangle}$ (19) as

$$\boldsymbol{A}'_{\langle k \rangle} = \hat{V}_k \boldsymbol{A}_{\langle k \rangle} \hat{V}_k^\sim = \left(\hat{V}_k \boldsymbol{a}_1 \hat{V}_k^\sim\right) \wedge \left(\hat{V}_k \boldsymbol{a}_2 \hat{V}_k^\sim\right) \wedge \ldots \wedge \left(\hat{V}_k \boldsymbol{a}_k \hat{V}_k^\sim\right), \tag{53}$$

which rotates or reflects the k-blade $\boldsymbol{A}_{\langle k \rangle}$ by rotating or reflecting each vector $\boldsymbol{a}_i$ in the k-blade. By the linearity of the versor operation (51), as a linear operator, a unimodular k-versor $\hat{V}_k$ can also transform any k-vector $A_{\langle k \rangle}$ or multivector A.

An STA unimodular 2-versor $\hat{V}_2$ is the geometric product of two unimodular vectors $\hat{\boldsymbol{a}}_i^2 = \pm 1$,

$$\hat{V}_2 = \hat{\boldsymbol{a}}_2\hat{\boldsymbol{a}}_1 = \hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1 + \hat{\boldsymbol{a}}_2 \wedge \hat{\boldsymbol{a}}_1 = \pm\exp(\alpha\hat{\boldsymbol{A}}), \tag{54}$$

which is a scalar $\hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1$ plus a 2-blade $\boldsymbol{A} = \hat{\boldsymbol{a}}_2 \wedge \hat{\boldsymbol{a}}_1$ with unit 2-blade $\hat{\boldsymbol{A}}$ by (35) in the direction of the plane of rotation. A unimodular 2-versor $\hat{V}_2$ is a *geometric number* [21] that is isomorphic to an elliptic complex number $a + bi$ (with $i^2 = -1$), or a parabolic complex number $a + b\varepsilon$ (with $\varepsilon^2 = 0$, $\varepsilon \neq 0$), or a hyperbolic complex number $a + bj$ (with $j^2 = 1$, $j \neq \pm 1$), and is generally a *rotation operator* that preserves the modulus $|\boldsymbol{a}_i|$ (31) of vectors $\boldsymbol{a}_i$ in (53). The angle α of rotation in (54) is given by

$$\alpha = \begin{cases} \theta = \operatorname{acos}(-\hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1) & : \hat{\boldsymbol{A}}^2 = -1 \quad (V_2 \cong \text{elliptic complex number}) \\ |\hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1|^{-1} \|\boldsymbol{A}\|_2 & : \hat{\boldsymbol{A}}^2 = 0 \quad\;\; (V_2 \cong \text{parabolic complex number}) \\ \varphi = \operatorname{atanh}(|\boldsymbol{A}|/(\hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1)) & : \hat{\boldsymbol{A}}^2 = +1 \quad (V_2 \cong \text{hyperbolic complex number}). \end{cases} \tag{55}$$

We use the notations $\theta = \operatorname{acos}(x)$ and $\varphi = \operatorname{atanh}(y/x)$, rather than $\theta = \operatorname{arccos}(x)$ and $\varphi = \operatorname{artanh}(y/x)$, since both angles, θ and φ, can be interpreted as arc lengths or as areas in the plane.

A unimodular 2-versor $\hat{V}_2$ operates on a vector $\boldsymbol{v}$ using the *versor "sandwich" operation*

$$\boldsymbol{v}' = \hat{V}_2 \boldsymbol{v} \hat{V}_2^{\sim}, \tag{56}$$

where the reverse $V_2^{\sim}$ corresponds to the *complex conjugate* $V_2^{-} = V_2^{\sim}$. For the unimodular exponential form $\hat{V}_2 = \exp(\alpha\hat{\boldsymbol{A}})$, which is our usual preferred form, we have $\hat{V}_2^{\sim} = \hat{V}_2^{-1}$ exactly.

In (56), the vector $\boldsymbol{v}$ is rotated by *twice* the angle α (i.e., by 2α). To rotate by angle α, we define a 2-versor $\hat{V}_\alpha$ for angle α as the square root

$$\hat{V}_\alpha = \sqrt{\hat{V}_2} = \sqrt{\hat{\boldsymbol{a}}_2\hat{\boldsymbol{a}}_1} = \sqrt{\hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1 + \hat{\boldsymbol{a}}_2 \wedge \hat{\boldsymbol{a}}_1} = \exp(\alpha\hat{\boldsymbol{A}}/2). \tag{57}$$

We usually assume that our 2-versors are unimodular $V_2 = \hat{V}_2$ and use $\hat{V}_2$ or its square root $\hat{V}_\alpha$ in operation (56), but for non-unimodular versors $V_k \neq \hat{V}_k$ the operation (52) could be used.

It can be useful to interpret the product of vectors $\hat{\boldsymbol{a}}_2\hat{\boldsymbol{a}}_1 = \hat{\boldsymbol{b}}/\hat{\boldsymbol{a}} = \hat{\boldsymbol{b}}\hat{\boldsymbol{a}}^{-1}$ as a certain ratio of vectors "$\hat{\boldsymbol{b}}$ by $\hat{\boldsymbol{a}}$" that transforms $\hat{\boldsymbol{a}}$ into $\hat{\boldsymbol{b}}$ as $(\hat{\boldsymbol{b}}/\hat{\boldsymbol{a}})\hat{\boldsymbol{a}} = (\hat{\boldsymbol{b}}/\hat{\boldsymbol{a}})^{\frac{1}{2}}\hat{\boldsymbol{a}}(\hat{\boldsymbol{b}}/\hat{\boldsymbol{a}})^{-\frac{1}{2}}$, and also transforms other vectors by the same proportion, which is by the same angle in the same plane of rotation.

In (54) and (55) for $\boldsymbol{A}^2 = 0$, the null unit 2-blade is $\hat{\boldsymbol{A}} = \boldsymbol{A}/\|\boldsymbol{A}\|_2$ by (35) and $|V_2| = \sqrt{|V_2 V_2^{-}|} = \sqrt{|(\hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1)^2|} = |\hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1|$ by (48), where by (49) $\hat{V}_2 = |V_2|^{-1}V_2 = |\hat{\boldsymbol{a}}_2 \cdot \hat{\boldsymbol{a}}_1|^{-1}V_2 = \pm\exp(\alpha\hat{\boldsymbol{A}}) = \pm(1 + \alpha\hat{\boldsymbol{A}})$ (n.b., $V_2 = \pm|V_2|\exp(\alpha\hat{\boldsymbol{A}})$, but in (56) $\pm$ is canceled). Then, $\hat{V}_2$ is a unimodular 2-versor for a special type of *translation* (rotation around the point at infinity), such that the modulus $|\boldsymbol{v}'| = |\boldsymbol{v}|$ is invariant. General translation does not preserve the modulus. For example, using $\hat{\boldsymbol{a}}_2 = \boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2$ and $\hat{\boldsymbol{a}}_1 = -\boldsymbol{\gamma}_2$, the product $\hat{\boldsymbol{a}}_2\hat{\boldsymbol{a}}_1$ is the ratio $\hat{\boldsymbol{a}}_2/\boldsymbol{\gamma}_2 = 1 + \boldsymbol{\gamma}_2 \wedge (\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1) = 1 + \boldsymbol{A} = \exp(\boldsymbol{A}) = \hat{V}_2$, and then the versor $\hat{V}_\alpha = \exp(\boldsymbol{A}/2)$ transforms $\boldsymbol{\gamma}_2$ into $\hat{\boldsymbol{a}}_2 = \hat{V}_\alpha \boldsymbol{\gamma}_2 \hat{V}_\alpha^{\sim}$, which is $\boldsymbol{\gamma}_2$ translated by the null vector $\boldsymbol{n} = \boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1$, where $|\boldsymbol{\gamma}_2 + \boldsymbol{n}| = |\boldsymbol{\gamma}_2| = 1$. Other vectors are transformed by $\hat{V}_\alpha$ by the same proportion, which are special translations by *various* null or non-null vectors (not a constant vector) in the plane of $\boldsymbol{A}$ that preserve the modulus. When STA $\mathcal{G}_{1,3}$ is viewed as the CGA $\mathcal{G}_{1,2+1}$ of the $\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$-plane (CSA in 2D), then the *translator* $T = 1 - \mathbf{d}(\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1)/2 = \exp(\mathbf{e}_\infty \mathbf{d}/2)$ translates CGA entities by $\mathbf{d}$ in the $\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$-plane (see CSA in [7] for details). In CGA $\mathcal{G}_{1,2+1}$, the translation of CGA point $\mathbf{P}$ into $\mathbf{P}' = T\mathbf{P}T^{-1}$ is a parabolic rotation of $\mathbf{P}$ into $\mathbf{P}'$ along the null parabola cut from the null cone of $\mathcal{G}_{1,2+1}$ that connects $\mathbf{P}$ to $\mathbf{P}'$.

If $\boldsymbol{a}$ is a null vector, then $\hat{\boldsymbol{a}}$ is Euclidean norm by (34), which is *not unimodular* since the modulus is $|\hat{\boldsymbol{a}}| = 0 \neq 1$. The reflection $\boldsymbol{v}' = \boldsymbol{a}\boldsymbol{v}\boldsymbol{a}$ of any vector $\boldsymbol{v}$ in any null vector $\boldsymbol{a}$ produces $\boldsymbol{v}'$ with modulus $|\boldsymbol{v}'| = 0$, which is a null vector or the zero vector 0. Therefore, if any vector $\boldsymbol{a}_i$ in the k-versor V_k is null $\boldsymbol{a}_i^2 = 0$, then the modulus is $|V_k| = 0$ by (48) and the operation $\boldsymbol{v}' = V_k \boldsymbol{v} V_k^{\sim}$ produces a vector with $|\boldsymbol{v}'| = 0$, which is a null vector or the zero vector 0. For a 2-versor $V_2 = \boldsymbol{a}_2\boldsymbol{a}_1$, if either vector $\boldsymbol{a}_i$ is a null vector then the modulus is $|V_2| = 0$ by (48), V_2 has no exponential form (thus, not a proper 2-versor), and all resulting vectors $\boldsymbol{v}' = V_2 \boldsymbol{v} V_2^{\sim}$ are zero modulus $|\boldsymbol{v}'| = 0$. A null vector $\boldsymbol{a}_i$ is not generally admitted in a proper k-versor V_k (46), except in the special case where V_k is to act only on null vectors or null blades (i.e., on CGA points or other null entities) such that their zero (or null) moduli are preserved. For example, using $\boldsymbol{a}_2 = \boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1$ and $\boldsymbol{a}_1 = -\boldsymbol{\gamma}_2$ might be interpreted as the ratio $\boldsymbol{a}_2/\boldsymbol{\gamma}_2$, but the attempt to transform as a versor operation gives $V_2 \boldsymbol{\gamma}_2 V_2^{\sim} = 0$.

The k-versor (47) for even $k=2m$ is a composition of m 2-versors, and the k-versor for odd $k=2m+1$ includes one more 1-versor for a final reflection in a vector or hyperplane.

An STA 1-versor $\boldsymbol{a}$ is any non-null STA vector $\boldsymbol{a}=\boldsymbol{a}_{\mathcal{M}}$ with an inverse $\boldsymbol{a}^{-1}=\boldsymbol{a}/\boldsymbol{a}^2$. The reflection $\boldsymbol{p}'$ of vector $\boldsymbol{p}$ in vector $\boldsymbol{a}$ is given by the *versor "sandwich" operation* (or *conjugation*)

$$\boldsymbol{p}' \;=\; \boldsymbol{a}\boldsymbol{p}\boldsymbol{a}^{-1}=\boldsymbol{p}^{||\boldsymbol{a}}-\boldsymbol{p}^{\perp\boldsymbol{a}}=\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{p})-\mathcal{P}^{\perp}_{\boldsymbol{a}}(\boldsymbol{p})=(\boldsymbol{p}\cdot\boldsymbol{a})\boldsymbol{a}^{-1}-(\boldsymbol{p}\wedge\boldsymbol{a})\boldsymbol{a}^{-1}. \tag{58}$$

Two successive reflections (58), in vector $\boldsymbol{a}$ and then in vector $\boldsymbol{b}$, forms the 2-versor $\boldsymbol{b}\boldsymbol{a}$. In general, in the geometric algebra of an nD vector space, k successive reflections (58) in $1\leq k\leq n$ vectors $\boldsymbol{a}_i$ forms a k-versor $\boldsymbol{a}_k....\boldsymbol{a}_2\boldsymbol{a}_1$ for an orthogonal transformation (CARTAN-DIEUDONNÉ theorem). All of the versors in this paper can be derived from successive vector reflections (58). The 2-versors are generalized rotation operators with unimodular exponential forms $\exp(A)=e^A$. The opposite orientation to (58), $\boldsymbol{p}'=-\boldsymbol{a}\boldsymbol{p}\boldsymbol{a}^{-1}$, is reflection in the hyperplane orthogonal to $\boldsymbol{a}$. The geometrical distinction between reflection in vectors or in hyperplanes is not very important in this paper since we will only use even k-versors as products of unimodular 2-versors in exponential forms, or transform *homogeneous entities* in CSTA and DCSTA that are equivalent up to any non-zero scalar multiple, where any changes in orientation ($\pm$ sign) or scale are usually of little significance.

The STA 2-versors include the spatial *rotor* R and the spacetime *hyperbolic rotor (boost)* B. The STA 2-versor spatial *rotor* R is defined as (n.b., $(\hat{\mathbf{n}}^*_{\mathcal{S}})^2=-1$, see (14))

$$R=R_{\mathbf{n}}=(\hat{\mathbf{b}}/\hat{\mathbf{a}})^{\frac{1}{2}}=\exp(\theta\hat{\mathbf{n}}^*_{\mathcal{S}}/2)=e^{\frac{1}{2}\theta\hat{\mathbf{n}}^*_{\mathcal{S}}}=\cos(\theta/2)+\sin(\theta/2)\hat{\mathbf{n}}^*_{\mathcal{S}} \tag{59}$$

for spatial rotation around the spatial unit vector axis $\hat{\mathbf{n}}_{\mathcal{S}}$ through the origin by the radians angle θ subtended from $\hat{\mathbf{a}}$ to $\hat{\mathbf{b}}$ (by right-hand rule) in the $\mathbf{ab}$-plane orthogonal to $\mathbf{n}$. Two successive reflections (58), in vector $\mathbf{a}$ then in vector $\mathbf{b}$, rotates by *twice* the angle $\theta=\angle\mathbf{ab}$, but R rotates by just θ. The ratio $\hat{\mathbf{b}}/\hat{\mathbf{a}}$ is isomorphic to HAMILTON's unit quaternion *versor* [13].

The STA 2-versor space-time *hyperbolic rotor (boost)* B is defined as (n.b., $(\hat{\mathbf{v}}\boldsymbol{\gamma}_0)^2=+1$)

$$B=B_{\mathbf{v}}=(\gamma\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}=\exp(\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0/2)=\cosh(\varphi/2)+\sinh(\varphi/2)\hat{\mathbf{v}}\boldsymbol{\gamma}_0, \tag{60}$$

where three-dimensional *spatial speed* v in physics is

$$v=\beta c=\|\mathbf{v}\|=\sqrt{v_x^2+v_y^2+v_z^2}, \tag{61}$$

light speed is c, *natural speed* β is

$$0\leq(\beta=\beta_{\mathbf{v}}=\|\mathbf{v}\|/\|\boldsymbol{o}\|=v/c)\leq 1, \tag{62}$$

space-time velocity is by (7)

$$\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}=c\boldsymbol{\gamma}_0+\beta c\hat{\mathbf{v}}, \tag{63}$$

and *rapidity* (hyperbolic angle in (60)) is

$$\varphi=\varphi_{\mathbf{v}}=\operatorname{atanh}(\beta). \tag{64}$$

The *Lorentz factor*

$$\gamma=\gamma_{\mathbf{v}}=\mathrm{d}t/\mathrm{d}\tau=\sqrt{\boldsymbol{o}^2/\boldsymbol{v}^2}=|\boldsymbol{o}|/|\boldsymbol{v}|=1/\sqrt{1-\beta_{\mathbf{v}}^2}=1/d \tag{65}$$

is related to special relativity *length contraction* (from L_0 to L) as

$$L=\sqrt{1-\beta^2}L_0=L_0/\gamma=dL_0, \tag{66}$$

where $\tau=t_{p\boldsymbol{v}}$ is the *proper time* of the observable with space-time velocity $\boldsymbol{v}$. The proper time of $\boldsymbol{o}$ is $t_{p\boldsymbol{o}}=t$, which is also the *coordinate time* $t_{c\boldsymbol{v}}=t$ of any vector $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ directly relative to an observer $\boldsymbol{o}$. The proper time of $\boldsymbol{v}^\dagger=\boldsymbol{o}+\mathbf{v}^\dagger=\boldsymbol{o}-\mathbf{v}$ (see (17) and (18)) is $t_{p\boldsymbol{v}^\dagger}$. Relative to $\boldsymbol{o}$ with coordinate time t, then $t_{p\boldsymbol{v}}=t_{p\boldsymbol{v}^\dagger}=\tau$ and $\gamma_{\mathbf{v}^\dagger}=\gamma_{\mathbf{v}}$, but these are not equal relative to another observer $\boldsymbol{u}\neq\boldsymbol{o}$. A space-time velocity $\boldsymbol{u}=\boldsymbol{o}+\mathbf{u}$ is a spatial velocity $\mathbf{u}$ relative to the spatially stationary observer $\boldsymbol{o}$. The coordinate time of $\boldsymbol{u}$ is $t_{c\boldsymbol{u}}=t_{p\boldsymbol{o}}$, and the proper time of $\boldsymbol{u}$ is $t_{p\boldsymbol{u}}$.

The unimodular boost operator B is either an active boost $B=B_{\mathbf{v}}$ by $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ into the rest frame of $\boldsymbol{v}$ or an equivalent passive boost $B=B_{\mathbf{v}^\dagger}^{-1}$ relative to $\boldsymbol{v}^\dagger$, boosting $\boldsymbol{o}$ as

$$B_{\mathbf{v}}\boldsymbol{o}B_{\mathbf{v}}^{-1}\tau=\boldsymbol{o}\oplus\boldsymbol{v}\tau \;=\; \gamma_{\mathbf{v}}\tau\boldsymbol{v}=\gamma_{\mathbf{v}}\tau(\boldsymbol{o}+\mathbf{v})=t(\boldsymbol{o}+\mathbf{v})=\boldsymbol{v}t \tag{67}$$

$$B_{\mathbf{v}^\dagger}^{-1}\boldsymbol{o}B_{\mathbf{v}^\dagger}t=\boldsymbol{o}\ominus\boldsymbol{v}^\dagger t \;=\; \gamma_{\mathbf{v}^\dagger}t\boldsymbol{v}=\gamma_{\mathbf{v}^\dagger}t(\boldsymbol{o}-\mathbf{v}^\dagger)=\tau(\boldsymbol{o}-\mathbf{v}^\dagger)=\boldsymbol{v}\tau. \tag{68}$$

The above active and passive boosts, while algebraically equivalent, have different interpretations, especially of the time transformations. For $B=B_{\mathbf{v}}$ then $\gamma_{\mathbf{v}}\tau=\gamma_{\mathbf{v}}t_{p\boldsymbol{v}}=t_{p\boldsymbol{o}}$, and for $B=B_{\mathbf{v}^\dagger}^{-1}$ the relatively corresponding (and numerically equivalent) times are $\gamma_{\mathbf{v}^\dagger}t=\gamma_{\mathbf{v}^\dagger}t_{p\boldsymbol{o}}=t_{p\boldsymbol{v}^\dagger}$. We also use the alternative notations

$$\gamma_{\mathbf{v}}=\gamma_{0\oplus\mathbf{v}}=\gamma_{\oplus\mathbf{v}}=\gamma_{\oplus\boldsymbol{v}}=\gamma_{\boldsymbol{o}\oplus\boldsymbol{v}}, \tag{69}$$

where $\gamma_{0\oplus\mathbf{v}}=\gamma_{\oplus\mathbf{v}}$ emphasizes the spatial velocity boost by $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ from velocity 0 or from an unspecified arbitrary initial velocity, and $\gamma_{\oplus\boldsymbol{v}}=\gamma_{\boldsymbol{o}\oplus\boldsymbol{v}}$ emphasizes the space-time boost by $\boldsymbol{v}$ and that $(\boldsymbol{o}\oplus\boldsymbol{v})\ominus\boldsymbol{o}=\boldsymbol{v}$ and $(\boldsymbol{o}\oplus\boldsymbol{v})\ominus\boldsymbol{v}=\boldsymbol{o}$ (and similarly for initial velocity $\boldsymbol{u}$ instead of $\boldsymbol{o}$).

The notation for the *active boost* of $\boldsymbol{u}$ by $\boldsymbol{v}$ is

$$B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{-1}=\boldsymbol{u}\oplus\boldsymbol{v}=\gamma_{\mathbf{u}\oplus\mathbf{v}}(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v}), \tag{70}$$

where the active dilation factor (including an alternative notation subscript) is

$$\gamma_{\mathbf{u}\oplus\mathbf{v}}=\gamma_{\boldsymbol{u}\oplus\boldsymbol{v}}=\gamma_{\mathbf{v}}\Big(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\Big) \quad\neq(\gamma_{0\oplus\mathbf{u}\oplus\mathbf{v}}=\gamma_{\oplus\mathbf{u}\oplus\mathbf{v}}=\gamma_{\boldsymbol{o}\oplus\boldsymbol{u}\oplus\boldsymbol{v}}), \tag{71}$$

the active time transformation $(\tau=t_{p\boldsymbol{v}})\rightarrow(t=t_{c\boldsymbol{u}}=t_{p\boldsymbol{o}}=t_{c\boldsymbol{o}})$ is

$$\gamma_{\mathbf{u}\oplus\mathbf{v}}\tau=t, \tag{72}$$

and the spatial *velocity addition* $\mathbf{u}\oplus\mathbf{v}$ ("$\mathbf{u}$ boosted by $\mathbf{v}$") is

$$(\mathbf{u}\oplus\mathbf{v})\gamma_{\mathbf{u}\oplus\mathbf{v}}\tau=\left(\frac{\gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}+\gamma_{\mathbf{v}}\mathbf{v}+\mathbf{u}^{\perp\hat{\mathbf{v}}}}{\gamma_{\mathbf{u}\oplus\mathbf{v}}}\right)\gamma_{\mathbf{u}\oplus\mathbf{v}}\tau=\left(\frac{\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v})\mathbf{v}^{-1}+\gamma_{\mathbf{v}}\mathbf{v}+(\mathbf{u}\wedge\mathbf{v})\mathbf{v}^{-1}}{\gamma_{\mathbf{u}\oplus\mathbf{v}}}\right)t. \tag{73}$$

The active time transformation is a relative time transformation of the proper time $t_{p\boldsymbol{v}}$ of $\boldsymbol{v}$ into the coordinate time $t_{c\boldsymbol{u}}$ of $\boldsymbol{u}$, which is the proper time $t_{p\boldsymbol{o}}=t$ of $\boldsymbol{o}$. Then, $(\boldsymbol{u}\oplus\boldsymbol{v})\tau=(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})t$ is a relative transformation of the velocity audition $\mathbf{u}\oplus\mathbf{v}$ back into the frame of $\boldsymbol{o}$ with coordinate time t. In general, $\boldsymbol{u}\oplus\boldsymbol{v}\neq\boldsymbol{v}\oplus\boldsymbol{u}$, otherwise the direction of length contraction would be ambiguous. The formulas for $\gamma_{\mathbf{u}\oplus\mathbf{v}}$ and $\mathbf{u}\oplus\mathbf{v}$ can be derived and verified by algebraically expanding the boost versor operation $B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{-1}$. For any boost $B_{\mathbf{v}}$, we must limit $\beta_{\mathbf{v}}$ to less than light speed $0\leq\beta_{\mathbf{v}}<1$ such that $1\leq\gamma_{\mathbf{v}}<\infty$ and $1\geq\gamma_{\mathbf{v}}^{-1}>0$.

Note that, in some other literature $\gamma_{\mathbf{u}\oplus\mathbf{v}}$ (71) is sometimes defined differently, as we define $\gamma_{\oplus\mathbf{u}\oplus\mathbf{v}}$ (81) for the composition of *two* boosts (two $\oplus$), by $\boldsymbol{u}$ and then by $\boldsymbol{v}$. Our definition of $\gamma_{\mathbf{u}\oplus\mathbf{v}}$ is for *one* boost (one $\oplus$) of $\boldsymbol{u}$ by $\boldsymbol{v}$, boosting $\boldsymbol{u}$ into the rest frame of $\boldsymbol{v}$ with new time $\tau=t_{p\boldsymbol{v}}$, which passively transforms back to t, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in the frame of $\boldsymbol{o}$ with coordinate time t. Also note that, due to the anti-Euclidean metric of spatial vectors $\mathbf{u}$ and $\mathbf{v}$, the sign on $\mathbf{u}\cdot\mathbf{v}$ is *negative* compared to some other literature that uses the positive Euclidean metric for spatial vectors. The expression $(\mathbf{u}\oplus\mathbf{v})=\frac{\gamma_{\mathbf{v}}^{-1}}{\gamma_{\mathbf{v}}^{-1}}(\mathbf{u}\oplus\mathbf{v})$ may appear more like some other literature.

The notation for the *passive boost* of $\boldsymbol{u}$ relative to $\boldsymbol{v}$ is

$$B_{\mathbf{v}}^{-1}\boldsymbol{u}B_{\mathbf{v}}=\boldsymbol{u}\ominus\boldsymbol{v}=\gamma_{\mathbf{u}\ominus\mathbf{v}}(\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v}), \tag{74}$$

where the passive dilation factor (including an alternative notation subscript) is

$$\gamma_{\mathbf{u}\ominus\mathbf{v}}=\gamma_{\boldsymbol{u}\ominus\boldsymbol{v}}=\gamma_{\mathbf{v}}\Big(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\Big) \quad\neq(\gamma_{0\oplus\mathbf{u}\ominus\mathbf{v}}=\gamma_{\oplus\mathbf{u}\ominus\mathbf{v}}=\gamma_{\boldsymbol{o}\oplus\boldsymbol{u}\ominus\boldsymbol{v}}), \tag{75}$$

the passive time transformation $(t=t_{c\boldsymbol{u}}=t_{p\boldsymbol{o}}=t_{c\boldsymbol{o}})\rightarrow(\tau=t_{p\boldsymbol{v}})$ is

$$\gamma_{\mathbf{u}\ominus\mathbf{v}}t=\tau, \tag{76}$$

and the spatial *relative velocity* $\mathbf{u}\ominus\mathbf{v}$ ("$\mathbf{u}$ relative to $\mathbf{v}$") is

$$(\mathbf{u}\ominus\mathbf{v})\gamma_{\mathbf{u}\ominus\mathbf{v}}t=\left(\frac{\gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}-\gamma_{\mathbf{v}}\mathbf{v}+\mathbf{u}^{\perp\hat{\mathbf{v}}}}{\gamma_{\mathbf{u}\ominus\mathbf{v}}}\right)\gamma_{\mathbf{u}\ominus\mathbf{v}}t=\left(\frac{\gamma_{\mathbf{v}}(\mathbf{u}\cdot\mathbf{v})\mathbf{v}^{-1}-\gamma_{\mathbf{v}}\mathbf{v}+(\mathbf{u}\wedge\mathbf{v})\mathbf{v}^{-1}}{\gamma_{\mathbf{u}\ominus\mathbf{v}}}\right)\tau. \tag{77}$$

The passive time transformation is a relative transformation of the coordinate time $t_{c\boldsymbol{u}}$ of $\boldsymbol{u}$, which is the proper time $t_{p\boldsymbol{o}}=t$ of $\boldsymbol{o}$, into the proper time $t_{p\boldsymbol{v}}=\tau$ of $\boldsymbol{v}$. Then, $(\boldsymbol{u}\ominus\boldsymbol{v})t=(\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})\tau$ is a relative transformation of the relative velocity $\mathbf{u}\ominus\mathbf{v}$ into (relative to) the rest frame of $\boldsymbol{v}$, where $(\boldsymbol{v}\ominus\boldsymbol{v})t=(\boldsymbol{o}/\gamma_{\mathbf{v}})t=\boldsymbol{o}\tau$ and $((\boldsymbol{o}\oplus\boldsymbol{v})\ominus\boldsymbol{v})\tau=\boldsymbol{o}\tau$. In the rest frame of $\boldsymbol{v}$ with proper time τ as coordinate time, the worldline $\boldsymbol{o}\tau$ represents observable $\boldsymbol{v}t$ that is in the frame of $\boldsymbol{o}$ (i.e., $\boldsymbol{o}$ is the observer space-time velocity within any rest frame). *Velocity subtraction* $(\mathbf{u}\oplus\mathbf{v}^{\dagger})t=(\mathbf{u}\ominus\mathbf{v})t$ in the frame of $\boldsymbol{o}$ uses the spatial velocity addition (73) with a negative velocity $\mathbf{v}^{\dagger}=-\mathbf{v}$, which is not actually the same as the spatial relative velocity (77) in the frame of $\boldsymbol{v}$ since the frames $\boldsymbol{o}$ and $\boldsymbol{v}$ and their proper times t and τ are different.

Our *notation* $\mathbf{u}\oplus\mathbf{v}$, which we define by (73) as "$\mathbf{u}$ boosted by $\mathbf{v}$," is also found in some other literature, where the notation "$\mathbf{u}\oplus\mathbf{v}$" is defined differently as "$\mathbf{v}$ boosted by $\mathbf{u}$" with a reversed sense of operator and operand (or some other definition). In the expression $\mathbf{u}\oplus\mathbf{v}$, our operator $\oplus\mathbf{v}$ is a RHS operator that acts on LHS operand $\mathbf{u}$, while some other literature may define the LHS operator $\mathbf{u}\oplus$ that acts on $\mathbf{v}$. By our definitions of RHS unary operators $\oplus\mathbf{v}$ (73) and $\ominus\mathbf{v}$ (77), we arguably write more intuitive expressions such as $\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}=(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v}=\mathbf{u}$ with conventional left to right precedence of operations, while in other literature defining LHS operators this may be written backwards as $\mathbf{v}\ominus(\mathbf{v}\oplus\mathbf{u})=\mathbf{u}$ or perhaps as $\ominus\mathbf{v}\oplus(\mathbf{v}\oplus\mathbf{u})=\mathbf{u}$, which requires the parentheses to order the operations as right to left. The form of a relative vector $\mathbf{u}-\mathbf{v}$, of $\mathbf{u}$ relative to $\mathbf{v}$, better agrees with our choice of notational definition, expressing $\mathbf{u}$ relative to $\mathbf{v}$ as $\mathbf{u}\ominus\mathbf{v}$ (77). However, in other literature using LHS operators this same expression would be written $\mathbf{v}\ominus\mathbf{u}$ or perhaps $\ominus\mathbf{v}\oplus\mathbf{u}$, which is misleading or less intuitive. Some other literature may try to work around this notational problem by defining our RHS operators with other notations, such as $\boxplus\mathbf{v}$ and $\boxminus\mathbf{v}$, but whatever the exact differences may be in our definitions of $\oplus\mathbf{v}$ (73) and $\ominus\mathbf{v}$ (77) compared to other literature, we stand by our more general definitions of $\oplus\boldsymbol{v}$ (70) and $\ominus\boldsymbol{v}$ (74), which can operate not only on vectors $\boldsymbol{u}$ but also on versors and geometric entities.

The notations for the relatively equivalent active and passive boosts are

$$B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{-1}=B_{\mathbf{v}^{\dagger}}^{-1}\boldsymbol{u}B_{\mathbf{v}^{\dagger}}=\boldsymbol{u}\oplus\boldsymbol{v}=\boldsymbol{u}\ominus\boldsymbol{v}^{\dagger}=\gamma_{\mathbf{u}\ominus\mathbf{v}^{\dagger}}(\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v}^{\dagger}), \tag{78}$$

with the relatively equivalent active $\gamma_{\mathbf{u}\oplus\mathbf{v}}t_{p\boldsymbol{v}}=t_{c\boldsymbol{u}}$ and passive $\gamma_{\mathbf{u}\ominus\mathbf{v}^{\dagger}}t_{c\boldsymbol{u}}=t_{p\boldsymbol{v}^{\dagger}}$ time transformations, which are numerically equivalent for $t_{p\boldsymbol{v}}=t_{c\boldsymbol{o}}$ since $\gamma_{\mathbf{u}\oplus\mathbf{v}}=\gamma_{\mathbf{u}\ominus\mathbf{v}^{\dagger}}$. An active boost can be viewed as the equivalent passive boost, and vice versa, but their interpretations are different.

For the composition of active boosts $B_{\mathbf{v}}B_{\mathbf{u}}$, as $\boldsymbol{o}\oplus\boldsymbol{u}\oplus\boldsymbol{v}=B_{\mathbf{v}}B_{\mathbf{u}}\boldsymbol{o}B_{\mathbf{u}}^{-1}B_{\mathbf{v}}^{-1}$, we use the following notations

$$\begin{aligned}
B_{\mathbf{u}}\boldsymbol{o}B_{\mathbf{u}}^{-1}=\boldsymbol{o}\oplus\boldsymbol{u} &= \gamma_{0\oplus\mathbf{u}}(\boldsymbol{o}+0\oplus\mathbf{u})=\gamma_{\oplus\mathbf{u}}(\boldsymbol{o}+\mathbf{u})=\gamma_{\mathbf{u}}(\boldsymbol{o}+\mathbf{u})=\gamma_{\boldsymbol{o}\oplus\boldsymbol{u}}(\boldsymbol{o}+\mathbf{u}) &(79)\\
B_{\mathbf{v}}(\boldsymbol{o}\oplus\boldsymbol{u})B_{\mathbf{v}}^{-1}=\boldsymbol{o}\oplus\boldsymbol{u}\oplus\boldsymbol{v} &= \gamma_{\oplus\mathbf{u}}\gamma_{\mathbf{u}\oplus\mathbf{v}}(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})=\gamma_{\oplus\mathbf{u}\oplus\mathbf{v}}(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v}) &(80)\\
\gamma_{\oplus\mathbf{u}\oplus\mathbf{v}}=\gamma_{\boldsymbol{o}\oplus\boldsymbol{u}\oplus\boldsymbol{v}} &= \gamma_{\oplus\mathbf{u}}\gamma_{\mathbf{u}\oplus\mathbf{v}}=\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) &(81)\\
\gamma_{\oplus\mathbf{u}\ominus\mathbf{v}}=\gamma_{\oplus\mathbf{u}\oplus\mathbf{v}^{\dagger}} &= \gamma_{\oplus\mathbf{u}}\gamma_{\mathbf{u}\ominus\mathbf{v}}=\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) &(82)\\
\gamma_{\boldsymbol{o}\oplus\boldsymbol{u}\oplus\boldsymbol{v}}t_{p\boldsymbol{v}} &= \gamma_{\boldsymbol{o}\oplus\boldsymbol{u}}t_{p\boldsymbol{u}}=t_{c\boldsymbol{o}}=t. &(83)
\end{aligned}$$

For the composition of the relatively equivalent passive boosts $B_{\mathbf{v}}B_{\mathbf{u}}=B_{\mathbf{v}^{\dagger}}^{-1}B_{\mathbf{u}^{\dagger}}^{-1}$, we use the following notations

$$\begin{aligned}
B_{\mathbf{u}^{\dagger}}^{-1}\boldsymbol{o}B_{\mathbf{u}^{\dagger}}=\boldsymbol{o}\ominus\boldsymbol{u}^{\dagger} &= \gamma_{\boldsymbol{o}\ominus\boldsymbol{u}^{\dagger}}(\boldsymbol{o}-\mathbf{u}^{\dagger})=\gamma_{0\ominus\mathbf{u}^{\dagger}}(\boldsymbol{o}-\mathbf{u}^{\dagger})=\gamma_{\ominus\mathbf{u}^{\dagger}}(\boldsymbol{o}-\mathbf{u}^{\dagger})=\gamma_{\oplus\mathbf{u}}\boldsymbol{u} &(84)\\
B_{\mathbf{v}^{\dagger}}^{-1}(\boldsymbol{o}\ominus\boldsymbol{u}^{\dagger})B_{\mathbf{v}^{\dagger}}=\boldsymbol{o}\ominus\boldsymbol{u}^{\dagger}\ominus\boldsymbol{v}^{\dagger} &= \gamma_{\ominus\mathbf{u}^{\dagger}}\gamma_{\mathbf{u}^{\dagger}\ominus\mathbf{v}^{\dagger}}(\boldsymbol{o}-\mathbf{u}^{\dagger}\ominus\mathbf{v}^{\dagger})=\gamma_{\boldsymbol{o}\ominus\boldsymbol{u}^{\dagger}\ominus\boldsymbol{v}^{\dagger}}(\boldsymbol{o}-\mathbf{u}^{\dagger}\ominus\mathbf{v}^{\dagger}) &(85)\\
&= \gamma_{\boldsymbol{o}\oplus\boldsymbol{u}\oplus\boldsymbol{v}}(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})=\boldsymbol{o}\oplus\boldsymbol{u}\oplus\boldsymbol{v}\\
t_{c\boldsymbol{o}}\gamma_{\boldsymbol{o}\ominus\boldsymbol{u}^{\dagger}\ominus\boldsymbol{v}^{\dagger}} &= t_{p\boldsymbol{u}^{\dagger}}\gamma_{\boldsymbol{u}^{\dagger}\ominus\boldsymbol{v}^{\dagger}}=t_{p\boldsymbol{v}^{\dagger}}. &(86)
\end{aligned}$$

Although $B_{\mathbf{v}}B_{\mathbf{u}}=B_{\mathbf{v}^{\dagger}}^{-1}B_{\mathbf{u}^{\dagger}}^{-1}$, their interpretations and time transformations are different.

For continued compositions of boosts $\boldsymbol{t}\oplus\boldsymbol{u}\oplus\ldots\oplus\boldsymbol{v}\oplus\boldsymbol{w}$ and $\boldsymbol{t}\ominus\boldsymbol{u}\ominus\ldots\ominus\boldsymbol{v}\ominus\boldsymbol{w}$ etc., the notations $\gamma_{\boldsymbol{t}\oplus\boldsymbol{u}\oplus\ldots\oplus\boldsymbol{v}\oplus\boldsymbol{w}}t_{p\boldsymbol{w}} = \gamma_{\boldsymbol{t}\oplus\boldsymbol{u}\oplus\ldots\oplus\boldsymbol{v}}t_{p\boldsymbol{v}} = \ldots = \gamma_{\boldsymbol{t}\oplus\boldsymbol{u}}t_{p\boldsymbol{u}} = t_{c\boldsymbol{t}}$ and $t_{c\boldsymbol{t}}\gamma_{\boldsymbol{t}\ominus\boldsymbol{u}\ominus\ldots\ominus\boldsymbol{v}\ominus\boldsymbol{w}} = t_{p\boldsymbol{u}}\gamma_{\boldsymbol{u}\ominus\ldots\ominus\boldsymbol{v}\ominus\boldsymbol{w}} = \ldots = t_{p\boldsymbol{v}}\gamma_{\boldsymbol{v}\ominus\boldsymbol{w}} = t_{p\boldsymbol{w}}$ etc. may be most intuitive for the time transformations from frame to frame. For a mixed active/passive composition $\gamma_{\mathbf{t}\ominus\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{w}} = \gamma_{\boldsymbol{t}\ominus\boldsymbol{u}\oplus\boldsymbol{v}\ominus\boldsymbol{w}}$ (as an example), we must have $t_{c\boldsymbol{t}}\gamma_{\boldsymbol{t}\ominus\boldsymbol{u}\oplus\boldsymbol{v}\ominus\boldsymbol{w}} = t_{p\boldsymbol{w}}$ since the last boost is passive into the frame of $\boldsymbol{w}$, and then we make an equivalent purely passive factor $\gamma_{\boldsymbol{t}\ominus\boldsymbol{u}\oplus\boldsymbol{v}\ominus\boldsymbol{w}} = \gamma_{\boldsymbol{t}\ominus\boldsymbol{u}\ominus\boldsymbol{v}^\dagger\ominus\boldsymbol{w}}$, where $t_{c\boldsymbol{t}}\gamma_{\boldsymbol{t}\ominus\boldsymbol{u}\ominus\boldsymbol{v}^\dagger\ominus\boldsymbol{w}} = t_{p\boldsymbol{u}}\gamma_{\boldsymbol{u}\ominus\boldsymbol{v}^\dagger\ominus\boldsymbol{w}} = t_{p\boldsymbol{v}}\gamma_{\boldsymbol{v}^\dagger\ominus\boldsymbol{w}} = t_{p\boldsymbol{w}}$, noting that $t_{p\boldsymbol{v}} = t_{p\boldsymbol{v}^\dagger}$ etc. Similarly, a continued boost that ends active $\gamma_{\boldsymbol{t}\oplus\boldsymbol{u}\ominus\boldsymbol{v}\oplus\boldsymbol{w}}$ must have $\gamma_{\boldsymbol{t}\oplus\boldsymbol{u}\ominus\boldsymbol{v}\oplus\boldsymbol{w}}t_{p\boldsymbol{w}} = t_{c\boldsymbol{t}}$, and then we make an equivalent purely active factor $\gamma_{\boldsymbol{t}\oplus\boldsymbol{u}\ominus\boldsymbol{v}\oplus\boldsymbol{w}} = \gamma_{\boldsymbol{t}\oplus\boldsymbol{u}\oplus\boldsymbol{v}^\dagger\oplus\boldsymbol{w}}$, where $\gamma_{\boldsymbol{t}\oplus\boldsymbol{u}\oplus\boldsymbol{v}^\dagger\oplus\boldsymbol{w}}t_{p\boldsymbol{w}} = \gamma_{\boldsymbol{t}\oplus\boldsymbol{u}\oplus\boldsymbol{v}^\dagger}t_{p\boldsymbol{v}} = \gamma_{\boldsymbol{t}\oplus\boldsymbol{u}}t_{p\boldsymbol{u}} = t_{c\boldsymbol{t}}$. A continued active boost has a time transformation taking a proper time $t_{p\boldsymbol{w}}$ of $\boldsymbol{w}$ through many proper times to a coordinate time $t_{c\boldsymbol{t}}$ of $\boldsymbol{t}$, and a continued passive boost takes a coordinate time $t_{c\boldsymbol{t}}$ through many proper times to the proper time $t_{p\boldsymbol{w}}$ of $\boldsymbol{w}$. Note that, $\boldsymbol{t}$ can be replaced with $\boldsymbol{o}\oplus\boldsymbol{t} = \boldsymbol{o}\ominus\boldsymbol{t}^\dagger$, and then the purely active boosts convert time into $t_{c\boldsymbol{o}} = t_{p\boldsymbol{o}}$, and the purely passive boosts convert from $t_{c\boldsymbol{o}} = t_{p\boldsymbol{o}} = t_{c\boldsymbol{t}}$, such that only proper times are transformed from frame to frame.

For the active boost $\boldsymbol{u}\oplus\boldsymbol{v}$, the spatial velocity $\mathbf{u}\oplus\mathbf{v}$ generally has a natural speed

$$\beta_{\mathbf{u}\oplus\mathbf{v}} = \|\mathbf{u}\oplus\mathbf{v}\|/c, \tag{87}$$

and for the special case $\mathbf{u}||\mathbf{v}$ of parallel velocities

$$\beta_{\mathbf{u}\oplus\mathbf{v}} = \frac{\beta_{\mathbf{u}}+\beta_{\mathbf{v}}}{1+\beta_{\mathbf{u}}\beta_{\mathbf{v}}}. \tag{88}$$

For the passive boost $\boldsymbol{u}\ominus\boldsymbol{v}$, relative to $\boldsymbol{v}$, the spatial velocity $\mathbf{u}\ominus\mathbf{v}$ generally has a natural speed

$$\beta_{\mathbf{u}\ominus\mathbf{v}} = \|\mathbf{u}\ominus\mathbf{v}\|/c, \tag{89}$$

and for the special case $\mathbf{u}||\mathbf{v}$ of parallel velocities

$$\beta_{\mathbf{u}\ominus\mathbf{v}} = \frac{\beta_{\mathbf{u}}-\beta_{\mathbf{v}}}{1-\beta_{\mathbf{u}}\beta_{\mathbf{v}}}. \tag{90}$$

For the special case $\mathbf{u}\perp\mathbf{v}$ of perpendicular velocities

$$\beta_{\mathbf{u}\oplus\mathbf{v}} = \beta_{\mathbf{u}\ominus\mathbf{v}} = \sqrt{(1-\beta_{\mathbf{v}}^2)\beta_{\mathbf{u}}^2+\beta_{\mathbf{v}}^2}. \tag{91}$$

The boost notations and formulas given above are derived directly from the boost operations. The notations can extend to further compositions of boosts.

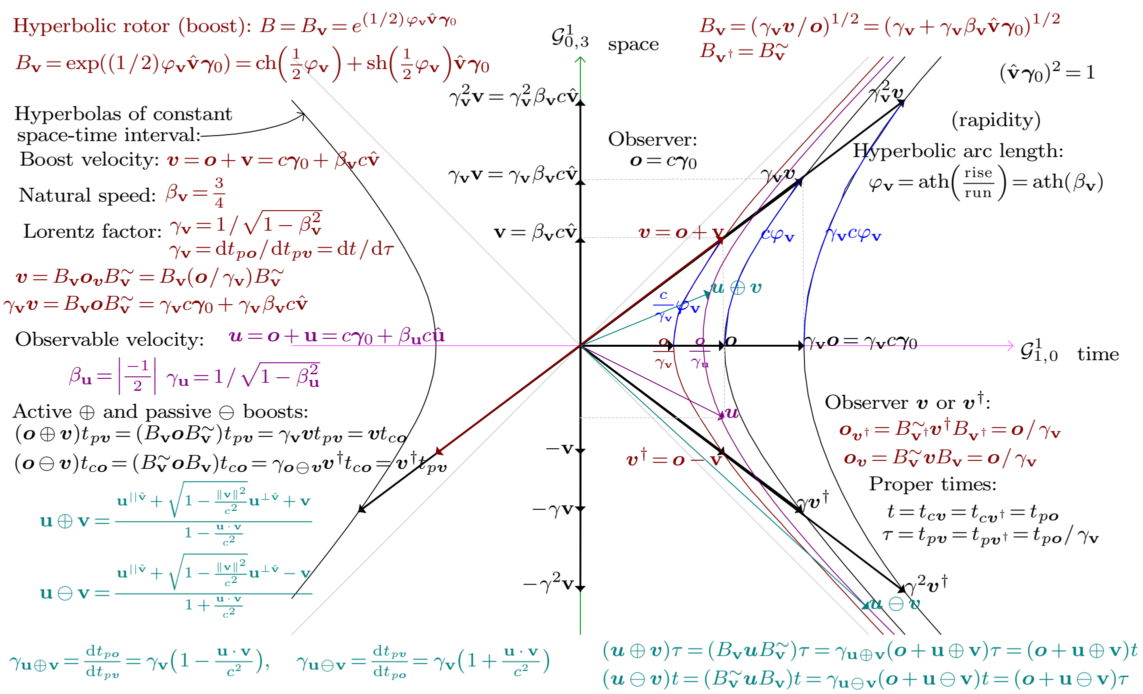

Figure 1. Space-time diagram of space-time boost B operations.

Figure 1 shows a space-time diagram of space-time boost B operations on space-time velocities as hyperbolic rotations. The orientation places the pseudospatial time axis $\boldsymbol{\gamma}_0 \in \mathcal{G}^1_{1,0}$ horizontal and the anti-Euclidean spatial axis $\hat{\mathbf{v}} \in \mathcal{G}^1_{0,3}$ vertical. The slope β of a space-time velocity $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v} = c\boldsymbol{\gamma}_0 + \beta c\hat{\mathbf{v}}$ is the natural speed $\beta = \beta_{\mathbf{v}} = \|\mathbf{v}\| / \|\boldsymbol{o}\| = v / c$. The Lorentz factor of $\boldsymbol{v}$ is $\gamma = \gamma_{\mathbf{v}} = |\boldsymbol{o}| / |\boldsymbol{v}| = c / \sqrt{c^2 - \beta^2_{\mathbf{v}} c^2} = 1 / \sqrt{1 - \beta^2_{\mathbf{v}}}$. In close analogy to elliptic trigonometry in a Euclidean plane where $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $\theta = \mathrm{atan}(y / x)$, in the Minkowski space-time plane of Figure 1, we have hyperbolic trigonometry where $x = r\cosh(\varphi)$, $y = r\sinh(\varphi)$, and $\varphi = \mathrm{atanh}(y / x) = \mathrm{atanh}(\beta)$. Here, r is the constant hyperbolic radius under hyperbolic rotations (r^2 is the constant interval), and φ is the hyperbolic angle (rapidity). The stationary observer has space-time velocity $\boldsymbol{o} = r\boldsymbol{\gamma}_0 = c\boldsymbol{\gamma}_0$. The observer worldline is $\boldsymbol{o}t$. The hyperbolic rotation $\boldsymbol{u}' = \boldsymbol{u} \oplus \boldsymbol{v} = B_{\mathbf{v}}\boldsymbol{u}B^{\sim}_{\mathbf{v}}$ of a space-time velocity $\boldsymbol{u}$ by angle $\varphi_{\mathbf{v}} = \mathrm{atanh}(\beta_{\mathbf{v}})$ is an active boost by spatial velocity $\mathbf{v} = \beta c\hat{\mathbf{v}}$ that transforms the slope $\beta_{\mathbf{u}}$ of the space-time velocity $\boldsymbol{u}$ into $\beta'_{\mathbf{u}} = \beta_{\mathbf{u}\oplus\mathbf{v}}$ of $\boldsymbol{u}' = \boldsymbol{u} \oplus \boldsymbol{v}$, while holding the interval $|\boldsymbol{u}|^2 = |\boldsymbol{u}'|^2$ constant. In the figure, the active boost of $\boldsymbol{o}$ by $\boldsymbol{v}$ is $\boldsymbol{o} \oplus \boldsymbol{v} = B_{\mathbf{v}}\boldsymbol{o}B^{\sim}_{\mathbf{v}} = \gamma_{\mathbf{v}}\boldsymbol{v}$, where the new time $\tau = t_{p\boldsymbol{v}}$ is the proper time of $\boldsymbol{v}$, where $\gamma_{\mathbf{v}}\tau = t$ and $\gamma_{\mathbf{v}}\boldsymbol{v}\tau = \boldsymbol{v}t = (\boldsymbol{o} + \mathbf{v})t$ is in the frame of $\boldsymbol{o}$. Note that, no boost B can ever boost a speed β to exactly $\beta' = 1$ since the hyperbolic rotation can only asymptotically approach, but never reach, the direction of a light-speed null vector $c\boldsymbol{\gamma}_0 \pm c\hat{\mathbf{v}}$ on the light-like null hypercone. The time-like hyperbolic (pseudo-Euclidean) length $|\boldsymbol{v}t| = |(\boldsymbol{o} + \mathbf{v})t| = \sqrt{(c\boldsymbol{\gamma}_0 + \beta c\hat{\mathbf{v}})^2 t^2} = \sqrt{1 - \beta^2}ct = ct / \gamma = c\tau = ct_{p\boldsymbol{v}}$ gives the proper time $\tau = t_{p\boldsymbol{v}}$ of $\boldsymbol{v}$. Proper time is the pseudo-Euclidean length of the worldline when using only natural speeds with $c = 1$. For $\beta = 0$, the observable $\boldsymbol{v}t$ coincides with the observer $\boldsymbol{o}t$, and the observer measures time $t = L_0$. For $0 \leq \beta \leq 1$, the observer computes the time $t_{p\boldsymbol{v}} = L = \sqrt{1 - \beta^2}L_0 = L_0 / \gamma$, which is the special relativity length contraction formula (66) for length (and time, or space-time) contraction in direction $\hat{\mathbf{v}}$ as experienced by the observable $\boldsymbol{v}$ relative to $\boldsymbol{o}$. In effect, $\boldsymbol{v}t = (\boldsymbol{v} \ominus \boldsymbol{o})t$, relative to $\boldsymbol{o}t$, experiences contraction $\boldsymbol{o}\tau = \boldsymbol{o}t / \gamma_{\mathbf{v}} = (\boldsymbol{v} \ominus \boldsymbol{v})t$.

A composition of two successive unimodular boosts $B_{\mathbf{v}}B_{\mathbf{u}} = \hat{B}_{\mathbf{v}}\hat{B}_{\mathbf{u}} = \hat{B}_{\mathbf{w}}\hat{R}_{\epsilon\hat{\mathbf{n}}}$, by velocities $\mathbf{u}$ then $\mathbf{v}$, is equivalent to a Thomas-Wigner rotation $\hat{R}_{\epsilon\hat{\mathbf{n}}}$ that is followed by a single resultant boost $\hat{B}_{\mathbf{w}}$ by a velocity $\mathbf{w} = 0 \oplus \mathbf{u} \oplus \mathbf{v} = \mathbf{u} \oplus \mathbf{v}$. The Thomas-Wigner rotation $R_{\epsilon\hat{\mathbf{n}}}$ is a spatial rotation in the plane of $\mathbf{u}$ and $\mathbf{v}$ (with normal $\mathbf{n} = \epsilon\hat{\mathbf{n}}$) by an angle ϵ. However, the factoring of $B_{\mathbf{v}}B_{\mathbf{u}}$ can be done in two different ways as $B_{\mathbf{v}}B_{\mathbf{u}} = \hat{B}_{\mathbf{w}_2}\hat{R}_{\mathbf{n}} = \hat{R}_{\mathbf{n}}\hat{B}_{\mathbf{w}_1}$, where $\hat{B}_{\mathbf{w}_2} = \hat{B}_{\mathbf{w}}$. The product of two unimodular boosts $B_{\mathbf{v}}B_{\mathbf{u}}$ expands as

$$\begin{aligned}
B_{\mathbf{v}}B_{\mathbf{u}} &= \exp\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right)\exp\left(\frac{1}{2}\varphi_{\mathbf{u}}\hat{\mathbf{u}}\boldsymbol{\gamma}_0\right) && (92)\\
&= \cosh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\cosh\left(\frac{1}{2}\varphi_{\mathbf{u}}\right) + \sinh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\sinh\left(\frac{1}{2}\varphi_{\mathbf{u}}\right)(-\hat{\mathbf{v}}\hat{\mathbf{u}}) + && (93)\\
&\quad \cosh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\sinh\left(\frac{1}{2}\varphi_{\mathbf{u}}\right)\hat{\mathbf{u}}\boldsymbol{\gamma}_0 + \sinh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\cosh\left(\frac{1}{2}\varphi_{\mathbf{u}}\right)\hat{\mathbf{v}}\boldsymbol{\gamma}_0\\
&= h_{\mathrm{cc}} + h_{\mathrm{ss}}(-\hat{\mathbf{v}}\hat{\mathbf{u}}) + (h_{\mathrm{cs}}\hat{\mathbf{u}} + h_{\mathrm{sc}}\hat{\mathbf{v}})\boldsymbol{\gamma}_0. && (94)
\end{aligned}$$

The part of the expanded product $B_{\mathbf{v}}B_{\mathbf{u}}$ that is purely a spatial rotation $R_{\mathbf{n}}$ is

$$\begin{aligned}
R_{\mathbf{n}} &= h_{\mathrm{cc}} + h_{\mathrm{ss}}(-\hat{\mathbf{v}}\hat{\mathbf{u}}) = h_{\mathrm{cc}} + h_{\mathrm{ss}}(-\hat{\mathbf{v}} \cdot \hat{\mathbf{u}} - \hat{\mathbf{v}} \wedge \hat{\mathbf{u}}) = h_{\mathrm{cc}} + h_{\mathrm{ss}}\left(\cos(\theta) - \sin(\theta)\widehat{\hat{\mathbf{v}}^{\perp\hat{\mathbf{u}}}} \wedge \hat{\mathbf{u}}\right) && (95)\\
\hat{R}_{\mathbf{n}} &= |R_{\mathbf{n}}|^{-1}R_{\mathbf{n}} = \exp\left(\frac{1}{2}\epsilon\hat{\mathbf{n}}^*_{\mathcal{S}}\right) = \cos\left(\frac{1}{2}\epsilon\right) + \sin\left(\frac{1}{2}\epsilon\right)\hat{\mathbf{n}}^*_{\mathcal{S}} = \hat{R}_{\epsilon\hat{\mathbf{n}}}, && (96)
\end{aligned}$$

where the angle $0 \leq \theta \leq \pi$ between $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is

$$\theta = \mathrm{acos}(-\hat{\mathbf{v}} \cdot \hat{\mathbf{u}}), \tag{97}$$

the modulus of $R_{\mathbf{n}}$, as an elliptic complex number with $\left(-\widehat{\hat{\mathbf{v}}^{\perp\hat{\mathbf{u}}}} \wedge \hat{\mathbf{u}}\right)^2 = -1$, is

$$|R_{\mathbf{n}}| = \sqrt{R^{-}_{\mathbf{n}}R_{\mathbf{n}}} = \sqrt{R^{\sim}_{\mathbf{n}}R_{\mathbf{n}}}, \tag{98}$$

and the spatial Thomas-Wigner rotation angle ϵ of $\hat{R}_{\mathbf{n}}$ is

$$\epsilon = 2\,\mathrm{acos}\left(\frac{\hat{R}_{\mathbf{n}} + \hat{R}^{\sim}_{\mathbf{n}}}{2}\right). \tag{99}$$

Using trigonometric identities, the Thomas-Wigner rotation angle ϵ is also expressed as [20]

$$1+\cos(\epsilon)=\frac{(1+\gamma_{\oplus\mathbf{u}\oplus\mathbf{v}}+\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}})^2}{(1+\gamma_{\oplus\mathbf{u}\oplus\mathbf{v}})(1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}})}>0. \tag{100}$$

The axis $\hat{\mathbf{n}}_{\mathcal{S}}$ of the Thomas-Wigner rotation $\hat{R}_{\mathbf{n}}=\hat{R}_{\epsilon\hat{\mathbf{n}}}$ is (by the "*undual*" operation),

$$\hat{\mathbf{n}}_{\mathcal{S}}=-(\hat{\mathbf{n}}_{\mathcal{S}}^*)\mathbf{I}_{\mathcal{S}}=-\left(\frac{\hat{R}_{\mathbf{n}}-\hat{R}_{\mathbf{n}}^{\sim}}{|\hat{R}_{\mathbf{n}}-\hat{R}_{\mathbf{n}}^{\sim}|}\right)\mathbf{I}_{\mathcal{S}}, \tag{101}$$

where the orientation of the axis $\hat{\mathbf{n}}_{\mathcal{S}}$ in (101) is such that the rotation is from $\hat{\mathbf{u}}$ toward $\hat{\mathbf{v}}$ by angle $0\leq\epsilon\leq\pi$ in (99). Note that, in the anti-Euclidean metric of SA $\mathcal{G}_{0,3}$, the unit bivector $\hat{\mathbf{n}}_{\mathcal{S}}^*$ (generator of rotation) is formed in reverse (or negative) to the similar unit bivector $\hat{\mathbf{n}}_{\mathcal{E}}^*$ in the Euclidean metric of $\mathcal{G}_3$.

Using the pure rotor $R_{\mathbf{n}}$, the composition of boosts $B_{\mathbf{v}}B_{\mathbf{u}}$ factorizes two different ways as

$$\begin{aligned}
B_{\mathbf{v}}B_{\mathbf{u}} &= R_{\mathbf{n}}(1+R_{\mathbf{n}}^{-1}(h_{\rm cs}\hat{\mathbf{u}}+h_{\rm sc}\hat{\mathbf{v}})\boldsymbol{\gamma}_0)=R_{\mathbf{n}}B_{\mathbf{w}_1} && (102)\\
&= (1+(h_{\rm cs}\hat{\mathbf{u}}+h_{\rm sc}\hat{\mathbf{v}})R_{\mathbf{n}}^{-1}\boldsymbol{\gamma}_0)R_{\mathbf{n}}=B_{\mathbf{w}_2}R_{\mathbf{n}} && (103)\\
\hat{B}_{\mathbf{w}_1} &= |B_{\mathbf{w}_1}|^{-1}B_{\mathbf{w}_1}=\exp\left(\frac{1}{2}\varphi_{\mathbf{w}_1}\hat{\mathbf{w}}_1\boldsymbol{\gamma}_0\right)=\cosh\left(\frac{1}{2}\varphi_{\mathbf{w}_1}\right)+\sinh\left(\frac{1}{2}\varphi_{\mathbf{w}_1}\right)\hat{\mathbf{w}}_1\boldsymbol{\gamma}_0 && (104)\\
\hat{B}_{\mathbf{w}_2} &= |B_{\mathbf{w}_2}|^{-1}B_{\mathbf{w}_2}=\exp\left(\frac{1}{2}\varphi_{\mathbf{w}_2}\hat{\mathbf{w}}_2\boldsymbol{\gamma}_0\right)=\cosh\left(\frac{1}{2}\varphi_{\mathbf{w}_2}\right)+\sinh\left(\frac{1}{2}\varphi_{\mathbf{w}_2}\right)\hat{\mathbf{w}}_2\boldsymbol{\gamma}_0, && (105)
\end{aligned}$$

where the inverse $R_{\mathbf{n}}^{-1}$ of spatial rotor $R_{\mathbf{n}}$ is

$$R_{\mathbf{n}}^{-1}=\frac{R_{\mathbf{n}}^{-}}{R_{\mathbf{n}}^{-}R_{\mathbf{n}}}=\frac{R_{\mathbf{n}}^{\sim}}{R_{\mathbf{n}}^{\sim}R_{\mathbf{n}}}=\frac{h_{\rm cc}+h_{\rm ss}(-\hat{\mathbf{u}}\hat{\mathbf{v}})}{|R_{\mathbf{n}}|^2}=\frac{\hat{R}_{\mathbf{n}}^{\sim}}{|R_{\mathbf{n}}|}, \tag{106}$$

and the modulus $|B_{\mathbf{w}}|$ of $B_{\mathbf{w}}=|B_{\mathbf{w}}|\exp\left(\frac{1}{2}\varphi_{\mathbf{w}}\hat{\mathbf{w}}\boldsymbol{\gamma}_0\right)$, as a hyperbolic complex number with $(\hat{\mathbf{w}}\boldsymbol{\gamma}_0)^2=1$, is

$$|B_{\mathbf{w}}|=\sqrt{|B_{\mathbf{w}}^{-}B_{\mathbf{w}}|}=\sqrt{|B_{\mathbf{w}}^{\sim}B_{\mathbf{w}}|}=\sqrt{|B_{\mathbf{w}}|^2}. \tag{107}$$

For the composition of boosts $B_{\mathbf{v}}B_{\mathbf{u}}$, we now have the two unimodular factorings

$$B_{\mathbf{v}}B_{\mathbf{u}}=\hat{B}_{\mathbf{v}}\hat{B}_{\mathbf{u}}=\hat{B}_{\mathbf{w}_2}\hat{R}_{\mathbf{n}}=\hat{R}_{\mathbf{n}}\hat{B}_{\mathbf{w}_1}. \tag{108}$$

The boosts $B_{\mathbf{w}_1}$ and $B_{\mathbf{w}_2}$ can also be written as

$$\begin{aligned}
B_{\mathbf{w}_1} &= 1+\frac{\hat{R}_{\mathbf{n}}^{\sim}(h_{\rm cs}\hat{\mathbf{u}}+h_{\rm sc}\hat{\mathbf{v}})}{|R_{\mathbf{n}}|}\boldsymbol{\gamma}_0=1+\alpha_1\hat{\mathbf{w}}_1\boldsymbol{\gamma}_0=1+\tanh\left(\frac{1}{2}\varphi_{\mathbf{w}_1}\right)\hat{\mathbf{w}}_1\boldsymbol{\gamma}_0 && (109)\\
B_{\mathbf{w}_2} &= 1+\frac{\hat{R}_{\mathbf{n}}(h_{\rm cs}\hat{\mathbf{u}}+h_{\rm sc}\hat{\mathbf{v}})}{|R_{\mathbf{n}}|}\boldsymbol{\gamma}_0=1+\alpha_2\hat{\mathbf{w}}_2\boldsymbol{\gamma}_0=1+\tanh\left(\frac{1}{2}\varphi_{\mathbf{w}_2}\right)\hat{\mathbf{w}}_2\boldsymbol{\gamma}_0, && (110)
\end{aligned}$$

and we see that $\|\alpha_1\hat{\mathbf{w}}_1\|=\|\alpha_2\hat{\mathbf{w}}_2\|=\alpha$. Therefore,

$$\begin{aligned}
\varphi_{\mathbf{w}} &= \varphi_{\mathbf{w}_1}=\varphi_{\mathbf{w}_2}=2\,\mathrm{atanh}(\alpha)=2\,\mathrm{atanh}\left(\frac{\|h_{\rm cs}\hat{\mathbf{u}}+h_{\rm sc}\hat{\mathbf{v}}\|}{|R_{\mathbf{n}}|}\right) && (111)\\
\beta_{\mathbf{w}} &= \tanh(\varphi_{\mathbf{w}})=\beta_{\mathbf{w}_1}=\beta_{\mathbf{w}_2}. && (112)
\end{aligned}$$

In the composition $B_{\mathbf{v}}B_{\mathbf{u}}=\hat{B}_{\mathbf{w}_2}\hat{R}_{\mathbf{n}}$, the purely spatial rotation $\hat{R}_{\mathbf{n}}$ is applied first to a purely spatial point (or other geometric entity) with zero velocity, and then the space-time boost $\hat{B}_{\mathbf{w}_2}$ is applied second (subscript 2). Therefore, the velocity addition direction $\widehat{\mathbf{u}\oplus\mathbf{v}}$ is

$$\hat{\mathbf{w}}=\hat{\mathbf{w}}_2=\widehat{\mathbf{u}\oplus\mathbf{v}}=\widehat{\hat{R}_{\mathbf{n}}(h_{\rm cs}\hat{\mathbf{u}}+h_{\rm sc}\hat{\mathbf{v}})}, \tag{113}$$

and the unimodular boost $\hat{B}_{\mathbf{w}}=\hat{B}_{\mathbf{w}_2}=\hat{B}_{\mathbf{u}\oplus\mathbf{v}}$ can be expressed as

$$\hat{B}_{\mathbf{w}}=\hat{B}_{\mathbf{u}\oplus\mathbf{v}}=\hat{B}_{\mathbf{v}}\hat{B}_{\mathbf{u}}\hat{R}_{\epsilon\hat{\mathbf{n}}}^{\sim}, \tag{114}$$

with boost space-time velocity

$$\boldsymbol{w} = \boldsymbol{o} + \mathbf{w} = \boldsymbol{o} + \mathbf{u} \oplus \mathbf{v} = c\boldsymbol{\gamma}_0 + \beta_{\mathbf{w}} c\hat{\mathbf{w}}. \tag{115}$$

A triad of boosts $\hat{B}_{\mathbf{w}^\dagger}\hat{B}_{\mathbf{v}}\hat{B}_{\mathbf{u}}$ that returns to zero boost velocity is a Thomas-Wigner rotation

$$\hat{B}_{\mathbf{w}^\dagger}\big(\hat{B}_{\mathbf{v}}\hat{B}_{\mathbf{u}}\big) = \hat{B}_{\mathbf{u}\oplus\mathbf{v}}^{\sim}\big(\hat{B}_{\mathbf{v}}\hat{B}_{\mathbf{u}}\big) = \hat{R}_{\epsilon\hat{\mathbf{n}}}\hat{B}_{\mathbf{u}}^{\sim}\hat{B}_{\mathbf{v}}^{\sim}\big(\hat{B}_{\mathbf{v}}\hat{B}_{\mathbf{u}}\big) = \hat{B}_0\hat{R}_{\epsilon\hat{\mathbf{n}}} = \hat{R}_{\epsilon\hat{\mathbf{n}}}. \tag{116}$$

However, a quad of boosts $\hat{B}_{\mathbf{u}^\dagger}\hat{B}_{\mathbf{v}^\dagger}\hat{B}_{\mathbf{v}}\hat{B}_{\mathbf{u}} = 1 = \hat{B}_0\hat{R}_0$ that returns to zero boost velocity also returns to zero rotation.

In the composition $B_{\mathbf{v}}B_{\mathbf{u}} = \hat{R}_{\epsilon\hat{\mathbf{n}}}\hat{B}_{\mathbf{w}_1}$, the space-time boost $B_{\mathbf{w}_1}$ is applied first (subscript 1), and then the spatial rotation $\hat{R}_{\epsilon\hat{\mathbf{n}}}$ is applied second, where $\hat{R}_{\epsilon\hat{\mathbf{n}}}$ rotates the boost velocity direction $\hat{\mathbf{w}}_1$ as

$$\hat{R}_{\epsilon\hat{\mathbf{n}}}\hat{\mathbf{w}}_1\hat{R}_{\epsilon\hat{\mathbf{n}}}^{\sim} = \widehat{\hat{R}_{\epsilon\hat{\mathbf{n}}}\hat{R}_{\epsilon\hat{\mathbf{n}}}^{\sim}(h_{\mathrm{cs}}\hat{\mathbf{u}} + h_{\mathrm{sc}}\hat{\mathbf{v}})\hat{R}_{\epsilon\hat{\mathbf{n}}}^{\sim}} = \widehat{\hat{R}_{\epsilon\hat{\mathbf{n}}}(h_{\mathrm{cs}}\hat{\mathbf{u}} + h_{\mathrm{sc}}\hat{\mathbf{v}})} = \hat{\mathbf{w}}_2 = \hat{\mathbf{w}}. \tag{117}$$

Applying the rotation $\hat{R}_{\epsilon\hat{\mathbf{n}}}$ second has the effect of making a trajectory correction by the rotation $\hat{R}_{\epsilon\hat{\mathbf{n}}}\hat{\mathbf{w}}_1\hat{R}_{\epsilon\hat{\mathbf{n}}}^{\sim} = \hat{\mathbf{w}}_2$ of the boost velocity direction $\hat{\mathbf{w}}_1$ into $\hat{\mathbf{w}}_2 = \hat{\mathbf{w}}$, as well as spatially rotating any spatial entity. Spatial rotation of a boosted point or entity also rotates its velocity direction.

To apply only an arbitrary spatial rotation $R_{\mathbf{n}}$ to an entity with space-time velocity $\boldsymbol{u} = \boldsymbol{o} + \mathbf{u}$, without changing the velocity $\mathbf{u}$ or changing from the frame of $\boldsymbol{u}$, we use the *boosted rotor*

$$R_{\mathbf{n}} \oplus \boldsymbol{u} = B_{\mathbf{u}} R_{\mathbf{n}} B_{\mathbf{u}}^{\sim}, \tag{118}$$

where $R_{\mathbf{n}} \oplus \boldsymbol{u}$ relative to $\boldsymbol{u}$ is the spatial rotation $(R_{\mathbf{n}} \oplus \boldsymbol{u}) \ominus \boldsymbol{u} = R_{\mathbf{n}}$ in the frame of $\boldsymbol{u}$. Applying $R_{\mathbf{n}}$ directly to an entity with velocity $\mathbf{u}$ will also rotate the velocity into $\mathbf{u}' = R_{\mathbf{n}}\mathbf{u}R_{\mathbf{n}}^{\sim}$, which rotates the entity as stationary in the frame of $\boldsymbol{u}' = \boldsymbol{o} + \mathbf{u}'$. Similarly, to apply an arbitrary boost $B_{\mathbf{v}}$ to an entity in the frame of $\boldsymbol{u}$, without changing from the frame of $\boldsymbol{u}$, we use the *boosted boost*

$$B_{\mathbf{v}} \oplus \boldsymbol{u} = B_{\mathbf{u}} B_{\mathbf{v}} B_{\mathbf{u}}^{\sim}, \tag{119}$$

where $B_{\mathbf{v}} \oplus \boldsymbol{u}$ relative to $\boldsymbol{u}$ is the boost $(B_{\mathbf{v}} \oplus \boldsymbol{u}) \ominus \boldsymbol{u} = B_{\mathbf{v}}$ in the frame of $\boldsymbol{u}$. For a stationary entity in the frame of $\boldsymbol{u}$, then $B_{\mathbf{v}} \oplus \boldsymbol{u}$ is effectively equal to applying $B_{\mathbf{u}}B_{\mathbf{v}}$ ($\neq B_{\mathbf{v}}B_{\mathbf{u}}$) to the entity when stationary in the frame of $\boldsymbol{o}$. Note that, $B_{\mathbf{u}}B_{\mathbf{v}}$ boosts into the frame of $\boldsymbol{u}$, while $B_{\mathbf{v}}B_{\mathbf{u}}$ boosts into the frame of $\boldsymbol{v}$.

As we will show, the boost effects of $B_{\mathbf{v}}B_{\mathbf{u}}$, including length contraction $L = \sqrt{1 - \beta_{\mathbf{w}}^2}L_0$ and Thomas-Wigner rotation $\hat{R}_{\epsilon\hat{\mathbf{n}}}$, are easy to demonstrate when boosting the DCSTA 2-vector quadric surface entities (see Figure 4).

3 Notation of Conformal STA (CSTA)

The basis of CSTA $\mathcal{G}_{2,4}\,\mathcal{C}$, isomorphic to ($\cong$) CSTA1 $\mathcal{G}_{2,4}\,\mathcal{C}^1$ (index $\gamma = 1$), is

$$\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \mathbf{e}_+, \mathbf{e}_-\} \cong \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}, \tag{120}$$

and for the second copy CSTA2 $\mathcal{G}_{2,4}\,\mathcal{C}^2$ (index $\gamma = 2$),

$$\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \mathbf{e}_+, \mathbf{e}_-\} \cong \{\mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{12}\}. \tag{121}$$

The six-dimensional CSTA unit pseudoscalar is

$$\mathbf{I}_{\mathcal{C}} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\mathbf{e}_+\mathbf{e}_-. \tag{122}$$

The $\mathcal{G}_{1,4}$ Conformal Space Algebra (CSA), subalgebra of $\mathcal{G}_{2,4}$ CSTA, omits the time-like basis vector $\boldsymbol{\gamma}_0$ and the time coordinate $w = ct = 0$, and then has only spatial entities and operations that are similar to those of $\mathcal{G}_{4,1}$ CGA. CSTA defines three *geometric inner product null space* (GIPNS) [18] 1-blade *entities*, as follows.

The CSTA GIPNS 1-blade *null hypercone* entity $\boldsymbol{K}_{\mathcal{C}}$ (growing sphere in time from a point), equal to the null *point* embedding $\boldsymbol{P}_{\mathcal{C}}$, is

$$\hat{\boldsymbol{K}}_{\mathcal{C}} = \hat{\boldsymbol{P}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \boldsymbol{p}_{\mathcal{M}} + \frac{1}{2}\boldsymbol{p}_{\mathcal{M}}^2\mathbf{e}_{\infty\gamma} + \mathbf{e}_{o\gamma}, \qquad \boldsymbol{P}_{\mathcal{C}}^2 = 0, \tag{123}$$

centered at vertex $\boldsymbol{p}_{\mathcal{M}}$ with null *infinity point* (representing the point at infinity)

$$\mathbf{e}_{\infty\gamma} = \mathbf{e}_+ + \mathbf{e}_-, \qquad \mathbf{e}_{\infty\gamma}^2 = 0, \tag{124}$$

and null *origin point* (representing the point at the origin)

$$\mathbf{e}_{o\gamma} = (\mathbf{e}_- - \mathbf{e}_+)/2, \qquad \mathbf{e}_{o\gamma}^2 = 0, \qquad \mathbf{e}_{o\gamma} \cdot \mathbf{e}_{\infty\gamma} = -1, \qquad \mathbf{e}_+\mathbf{e}_- = \mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}. \tag{125}$$

A normalized point entity $\hat{\boldsymbol{P}}_{\mathcal{C}}$ has unit scale on the homogeneous term $\mathbf{e}_{o\gamma}$ as

$$\hat{\boldsymbol{P}}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}}/(-\boldsymbol{P}_{\mathcal{C}} \cdot \mathbf{e}_{\infty\gamma}). \tag{126}$$

The vector $\boldsymbol{p}_{\mathcal{M}}$ and its embedding $\hat{\mathbf{P}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ represent a specific position point $(p_w,\ p_x,\ p_y,\ p_z)$. The symbolic vector $\boldsymbol{t}_{\mathcal{M}}$ and its embedding $\hat{\boldsymbol{T}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ represent the symbolic variable "test" point $(w = ct, x, y, z)$.

The *projection* (inverse of embedding) of a point $\hat{\boldsymbol{P}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ to its embedded STA vector is

$$\boldsymbol{p}_{\mathcal{M}} \;=\; \mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{C}}) = (\hat{\boldsymbol{P}}_{\mathcal{C}} \cdot \mathbf{I}_{\mathcal{M}})\mathbf{I}_{\mathcal{M}}^{-1} = (\hat{\boldsymbol{P}}_{\mathcal{C}} \wedge \mathbf{e}_+ \wedge \mathbf{e}_-)(\mathbf{e}_+ \wedge \mathbf{e}_-), \tag{127}$$

which is geometrically *projection* onto $\mathbf{I}_{\mathcal{M}}$ or *rejection* from $\mathbf{e}_+\mathbf{e}_- = \mathbf{e}_+ \wedge \mathbf{e}_- = \mathbf{e}_{\infty\gamma} \wedge \mathbf{e}_{o\gamma}$.

The inner product of two normalized points $\hat{\boldsymbol{P}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ and $\hat{\boldsymbol{Q}} = \mathcal{C}(\boldsymbol{q}_{\mathcal{M}})$ is

$$\hat{\boldsymbol{P}}_{\mathcal{C}} \cdot \hat{\boldsymbol{Q}}_{\mathcal{C}} = \boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{q}_{\mathcal{M}} - \frac{1}{2}\boldsymbol{p}_{\mathcal{M}}^2 - \frac{1}{2}\boldsymbol{q}_{\mathcal{M}}^2 = -\frac{1}{2}(\boldsymbol{p}_{\mathcal{M}} - \boldsymbol{q}_{\mathcal{M}})^2, \tag{128}$$

which is $-1/2$ the space-time interval $(\boldsymbol{p}_{\mathcal{M}} - \boldsymbol{q}_{\mathcal{M}})^2$ between $\boldsymbol{p}_{\mathcal{M}}$ and $\boldsymbol{q}_{\mathcal{M}}$.

The reciprocals of $\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \mathbf{e}_+, \mathbf{e}_-\}$ are $\{\boldsymbol{\gamma}_0, -\boldsymbol{\gamma}_1, -\boldsymbol{\gamma}_2, -\boldsymbol{\gamma}_3, \mathbf{e}_+, -\mathbf{e}_-\}$, and the reciprocals of $\{\mathbf{e}_{o\gamma}, \mathbf{e}_{\infty\gamma}\}$ are $\{-\mathbf{e}_{\infty\gamma}, -\mathbf{e}_{o\gamma}\}$. Using these reciprocals, we define the CSTA *extraction operators*

$$T_{\mathcal{C}}^s \in T_{\mathcal{C}} = \{T_{\mathcal{C}}^w, T_{\mathcal{C}}^t, T_{\mathcal{C}}^x, T_{\mathcal{C}}^y, T_{\mathcal{C}}^z, T_{\mathcal{C}}^1, T_{\mathcal{C}}^{\boldsymbol{t}_{\mathcal{M}}^2}\} = \{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_0/c, -\boldsymbol{\gamma}_1, -\boldsymbol{\gamma}_2, -\boldsymbol{\gamma}_3, -\mathbf{e}_{\infty\gamma}, -2\mathbf{e}_{o\gamma}\}, \tag{129}$$

which are for the extractions of the corresponding coefficients $s \in \{w = ct, t, x, y, z, 1, \boldsymbol{t}_{\mathcal{M}}^2\}$ from any point $\hat{\boldsymbol{T}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ as the inner product $s = \hat{\boldsymbol{T}}_{\mathcal{C}} \cdot T_{\mathcal{C}}^s$. Linear combinations of the extraction operators $T_{\mathcal{C}}^s \in T_{\mathcal{C}}$ (extracting values s) can form the CSTA GIPNS 1-blade entities for hypercones $\boldsymbol{K}_{\mathcal{C}}$, hyperplanes $\boldsymbol{E}_{\mathcal{C}}$, and hyperpseudospheres $\boldsymbol{\Sigma}_{\mathcal{C}}$ in terms of their algebraic polynomial implicit surface functions $F(w, x, y, z) = 0$ in space-time. The inner product of the symbolic test point $\hat{\boldsymbol{T}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ with point $\hat{\boldsymbol{P}}_{\mathcal{C}}$ is

$$\hat{\boldsymbol{T}}_{\mathcal{C}} \cdot \hat{\boldsymbol{P}}_{\mathcal{C}} \;=\; \boldsymbol{t}_{\mathcal{M}} \cdot \boldsymbol{p}_{\mathcal{M}} - \frac{1}{2}\boldsymbol{t}_{\mathcal{M}}^2 - \frac{1}{2}\boldsymbol{p}_{\mathcal{M}}^2 = -\frac{1}{2}(\boldsymbol{t}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}})^2 \tag{130}$$

$$\;=\; -\frac{1}{2}((w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2), \tag{131}$$

which represents the implicit surface function $F(w, x, y, z) = \hat{\boldsymbol{T}}_{\mathcal{C}} \cdot \hat{\boldsymbol{P}}_{\mathcal{C}}$ of a hypercone $F = 0$ with vertex $\boldsymbol{p}_{\mathcal{M}}$. A point $\boldsymbol{T}_{\mathcal{C}}$ is on the hypercone surface presented by $\boldsymbol{P}_{\mathcal{C}}$ iff (if and only if) $\boldsymbol{T}_{\mathcal{C}} \cdot \boldsymbol{P}_{\mathcal{C}} = 0$. As an IPNS entity, the conformal point embedding $\hat{\boldsymbol{P}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ represents the hypercone implicit surface function (131) for a hypercone with vertex $\boldsymbol{p}_{\mathcal{M}}$, not just the embedded point $\boldsymbol{p}_{\mathcal{M}}$. However $\boldsymbol{T}_{\mathcal{C}} \wedge \boldsymbol{P}_{\mathcal{C}} = 0$ iff $\boldsymbol{T}_{\mathcal{C}} \simeq \boldsymbol{P}_{\mathcal{C}}$, and we call $\boldsymbol{P}_{\mathcal{C}}$ a geometric outer product null space (GOPNS) point entity. The relation $\simeq$ denotes the equality, up to a non-zero scalar multiple, of homogeneous entities representing the same geometry.

The IPNS hypercone entity $\boldsymbol{P}_{\mathcal{C}} = \boldsymbol{K}_{\mathcal{C}}$ can be written in terms of the CSTA extraction operators $T_{\mathcal{C}}^s$ as

$$-2\hat{\boldsymbol{K}}_{\mathcal{C}} \;=\; T_{\mathcal{C}}^{\boldsymbol{t}_{\mathcal{M}}^2} - 2p_w T_{\mathcal{C}}^w + 2p_x T_{\mathcal{C}}^x + 2p_y T_{\mathcal{C}}^y + 2p_z T_{\mathcal{C}}^z + (p_w^2 - p_x^2 - p_y^2 - p_z^2)T_{\mathcal{C}}^1 \tag{132}$$

$$\;=\; -2\mathbf{e}_{o\gamma} - 2\boldsymbol{p}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}}^2\mathbf{e}_{\infty\gamma}. \tag{133}$$

It can then be verified that $\hat{\boldsymbol{T}}_{\mathcal{C}} \cdot \hat{\boldsymbol{K}}_{\mathcal{C}} = F(w,\ x,\ y,\ z)$ of (131). Linear combinations of the extraction operators $T_{\mathcal{C}}^s$ construct IPNS entities that directly correspond to certain polynomial functions $F(w, x, y, z)$ that can be formed as linear combinations of the available terms s. While the $T_{\mathcal{C}}^s$ represent symbolic variables and constants of a polynomial function $F(w,\ x,\ y,\ z)$, their linear combinations form specific elements of CSTA, which we call geometric inner product null space (GIPNS) *entities*.

If points $\boldsymbol{p}_{\mathcal{M}}$ are restricted to spatial points $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$ with no time $(w = ct = 0)$, the conformal embedding is the CSA null 1-blade spatial *point* entity $\hat{\mathbf{P}}_{\mathcal{CS}} = \mathcal{C}(\mathbf{p}_{\mathcal{S}})$ of the CSA $\mathcal{CS}$ subalgebra of CSTA $\mathcal{C}$. Both the IPNS and OPNS of the CSA spatial point $\mathbf{P}_{\mathcal{CS}}$ represent only the point $\mathbf{p}_{\mathcal{S}}$ when tested against the CSA test point $\hat{\mathbf{T}}_{\mathcal{CS}} = \mathcal{C}(\mathbf{t}_{\mathcal{S}})$ (i.e., $\mathbf{T}_{\mathcal{CS}}\mathbf{P}_{\mathcal{CS}} = 0$ iff $\mathbf{T}_{\mathcal{CS}} \simeq \mathbf{P}_{\mathcal{CS}}$).

The CSTA GIPNS 1-blade space-time *hyperplane* (3D subspace of 4D ST) entity $\boldsymbol{E}_{\mathcal{C}}$ is

$$\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{n}_{\mathcal{M}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma}, \tag{134}$$

which represents the 3D subspace orthogonal to $\boldsymbol{n}_{\mathcal{M}}$ and passing through space-time position $\boldsymbol{p}_{\mathcal{M}}$. A normalized hyperplane $\hat{\boldsymbol{E}}_{\mathcal{C}}$ has $\boldsymbol{n}_{\mathcal{M}} = \hat{\boldsymbol{n}}_{\mathcal{M}}$ by (34). When $\boldsymbol{n}_{\mathcal{M}}$ is a null vector ($\boldsymbol{n}_{\mathcal{M}}^2 = 0$), then $\boldsymbol{E}_{\mathcal{C}} = \mathbb{L}_{\mathcal{C}}$ is the CSA GIPNS 1-blade null *line* (*light-line*) entity $\mathbb{L}_{\mathcal{C}}$ through the point $\boldsymbol{p}_{\mathcal{M}}$ in the null direction $\boldsymbol{n}_{\mathcal{M}}$ that includes point $\mathbf{e}_{\infty\gamma}$ on the null line.

The inner product of the symbolic test point $\hat{\boldsymbol{T}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ with $\boldsymbol{E}_{\mathcal{C}}$ is

$$\begin{aligned}
\hat{\boldsymbol{T}}_{\mathcal{C}} \cdot \boldsymbol{E}_{\mathcal{C}} &= \boldsymbol{t}_{\mathcal{M}} \cdot \boldsymbol{n}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{n}_{\mathcal{M}} = (\boldsymbol{t}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}}) \cdot \boldsymbol{n}_{\mathcal{M}} && (135)\\
&= (w - p_w)n_w - (x - p_x)n_x - (y - p_y)n_y - (z - p_z)n_z, && (136)
\end{aligned}$$

which represents the implicit surface function $F(w, x, y, z) = \hat{\boldsymbol{T}}_{\mathcal{C}} \cdot \boldsymbol{E}_{\mathcal{C}}$ of a hyperplane $F = 0$ orthogonal to $\boldsymbol{n}_{\mathcal{M}}$ and passing through $\boldsymbol{p}_{\mathcal{M}}$ (by translation). The hyperplane entity $\boldsymbol{E}_{\mathcal{C}}$ can be written in terms of the CSTA extraction operators $T_{\mathcal{C}}^s$ as

$$\boldsymbol{E}_{\mathcal{C}} = n_w T_{\mathcal{C}}^w - n_x T_{\mathcal{C}}^x - n_y T_{\mathcal{C}}^y - n_z T_{\mathcal{C}}^z - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{n}_{\mathcal{M}}) T_{\mathcal{C}}^1. \tag{137}$$

For position $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$ and normal $\boldsymbol{n}_{\mathcal{M}} = \mathbf{n}_{\mathcal{S}}$ restricted to spatial vectors, the entity $\boldsymbol{E}_{\mathcal{C}}$ is the CSA GIPNS 1-blade spatial *plane* entity $\boldsymbol{\Pi}_{\mathcal{CS}} = \boldsymbol{E}_{\mathcal{C}}$ (i.e., holding $w = ct = 0$ removes the $\boldsymbol{\gamma}_0$ dimension), where the CSA point $\hat{\mathbf{T}}_{\mathcal{CS}} = \mathcal{C}(\mathbf{t}_{\mathcal{S}})$ is on the plane $\boldsymbol{\Pi}_{\mathcal{CS}}$ iff $\mathbf{T}_{\mathcal{CS}} \cdot \boldsymbol{\Pi}_{\mathcal{CS}} = 0$. In CSTA, $\boldsymbol{\Pi}_{\mathcal{CS}}$ is a purely spatial plane entity at zero velocity (no time t dependency), which can be boosted $B_{\mathbf{v}} \boldsymbol{\Pi}_{\mathcal{CS}} B_{\mathbf{v}}^{\sim}$ into a velocity $\mathbf{v}$.

The CSTA GIPNS 1-blade *hyperpseudosphere* entity $\boldsymbol{\Sigma}_{\mathcal{C}}$ centered at point $\hat{\boldsymbol{P}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with initial radius r_0, or through point $\hat{\boldsymbol{Q}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{q}_{\mathcal{M}})$, is

$$\hat{\boldsymbol{\Sigma}}_{\mathcal{C}} = \hat{\boldsymbol{P}}_{\mathcal{C}} + (1/2) r_0^2 \mathbf{e}_{\infty\gamma} = \hat{\boldsymbol{P}}_{\mathcal{C}} + \big(\hat{\boldsymbol{P}}_{\mathcal{C}} \cdot \hat{\boldsymbol{Q}}_{\mathcal{C}}\big)\mathbf{e}_{\infty\gamma}, \tag{138}$$

where the initial radius r_0 can be real or imaginary. For spatial points $\hat{\boldsymbol{P}}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{S}})$ and $\hat{\boldsymbol{Q}}_{\mathcal{C}} = \mathcal{C}(\mathbf{q}_{\mathcal{S}})$, r_0 is the initial radius of a spatial sphere that grows with time in space-time, and then $\boldsymbol{\Sigma}_{\mathcal{D}}$ is a (hyper)hyperboloid of one sheet in space-time. More generally $r_0^2 = -(\boldsymbol{p}_{\mathcal{M}} - \boldsymbol{q}_{\mathcal{M}})^2$ (cf.(128)), and for a space-like interval $(\boldsymbol{p}_{\mathcal{M}} - \boldsymbol{q}_{\mathcal{M}})^2 < 0$ then r_0 is real and $\boldsymbol{\Sigma}_{\mathcal{C}}$ is a (hyper)hyperboloid of one sheet, for a time-like interval $(\boldsymbol{p}_{\mathcal{M}} - \boldsymbol{q}_{\mathcal{M}})^2 > 0$ then r_0 is imaginary and $\boldsymbol{\Sigma}_{\mathcal{C}}$ is a (hyper)hyperboloid of two sheets, and when $\hat{\boldsymbol{P}}_{\mathcal{C}} = \hat{\boldsymbol{Q}}_{\mathcal{C}}$ then $r_0 = 0$ and $\hat{\boldsymbol{\Sigma}}_{\mathcal{C}} = \hat{\boldsymbol{P}}_{\mathcal{C}}$ is a null hypercone. The inner product of the symbolic test point $\hat{\boldsymbol{T}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ with $\hat{\boldsymbol{\Sigma}}_{\mathcal{C}}$ is

$$\begin{aligned}
-2\hat{\boldsymbol{T}}_{\mathcal{C}} \cdot \hat{\boldsymbol{\Sigma}}_{\mathcal{C}} &= -2\bigg(-\frac{1}{2}(\boldsymbol{t}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}})^2 - \frac{1}{2} r_0^2\bigg) && (139)\\
&= r_0^2 + (w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2, && (140)
\end{aligned}$$

which represents the implicit surface function $F(w,\ x,\ y,\ z)$ of a pseudosphere (space-time circular hyperboloid) in the pseudospatial time w dimension with any two space dimensions, and a *hyper*pseudosphere $F = 0$ in all four STA dimensions. For $\hat{\boldsymbol{P}}_{\mathcal{C}} = \hat{\mathbf{P}}_{\mathcal{CS}} = \mathcal{C}(\mathbf{p}_{\mathcal{S}})$ restricted to a spatial center point $\mathbf{p}_{\mathcal{S}}$ with $w = ct = 0$, the hyperpseudosphere $\hat{\boldsymbol{\Sigma}}_{\mathcal{C}}$ is the CSA GIPNS 1-blade spatial *sphere* entity $\hat{\mathbf{S}}_{\mathcal{CS}} = \hat{\boldsymbol{\Sigma}}_{\mathcal{C}}$ with radius $r = r_0$, where the CSA null 1-blade spatial point $\hat{\mathbf{T}}_{\mathcal{CS}} = \mathcal{C}(\mathbf{t}_{\mathcal{S}})$ is on the spatial sphere $\mathbf{S}_{\mathcal{CS}}$ iff $\mathbf{T}_{\mathcal{CS}} \cdot \mathbf{S}_{\mathcal{CS}} = 0$. The hyperpseudosphere entity $\boldsymbol{\Sigma}_{\mathcal{C}}$ can be written in terms of the CSTA extraction operators $T_{\mathcal{C}}^s$ as

$$-2\hat{\boldsymbol{\Sigma}}_{\mathcal{C}} = r_0^2 T_{\mathcal{C}}^1 + T_{\mathcal{C}}^{\boldsymbol{t}_{\mathcal{M}}^2} - 2p_w T_{\mathcal{C}}^w + 2p_x T_{\mathcal{C}}^x + 2p_y T_{\mathcal{C}}^y + 2p_z T_{\mathcal{C}}^z + \boldsymbol{p}_{\mathcal{M}}^2 T_{\mathcal{C}}^1. \tag{141}$$

The *quasi-sphere* implicit surface is defined by $a\boldsymbol{t}_{\mathcal{M}}^2 + b\boldsymbol{n}_{\mathcal{M}} \cdot \boldsymbol{t}_{\mathcal{M}} + c = 0$, which can be represented as the linear combination $a\boldsymbol{\Sigma}_{\mathcal{C}} + b\boldsymbol{E}_{\mathcal{C}} + cT_{\mathcal{C}}^1$. The quasi-sphere is a hyperpseudosphere for $a \neq 0$, and is a hyperplane for $a = 0$ and $b \neq 0$. The quasi-sphere generalizes the hyperpseudosphere and hyperplane. The hyperpseudosphere entity $\hat{\boldsymbol{\Sigma}}_{\mathcal{C}} / |r_0|$ through point $\boldsymbol{q}_{\mathcal{M}}$ with center $\boldsymbol{p}_{\mathcal{M}} = \boldsymbol{q}_{\mathcal{M}} + |r_0|\hat{\boldsymbol{n}}$ becomes, in the limit $|r_0| \rightarrow \infty$, the normalized hyperplane $\hat{\boldsymbol{E}}_{\mathcal{C}}$ with normal $\hat{\boldsymbol{n}}$ through $\boldsymbol{q}_{\mathcal{M}}$,

$$\begin{aligned}
\hat{\boldsymbol{E}}_{\mathcal{C}} &= \lim_{|r_0| \rightarrow \infty} \frac{\hat{\boldsymbol{\Sigma}}_{\mathcal{C}}}{|r_0|} = \lim_{|r_0| \rightarrow \infty} \frac{\boldsymbol{q}_{\mathcal{M}} + |r_0|\hat{\boldsymbol{n}} + \frac{1}{2}((\boldsymbol{q}_{\mathcal{M}} + |r_0|\hat{\boldsymbol{n}})^2 - (|r_0|\hat{\boldsymbol{n}})^2)\mathbf{e}_{\infty\gamma} + \mathbf{e}_{o\gamma}}{|r_0|} && (142)\\
&= \hat{\boldsymbol{n}} + (\boldsymbol{q}_{\mathcal{M}} \cdot \hat{\boldsymbol{n}})\mathbf{e}_{\infty\gamma}. && (143)
\end{aligned}$$

Thus, the hyperpseudosphere entity generalizes to the hyperplane entity, like the quasi-sphere.

The extraction operators (129) can also form the following CSTA *differential operators*

$$\begin{aligned}
D_w &= T_{\mathcal{C}}^1 (T_{\mathcal{C}}^w)^{-1} = -\mathbf{e}_{\infty\gamma}\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_0 \wedge \mathbf{e}_{\infty\gamma} && (144)\\
D_t &= T_{\mathcal{C}}^1 (T_{\mathcal{C}}^t)^{-1} = -c\mathbf{e}_{\infty\gamma}\boldsymbol{\gamma}_0 = c\boldsymbol{\gamma}_0 \wedge \mathbf{e}_{\infty\gamma} && (145)\\
D_x &= T_{\mathcal{C}}^1 (T_{\mathcal{C}}^x)^{-1} = -\mathbf{e}_{\infty\gamma}\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_1 \wedge \mathbf{e}_{\infty\gamma} && (146)\\
D_y &= T_{\mathcal{C}}^1 (T_{\mathcal{C}}^y)^{-1} = -\mathbf{e}_{\infty\gamma}\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_2 \wedge \mathbf{e}_{\infty\gamma} && (147)\\
D_z &= T_{\mathcal{C}}^1 (T_{\mathcal{C}}^z)^{-1} = -\mathbf{e}_{\infty\gamma}\boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_3 \wedge \mathbf{e}_{\infty\gamma}. && (148)
\end{aligned}$$

The CSTA differential elements are *free vectors* [3], which represent directions without locations, and are invariant by the translation operator (159). The $\boldsymbol{n}$-directional ($\|\boldsymbol{n}\|_2 = \sqrt{\boldsymbol{n}\cdot\boldsymbol{n}^\dagger} = 1$) derivative of any CSTA GIPNS 1-blade entity $\boldsymbol{A}$ is, by the commutator product $\times$ (186) with a differential operator,

$$\partial_{\boldsymbol{n}}\boldsymbol{A} = (n_w D_w + n_x D_x + n_y D_y + n_z D_z) \times \boldsymbol{A}. \tag{149}$$

The outer product of two to six of the above three CSTA GIPNS 1-blade entities (null-hypercones $\boldsymbol{K}_{\mathcal{C}}$, hyperplanes $\boldsymbol{E}_{\mathcal{C}}$, hyperpseudospheres $\boldsymbol{\Sigma}_{\mathcal{C}}$) forms, by intersection, more CSTA GIPNS entities of higher grades.

The CSTA GIPNS 2-blade space-time *plane* entity

$$\boldsymbol{\Pi}_{\mathcal{C}} = \boldsymbol{D}_{\mathcal{M}}^* - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}_{\mathcal{M}}^*)\mathbf{e}_{\infty\gamma}, \tag{150}$$

in direction of 2-blade $\boldsymbol{D}_{\mathcal{M}}$ through point $\boldsymbol{p}_{\mathcal{M}}$, is the intersection (wedge) of two space-time hyperplanes (134). A normalized plane $\hat{\boldsymbol{\Pi}}_{\mathcal{C}}$ has a unit 2-blade direction $\boldsymbol{D}_{\mathcal{M}} = \hat{\boldsymbol{D}}_{\mathcal{M}}$ by (35). Note that, the CSA GIPNS 2-blade spatial *line* entity $\mathbf{L}_{\mathcal{CS}} = \hat{\mathbf{d}}^{*\mathcal{S}} - \big(\mathbf{p}\cdot\hat{\mathbf{d}}^{*\mathcal{S}}\big)\mathbf{e}_{\infty\gamma}$, viewed as a CSTA entity that is tested against the CSTA point $\boldsymbol{T}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ instead of the CSA point $\mathbf{T}_{\mathcal{CS}} = \mathcal{C}(\mathbf{t}_{\mathcal{S}})$, gains the span of the pseudospatial direction $\boldsymbol{\gamma}_0$ and is the CSTA plane $\boldsymbol{\Pi}_{\mathcal{C}}$ with 2-blade direction $\boldsymbol{D}_{\mathcal{M}} = \mathbf{d}\wedge\boldsymbol{\gamma}_0$ through point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$.

The CSTA GIPNS 3-blade space-time *line* entity

$$\boldsymbol{L}_{\mathcal{C}} = \boldsymbol{d}_{\mathcal{M}}^* + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{d}_{\mathcal{M}}^*) \wedge \mathbf{e}_{\infty\gamma}, \tag{151}$$

in the direction $\boldsymbol{d}_{\mathcal{M}}$ through point $\boldsymbol{p}_{\mathcal{M}} = p_w\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}}$, is the intersection (wedge) of three space-time hyperplanes (134). A normalized line $\hat{\boldsymbol{L}}_{\mathcal{C}}$ has a unit direction $\boldsymbol{d}_{\mathcal{M}} = \hat{\boldsymbol{d}}_{\mathcal{M}}$ by (34). If the line direction vector $\boldsymbol{d}_{\mathcal{M}}$ is a null vector, then the line entity $\boldsymbol{L}_{\mathcal{C}}$ is a null 3-blade representing a null line (light-line). The point at infinity $\mathbf{e}_{\infty\gamma}$ is a point of any line $\boldsymbol{L}_{\mathcal{C}}$. The line $\boldsymbol{L}_{\mathcal{C}}$ can represent the worldline of an observable with STA velocity $\boldsymbol{v} = \boldsymbol{d}_{\mathcal{M}} = c\boldsymbol{\gamma}_0 + \beta c\hat{\mathbf{v}}$ with initial spatial position $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_0$. The initial position $\mathbf{p}_0$ can also be found as the CSTA GOPNS 2-blade *flat point*[2] [3] position

$$\mathcal{C}(\mathbf{p}_0) \wedge \mathbf{e}_{\infty\gamma} \simeq (\boldsymbol{\gamma}_0 \wedge \boldsymbol{L}_{\mathcal{C}})\mathbf{I}_{\mathcal{C}}^{-1} \tag{152}$$

at $t=0$, where $\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_0$ is the $t=0$ hyperplane (134) and $\boldsymbol{\gamma}_0 \wedge \boldsymbol{L}_{\mathcal{C}} = \mathbb{P}_{\mathcal{C}}$ is a CSTA GIPNS 4-blade flat point entity $\mathbb{P}_{\mathcal{C}}$, which is the intersection of four hyperplanes (134). By CSTA dualization (154), $(\boldsymbol{\gamma}_0 \wedge \boldsymbol{L}_{\mathcal{C}})\mathbf{I}_{\mathcal{C}}^{-1}$ is a CSTA GOPNS 2-blade flat point entity $\mathbb{P}_{\mathcal{C}}^* = \boldsymbol{P}_{\mathcal{C}} \wedge \mathbf{e}_{\infty\gamma}$. The point $\hat{\boldsymbol{P}}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ of a CSTA GOPNS 2-blade flat point $\mathbb{P}_{\mathcal{C}}^*$ is *projected* as

$$\boldsymbol{p}_{\mathcal{M}} = \frac{(\mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}) \cdot (\mathbf{e}_{o\gamma} \wedge \mathbb{P}_{\mathcal{C}}^*)}{-(\mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}) \cdot \mathbb{P}_{\mathcal{C}}^*}. \tag{153}$$

The boost B, and the other CSTA versors, can operate on the line $\boldsymbol{L}_{\mathcal{C}}$ to implement space-time transformations of a worldline representation. Intersecting $\boldsymbol{L}_{\mathcal{C}}$ with the time t hyperplane $\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_0 + ct\mathbf{e}_{\infty\gamma}$ finds the spatial position at coordinate time t in the current frame as the resulting CSTA GIPNS 4-blade flat point $\mathbb{P}_{\mathcal{C}} = \boldsymbol{L}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}}$. A passive boost changes the coordinate time t to be the proper time τ in the new frame.

2. *Flat point* $\boldsymbol{P}_{\mathcal{C}} \wedge \mathbf{e}_{\infty\gamma}$ in [3] is called *homogeneous point* $\boldsymbol{p}_{\mathcal{M}} \wedge \mathbf{e}_{\infty\gamma} + \mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}$ in [18].

CSTA *dualization* of a CSTA GIPNS k-blade entity $\boldsymbol{X}_{\mathcal{C}}$ gives its dual CSTA *geometric outer product null space* (GOPNS) [18] $(6-k)$-blade entity

$$\boldsymbol{X}_{\mathcal{C}}^{*}=\boldsymbol{X}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1}. \tag{154}$$

A CSTA point $\boldsymbol{P}_{\mathcal{C}}$ is on CSTA GIPNS entity $\boldsymbol{X}_{\mathcal{C}}$ iff

$$\boldsymbol{P}_{\mathcal{C}}\cdot\boldsymbol{X}_{\mathcal{C}}=0. \tag{155}$$

A CSTA point $\boldsymbol{P}_{\mathcal{C}}$ is on the corresponding dual CSTA GOPNS entity $\boldsymbol{X}_{\mathcal{C}}^{*}$ iff

$$\boldsymbol{P}_{\mathcal{C}}\wedge\boldsymbol{X}_{\mathcal{C}}^{*}=0. \tag{156}$$

The outer product of up to six well-chosen CSTA points $\boldsymbol{P}_{\mathcal{C}_i}$ produces various CSTA GOPNS (1...6)-blade space-time surface entities $\boldsymbol{X}_{\mathcal{C}}^{*}=\bigwedge\boldsymbol{P}_{\mathcal{C}_i}$ for surfaces that the points span as surface points. The CSTA GOPNS null 1-blade *point* (embedding) $\boldsymbol{P}_{\mathcal{C}}$ equals the CSTA GIPNS null 1-blade *hypercone* $\boldsymbol{P}_{\mathcal{C}}=\boldsymbol{K}_{\mathcal{C}}$.

CSTA inherits the STA 2-versor spatial *rotor*

$$R_{\mathcal{C}}=R=\exp(\theta\hat{\mathbf{n}}_{\mathcal{S}}^{*}/2)=\cos(\theta/2)+\sin(\theta/2)\hat{\mathbf{n}}_{\mathcal{S}}^{*}, \tag{157}$$

and STA 2-versor space-time *hyperbolic rotor (boost)*

$$B_{\mathcal{C}}=B=\exp(\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0/2)=\cosh(\varphi/2)+\sinh(\varphi/2)\hat{\mathbf{v}}\boldsymbol{\gamma}_0. \tag{158}$$

Compositions of rotor and boost, such as the boosted rotor and boosted boost are also inherited.

CSTA introduces the CSTA 2-versor space-time *translator*

$$T_{\mathcal{C}}=\exp(\mathbf{e}_{\infty\gamma}\boldsymbol{d}_{\mathcal{M}}/2)=1+\mathbf{e}_{\infty\gamma}\boldsymbol{d}_{\mathcal{M}}/2, \tag{159}$$

which translates by $\boldsymbol{d}_{\mathcal{M}}$. As versor compositions, CSTA also introduces the following three translated 2-versors. The translator $T_{\mathcal{C}}$ with $(\mathbf{e}_{\infty}\boldsymbol{d}_{\mathcal{M}})^2=0$ is a geometric number form of unimodular $|T_{\mathcal{C}}|=1$ *parabolic complex number* (dual number) $a+b\varepsilon$ with $\varepsilon^2=0$.

The CSTA 2-versor spatial *translated-rotor* is

$$L_{\mathcal{C}}=T_{\mathcal{C}}R_{\mathcal{C}}T_{\mathcal{C}}^{-1}=\exp\big(-\theta\boldsymbol{\gamma}_0\cdot\hat{\boldsymbol{L}}_{\mathcal{C}}/2\big)=\cos(\theta/2)+\sin(\theta/2)\hat{\mathbf{L}}_{\mathcal{CS}}, \tag{160}$$

which rotates by angle θ anticlockwise (by right-hand rule) around the spatial CSA line $\hat{\mathbf{L}}_{\mathcal{CS}}=\hat{\mathbf{n}}_{\mathcal{S}}^{*}-(\mathbf{d}_{\mathcal{S}}\cdot\hat{\mathbf{n}}_{\mathcal{S}}^{*})\mathbf{e}_{\infty\gamma}$ through point $\mathbf{d}_{\mathcal{S}}$ in the rotor axis direction $\hat{\mathbf{n}}_{\mathcal{S}}$.

The CSTA 2-versor *translated-boost* is

$$B_{\mathcal{C}}^{\boldsymbol{d}}=\exp(\varphi(\hat{\mathbf{v}}\boldsymbol{\gamma}_0-(\boldsymbol{d}_{\mathcal{M}}\cdot(\hat{\mathbf{v}}\boldsymbol{\gamma}_0))\mathbf{e}_{\infty\gamma})/2)=\exp\big(\varphi\hat{\mathbf{\Pi}}_{\mathcal{C}}/2\big)=\cosh(\varphi/2)+\sinh(\varphi/2)\hat{\mathbf{\Pi}}_{\mathcal{C}}, \tag{161}$$

centered on point $\boldsymbol{d}_{\mathcal{M}}$ and with plane direction $\hat{\boldsymbol{D}}_{\mathcal{M}}=(\hat{\mathbf{v}}\boldsymbol{\gamma}_0)\mathbf{I}_{\mathcal{M}}$.

The CSTA 2-versor *translated-isotropic dilator* is

$$D_{\mathcal{C}}=\exp\big(\ln(d)\hat{\boldsymbol{P}}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma}/2\big)=\cosh(\ln(d)/2)+\sinh(\ln(d)/2)\hat{\boldsymbol{P}}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma} \tag{162}$$

for isotropic dilation by factor $d>0$ relative to normalized center point $\hat{\boldsymbol{P}}_{\mathcal{C}}$, i.e. $\hat{\boldsymbol{P}}_{\mathcal{C}}\cdot\mathbf{e}_{\infty\gamma}=-1$. By *versor outermorphism* [18], all CSTA versors correctly transform all CSTA GIPNS and dual CSTA GOPNS entities.

4 Construction of Double CSTA (DCSTA)

In double conformal space-time algebra (DCSTA) $\mathcal{D}$, CSTA1 $\mathcal{C}^1$ and CSTA2 $\mathcal{C}^2$ are orthogonal subalgebras and their geometric or outer product is a doubling extension. Any CSTA1 entity or versor $A_{\mathcal{C}^1}$ and its double $A_{\mathcal{C}^2}$ in CSTA2 (with the same scalar coefficients on corresponding basis blades) can be multiplied to form the corresponding DCSTA entity or versor

$$A_{\mathcal{D}}=A_{\mathcal{C}^1}A_{\mathcal{C}^2}=A_{\mathcal{C}^1}\wedge A_{\mathcal{C}^2}. \tag{163}$$

By versor outermorphism, the DCSTA versors operate correctly on all DCSTA entities.

The DCSTA null 2-blade "standard" *point* entity

$$\boldsymbol{T}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{t}_{\mathcal{M}}) = \boldsymbol{T}_{\mathcal{C}^1} \wedge \boldsymbol{T}_{\mathcal{C}^2} = \mathcal{C}^1(\boldsymbol{t}_{\mathcal{M}^1}) \wedge \mathcal{C}^2(\boldsymbol{t}_{\mathcal{M}^2}) \tag{164}$$

is an extended, doubled form of the CSTA point embedding $\boldsymbol{T}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ (123). Note that, as in CSTA, the DCSTA point is a geometric OPNS (GOPNS) null point, but is a GIPNS null hypercone.

The construction method is further extensible to an Extended CGA (k-CGA) $\mathcal{K}$, which is using not just a double $k=2$, but some k corresponding orthogonal CGAi $\mathcal{G}_{p+1,q+1}$ $\mathcal{C}^i$ of a vector space $\mathbb{R}^{p,q}$ $\mathcal{V}^i$, $1 \le i \le k$, where the corresponding k-CGA entity or versor is

$$A_{\mathcal{K}} = A_{\mathcal{C}^1} A_{\mathcal{C}^2} \ldots A_{\mathcal{C}^k} = A_{\mathcal{C}^1} \wedge A_{\mathcal{C}^2} \wedge \ldots \wedge A_{\mathcal{C}^k}. \tag{165}$$

The Extended CGA (k-CGA) null k-blade *point* is

$$\boldsymbol{T}_{\mathcal{K}} = \mathcal{K}(\boldsymbol{t}_{\mathcal{V}}) = \boldsymbol{T}_{\mathcal{C}^1} \boldsymbol{T}_{\mathcal{C}^2} \ldots \boldsymbol{T}_{\mathcal{C}^k} = \mathcal{C}^1(\boldsymbol{t}_{\mathcal{V}^1}) \mathcal{C}^2(\boldsymbol{t}_{\mathcal{V}^2}) \ldots \mathcal{C}^k(\boldsymbol{t}_{\mathcal{V}^k}). \tag{166}$$

The k-CGA k-blade entities are called the "standard" entities, corresponding to CGA entities and representing the same hypersurfaces. In addition to the standard k-blade entities, there are also new k-vector entities that represent general degree k polynomial implicit hypersurfaces and specific other implicit hypersurfaces of polynomial degrees $k < l \le 2k$.

Similar to the DCSTA null 2-blade *point* entity $\boldsymbol{T}_{\mathcal{D}}$ (164), other "standard" doubled 2-blade entities are formed as the product of corresponding CSTA1 and CSTA2 GIPNS 1-blade entities, which include the DCSTA GIPNS 2-blade *hyperplane* $\boldsymbol{E}_{\mathcal{D}} = \boldsymbol{E}_{\mathcal{C}^1} \boldsymbol{E}_{\mathcal{C}^2}$, the DCSTA GIPNS 2-blade *hyperpseudosphere* $\boldsymbol{\Sigma}_{\mathcal{D}} = \boldsymbol{\Sigma}_{\mathcal{C}^1} \boldsymbol{\Sigma}_{\mathcal{C}^2}$, and the DCSTA GIPNS null 2-blade *hypercone* $\boldsymbol{K}_{\mathcal{D}} = \boldsymbol{K}_{\mathcal{C}^1} \boldsymbol{K}_{\mathcal{C}^2} = \boldsymbol{T}_{\mathcal{C}^1} \boldsymbol{T}_{\mathcal{C}^2}$. The CSTA GIPNS intersection entities of grades 2, 3, 4, and 5 can also be doubled into their corresponding "standard" DCSTA GIPNS *intersection* entities of even grades 4, 6, 8, or 10, respectively. The same holds that, the CSTA GOPNS entities can be doubled into DCSTA GOPNS entities, or obtained from DCSTA GIPNS entities by using the DCSTA dualization operation (177).

The doublings of the CSTA versors include the DCSTA 4-versor *translator* $T_{\mathcal{D}} = T_{\mathcal{C}^1} T_{\mathcal{C}^2}$, the DCSTA 4-versor *rotor* $R_{\mathcal{D}} = R_{\mathcal{C}^1} R_{\mathcal{C}^1}$ and its translated form $R_{\mathcal{D}}^{\mathbf{d}} = R_{\mathcal{C}^1}^{\mathbf{d}} R_{\mathcal{C}^2}^{\mathbf{d}} = L_{\mathcal{C}^1} L_{\mathcal{C}^2}$, and the DCSTA 4-versor *hyperbolic rotor (boost)* $B_{\mathcal{D}} = B_{\mathcal{C}^1} B_{\mathcal{C}^2}$ and its translated form $B_{\mathcal{D}}^{\boldsymbol{d}} = B_{\mathcal{C}^1}^{\boldsymbol{d}} B_{\mathcal{C}^2}^{\boldsymbol{d}}$. The DCSTA GIPNS 2-blade *hyperplane* $\boldsymbol{E}_{\mathcal{D}} = \boldsymbol{E}_{\mathcal{C}^1} \boldsymbol{E}_{\mathcal{C}^2}$ is also the DCSTA 2-versor *reflector* in the hyperplane. The DCSTA GIPNS 2-blade *hyperpseudosphere* $\boldsymbol{\Sigma}_{\mathcal{D}} = \boldsymbol{\Sigma}_{\mathcal{C}^1} \boldsymbol{\Sigma}_{\mathcal{C}^2}$ is also the DCSTA 2-versor *inversor* in the hyperpseudosphere. When time is fixed as $t=0$, DCSTA $\mathcal{D}$ effectively becomes the DCSA $\mathcal{DS}$ subalgebra, where the DCSA null point $\mathbf{T}_{\mathcal{DS}} = \boldsymbol{T}_{\mathcal{D}}$ represents only the point by both IPNS and OPNS (i.e., $\mathbf{P}_{\mathcal{DS}} \mathbf{T}_{\mathcal{DS}} = 0$ iff $\mathbf{P}_{\mathcal{DS}} = \mathbf{T}_{\mathcal{DS}}$), the DCSA 2-blade *sphere* $\mathbf{S}_{\mathcal{DS}} = \boldsymbol{\Sigma}_{\mathcal{D}}$ is the DCSA 2-versor *inversor* in the sphere, and the DCSA 2-blade *plane* $\mathbf{\Pi}_{\mathcal{DS}} = \boldsymbol{E}_{\mathcal{D}}$ is the DCSA 2-versor *reflector* in the plane.

From a DCSTA point $\boldsymbol{T}_{\mathcal{D}}$, certain scalar polynomial terms (monomials), or values s, in variables x, y, z, and $w = ct$ can be extracted from the basis 2-blade coefficients in $\boldsymbol{T}_{\mathcal{D}}$ by inner products $s = \boldsymbol{T}_{\mathcal{D}} \cdot T_s$ with certain corresponding value s *extraction operators* T_s (see Table 1), which are each a certain bivector that is an averaged sum of up to $m = k! = 2$ reciprocal (or pseudoinverse) basis 2-blades that extract the same coefficient value s. The DCSTA 2-blade extraction operator $T_{\mathcal{D}_i}^s : 1 \le i \le m$ for value s is the product

$$T_{\mathcal{D}_i}^s = T_{\mathcal{C}_i^2}^{s_2} T_{\mathcal{C}_i^1}^{s_1} = T_{\mathcal{C}_i^2}^{s_2} \wedge T_{\mathcal{C}_i^1}^{s_1} \tag{167}$$

of CSTA2 and CSTA1 extraction operators $T_{\mathcal{C}_i^2}^{s_2}$ and $T_{\mathcal{C}_i^1}^{s_1}$ (129), respectively, such that $s = s_2 s_1$. Note that, the reciprocal 2-blade $T_{\mathcal{D}_i}^s$ is formed by using the *reverse* order of multiplication of CGAi $\mathcal{C}^i$ elements as compared to the order that forms a point $\boldsymbol{T}_{\mathcal{D}} = \boldsymbol{T}_{\mathcal{C}^1} \boldsymbol{T}_{\mathcal{C}^2}$ (164). The monomial value s is extracted from a DCSTA null 2-blade point $\boldsymbol{T}_{\mathcal{D}}$ by contraction (inner product) with any one of the 2-blade extraction operators $T_{\mathcal{D}_i}^s$ as

$$s = \boldsymbol{T}_{\mathcal{D}} \cdot T_{\mathcal{D}_i}^s = T_{\mathcal{D}_i}^s \cdot \boldsymbol{T}_{\mathcal{D}}. \tag{168}$$

For example, in DCSTA $\mathcal{D}$, which is a 2-CGA, we can form a DCSTA 2-blade extraction operator $T_{\mathcal{D}_i}^x$ for value x in two ways, as (see (129))

$$T_{\mathcal{D}_1}^x = T_{\mathcal{C}_1^2}^1 T_{\mathcal{C}_1^1}^x = \mathbf{e}_{\infty 2} \wedge \mathbf{e}_2 \quad \text{and} \quad T_{\mathcal{D}_2}^x = T_{\mathcal{C}_2^2}^x T_{\mathcal{C}_2^1}^1 = \mathbf{e}_8 \wedge \mathbf{e}_{\infty 1}. \tag{169}$$

Then, the DCSTA bivector (2-vector) extraction operator $T_{\mathcal{D}}^{x} = T_x$ for value x is

$$T_{\mathcal{D}}^{x} = T_x = \frac{1}{2}(T_{\mathcal{D}_1}^{x} + T_{\mathcal{D}_2}^{x}) = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{\infty 1}), \tag{170}$$

which is the average of the reciprocal 2-blades $T_{\mathcal{D}_1}^{x}$ and $T_{\mathcal{D}_2}^{x}$. The value x is then extracted from a DCSTA point $\boldsymbol{T}_{\mathcal{D}}$ as

$$x = \boldsymbol{T}_{\mathcal{D}} \cdot T_x = T_x \cdot \boldsymbol{T}_{\mathcal{D}}. \tag{171}$$

Other extraction operators T_s are formed by the same method (see Table 1 and Eq. 173).

A DCSTA bivector extraction operator $T_s = T_{\mathcal{D}}^{s}$ is generally the sum of up to two 2-blade extraction operators $T_{\mathcal{D}_i}^{s}$. The bivector extraction operators $T_{\mathcal{D}}^{s}$ are usually employed, rather than just one of the 2-blade extraction operators $T_{\mathcal{D}_i}^{s}$, such that all relevant geometric terms are present and then all entities and operations remain consistent.

In Extended CGA (k-CGA) $\mathcal{G}_{k(p+1),k(q+1)}$ $\mathcal{K}$ of dimension $n = k(p+q+2)$, up to $m_s : 1 \leq m_s \leq k!$ reciprocal basis k-blades $T_{\mathcal{K}_i}^{s} = T_{\mathcal{C}_i^k}^{s_k} \ldots T_{\mathcal{C}_i^2}^{s_2} T_{\mathcal{C}_i^1}^{s_1} : 1 \leq i \leq m_s$ can extract the same monomial value $s = s_k \ldots s_2 s_1$ from a k-CGA point $\boldsymbol{T}_{\mathcal{K}}$ as the contraction $s = \boldsymbol{T}_{\mathcal{K}} \cdot T_{\mathcal{K}_i}^{s}$. Let $C(n,k)$, for $k \leq n$, denote the number of *combinations* of k items chosen from n items. Each monomial value s is of degree k, or is of degree $k-p$ when $p : 0 \leq p \leq k$ of the s_j are $s_j = 1$. If each of the first t distinct factors $s_j : 1 \leq j \leq t \leq k$ (including any $s_j = 1$) occurs as a factor of $s = s_k \ldots s_2 s_1$ a multiplicity of p_j times (or of power p_j) where $k = \sum p_j$ such that $s = \prod s_j^{p_j}$, then there are exactly $m_s = \prod_{j=1}^{t} C(k - \sum_{i=1}^{j-1} p_i, p_j) = k! / \prod_{j=1}^{t} p_j!$ reciprocal basis k-blades $T_{\mathcal{K}_i}^{s} = T_{\mathcal{C}_i^k}^{s_k} \ldots T_{\mathcal{C}_i^2}^{s_2} T_{\mathcal{C}_i^1}^{s_1} : 1 \leq i \leq m_s$, by distinguishable permutations of $\{s_k, \ldots, s_2, s_1\}$, that extract the same monomial value $s = \prod s_j^{p_j}$ from a k-CGA point $\boldsymbol{T}_{\mathcal{K}}$ as the contraction $s = \boldsymbol{T}_{\mathcal{K}} \cdot T_{\mathcal{K}_i}^{s}$. The k-vector extraction operator $T_{\mathcal{K}}^{s}$ for the monomial value s is then $T_s = T_{\mathcal{K}}^{s} = (1/m_s)\sum_{i=1}^{m_s} T_{\mathcal{K}_i}^{s} = (1/k!)\sum_{\sigma} T_{\mathcal{C}^k}^{\sigma(s_k)} \ldots T_{\mathcal{C}^2}^{\sigma(s_2)} T_{\mathcal{C}^1}^{\sigma(s_1)}$, which sums over all permutations σ of $\{s_k, \ldots, s_2, s_1\}$. A k-vector geometric entity $\boldsymbol{\Omega}$, representing a polynomial and its implicit hypersurface, is a linear combination of k-vector monomial extraction operators $\boldsymbol{\Omega} = \sum \alpha_s T_s$, with scalars α_s. The Extended CGA (k-CGA) k-vector entities, in the form of $\boldsymbol{\Omega}$, can represent general degree k polynomials and also specific degree $k < l \leq 2k$ polynomials, which in turn represent general degree k implicit hypersurfaces and their inversions in hyperpseudospheres (that generalize to reflections in hyperplanes). In DCGA [10][6] and DCSA [7], entities for general quadrics can be reflected in standard 2-blade sphere entities (inversions in spheres) to produce general Darboux cyclide entities (see Figure 2) of up to polynomial degree $2k = 4$. The new k-vector entities $\boldsymbol{\Omega}$ in k-CGA are not necessarily k-blades. However, $\boldsymbol{\Omega}$ may be a k-blade when it can be factored into the product $\boldsymbol{\Omega} = A_{\mathcal{C}^1} \ldots A_{\mathcal{C}^k}$ of k CGAi 1-blade entities $A_{\mathcal{C}^i}$. When the $A_{\mathcal{C}^i}$ are k corresponding entities, then the k-blade entity $\boldsymbol{\Omega} = A_{\mathcal{K}} = A_{\mathcal{C}^1} \ldots A_{\mathcal{C}^k}$ is a "standard" k-blade entity. Otherwise, when some of the $A_{\mathcal{C}^i}$ are not corresponding entities, then the k-blade entity $\boldsymbol{\Omega} = A_{\mathcal{C}^1} \ldots A_{\mathcal{C}^k}$ represents the *union* of the distinct hypersurfaces represented by the distinct hypersurface entities in $A_{\mathcal{C}^1} \ldots A_{\mathcal{C}^k}$. The contraction of the k-CGA k-vector entity $\boldsymbol{\Omega}_k = \boldsymbol{\Omega}$ by any CGA 1-blade (vector) surface point $\mathbf{P}_{\mathcal{C}^k}$ of $\boldsymbol{\Omega}_k$, where $\mathbf{P}_{\mathcal{K}} \cdot \boldsymbol{\Omega} = 0$, produces the *tangent hypersurface* $\boldsymbol{\Omega}_{k-1} = \mathbf{P}_{\mathcal{C}^k} \cdot \boldsymbol{\Omega}_k$ at point $\mathbf{P}$ of lower polynomial degree in the $(k-1)$-CGA subalgebra. The result $\boldsymbol{\Omega}_{k-1} = \mathbf{P}_{\mathcal{C}^k} \cdot \boldsymbol{\Omega}_k$ is zero if the tangent hypersurface does not exist, which is possible when the point $\mathbf{P}$ of the surface is not a regular point of the surface $\boldsymbol{\Omega}_k$. Repeated contraction of surface $\boldsymbol{\Omega}_k$ at a CGA surface point $\mathbf{P}_{\mathcal{C}}$ can give $\boldsymbol{\Omega}_1 = (\mathbf{P}_{\mathcal{C}^2} \ldots \mathbf{P}_{\mathcal{C}^k}) \cdot \boldsymbol{\Omega}_k$, which (if it exists) represents the tangent plane or the tangent sphere to the surface $\boldsymbol{\Omega}_k$ at point $\mathbf{P}$; the special tangent sphere $\boldsymbol{\Omega}_1 = \mathbf{S}$ (CGA 1-blade sphere) is obtained as the tangent surface to special polynomial degree $k < l \leq 2k$ surfaces, and the tangent plane $\boldsymbol{\Omega}_1 = \boldsymbol{\Pi}$ (CGA 1-blade plane) is obtained as the tangent surface to general polynomial degree k surfaces in k-CGA.

In k-CGA $\mathcal{K}$, the CGAi $\mathcal{C}^i$ $\{1,2\}$-versors $V_{\mathcal{C}^i} \in \{\boldsymbol{E}_{\mathcal{C}^i}, \boldsymbol{\Sigma}_{\mathcal{C}^i}, R_{\mathcal{C}^i}, T_{\mathcal{C}^i}, D_{\mathcal{C}^i}, B_{\mathcal{C}^i}\}$ have corresponding $\{k, 2k\}$-versors $V_{\mathcal{K}} = V_{\mathcal{C}^i} \ldots V_{\mathcal{C}^k}$ that perform the same corresponding operations on all k-CGA entities $\boldsymbol{\Omega}$ as versor "sandwich" operations $\boldsymbol{\Omega}' = V_{\mathcal{K}} \boldsymbol{\Omega} V_{\mathcal{K}}^{-1}$. The versor operation performs the *outermorphism*

$$\boldsymbol{\Omega}' = V_{\mathcal{K}} \boldsymbol{\Omega} V_{\mathcal{K}}^{-1} = \sum_{s} \alpha_s V_{\mathcal{K}} T_{\mathcal{K}}^{s} V_{\mathcal{K}}^{-1} = \sum_{s, 1 \leq i \leq m_s} \frac{\alpha_s}{m_s} V_{\mathcal{C}^k} T_{\mathcal{C}_i^k}^{s_k} V_{\mathcal{C}^k}^{-1} \ldots V_{\mathcal{C}^2} T_{\mathcal{C}_i^2}^{s_2} V_{\mathcal{C}^2}^{-1} V_{\mathcal{C}^1} T_{\mathcal{C}_i^1}^{s_1} V_{\mathcal{C}^1}^{-1} \tag{172}$$

on the CGAi $\mathcal{C}^i$ factors $T_{\mathcal{C}^i}^{s_i}$ (extraction operators) in the terms of the entities $\boldsymbol{\Omega}$.

$T_x = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{\infty 1})$	$T_y = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{\infty 1})$	$T_z = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{\infty 1})$
$T_{x^2} = \mathbf{e}_8 \wedge \mathbf{e}_2$	$T_{y^2} = \mathbf{e}_9 \wedge \mathbf{e}_3$	$T_{z^2} = \mathbf{e}_{10} \wedge \mathbf{e}_4$
$T_{xy} = \frac{1}{2}(\mathbf{e}_9 \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_3)$	$T_{yz} = \frac{1}{2}(\mathbf{e}_{10} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_4)$	$T_{zx} = \frac{1}{2}(\mathbf{e}_8 \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_2)$
$T_{x\boldsymbol{t}_{\mathcal{M}}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{o1}$	$T_{y\boldsymbol{t}_{\mathcal{M}}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{o1}$	$T_{z\boldsymbol{t}_{\mathcal{M}}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{o1}$
$T_1 = -\mathbf{e}_\infty = -\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}$	$T_{\boldsymbol{t}_{\mathcal{M}}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} \wedge \mathbf{e}_{o1}$	$T_{\boldsymbol{t}_{\mathcal{M}}^4} = -4\mathbf{e}_o = -4\mathbf{e}_{o1} \wedge \mathbf{e}_{o2}$
$T_w = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_7)$	$T_{w^2} = \mathbf{e}_7 \wedge \mathbf{e}_1$	$T_{w\boldsymbol{t}_{\mathcal{M}}^2} = \mathbf{e}_1 \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_7$
$T_{wx} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_8 + \mathbf{e}_2 \wedge \mathbf{e}_7)$	$T_{wy} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_9 + \mathbf{e}_3 \wedge \mathbf{e}_7)$	$T_{wz} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{10} + \mathbf{e}_4 \wedge \mathbf{e}_7)$
$T_t = \frac{1}{c} T_w$	$T_{t^2} = \frac{1}{c^2} T_{w^2}$	$T_{t\boldsymbol{t}_{\mathcal{M}}^2} = \frac{1}{c} T_{w\boldsymbol{t}_{\mathcal{M}}^2}$
$T_{tx} = \frac{1}{c} T_{wx}$	$T_{ty} = \frac{1}{c} T_{wy}$	$T_{tz} = \frac{1}{c} T_{wz}$

Table 1. DCSTA bivector *extraction elements* T_s.

Table 1 gives all 27 of the DCSTA bivector *extraction operators* (or *elements*) T_s for the extractions of scalar values s (indicated by the indices $x, ..., tz$) from any DCSTA point $\boldsymbol{T}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{t}_{\mathcal{M}})$ by the inner products

$$s = T_s \cdot \boldsymbol{T}_{\mathcal{D}} = \boldsymbol{T}_{\mathcal{D}} \cdot T_s. \tag{173}$$

Note that, in Table 1, the scalar time $t = w/c$ is not the vector $\boldsymbol{t}$ (in ***bold italic***), where $\boldsymbol{t} = \boldsymbol{t}_{\mathcal{M}}$ is the STA symbolic "test" position vector (5) with square (space-time interval) $\boldsymbol{t}_{\mathcal{M}}^2$ (11).

A linear combination of the DCSTA extraction operators T_s forms a DCSTA GIPNS bivector geometric entity $\boldsymbol{\Omega}$ that represents a polynomial function $F(w,x,y,z)$, which in turn represents a Darboux cyclide implicit surface $F(w,x,y,z) = 0$ in space-time, where

$$\boldsymbol{\Omega} = \sum_s \alpha_s T_s \tag{174}$$

and

$$F(w,x,y,z) = \boldsymbol{T}_{\mathcal{D}} \cdot \boldsymbol{\Omega} = \sum_s \alpha_s s, \tag{175}$$

with real scalar coefficients α_s.

For $w = ct = 0$, then $\boldsymbol{t}_{\mathcal{M}} = \mathbf{t}_{\mathcal{S}}$ and $\boldsymbol{T}_{\mathcal{D}} = \mathbf{T}_{\mathcal{DS}} = \mathcal{D}(\mathbf{t}_{\mathcal{S}})$ is a DCSA spatial point $\mathbf{T}_{\mathcal{DS}}$ [7], and then the first five rows in Table 1 are the DCSA $\mathcal{G}_{2(1),2(3+1)}$ extraction operators T_s for spatial Darboux cyclide surfaces in the anti-Euclidean space $\mathbb{R}^{0,3}$. DCSA $\mathcal{G}_{2,8}$ [7] is similar to the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) $\mathcal{G}_{8,2}$ [4][5][6][8][9][10] with opposite signature.

Darboux cyclides are quartic (polynomial degree 4) surfaces that include quartic Dupin cyclides (including tori), quartic Blum cyclides, cubic (polynomial degree 3) parabolic cyclides, and *general quadric* (polynomial degree 2) *surfaces*. In Extended CGA (k-CGA), linear combinations of the k-vector extraction operators $\boldsymbol{T}_{\mathcal{K}}^s$ form k-vector entities that represent a further generalization of the Darboux cyclide polynomial function F that includes general degree k implicit surfaces and certain other specific implicit surfaces of degrees $k < l \leq 2k$ of inversive geometry.

The DCSTA GIPNS bivector entities for quadrics and cyclides can be directly written as linear combinations of the extraction operators T_s. For example, an ellipsoid (centered at the origin, aligned along the SA axes $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$) is

$$\mathbf{E} = T_{x^2}/a^2 + T_{y^2}/b^2 + T_{z^2}/c^2 - T_1, \tag{176}$$

and a general point $\boldsymbol{P}_{\mathcal{D}}$ is on the ellipsoid iff $\boldsymbol{P}_{\mathcal{D}} \cdot \mathbf{E} = 0$. The DCSTA dualization of the bivector $\mathbf{E}$,

$$\mathbf{E}^{*\mathcal{D}} = \mathbf{E}\mathbf{I}_{\mathcal{D}}^{-1} = \mathbf{E}(\mathbf{I}_{\mathcal{C}^1}\mathbf{I}_{\mathcal{C}^2})^{-1}, \tag{177}$$

is a valid GOPNS 10-vector entity where $\boldsymbol{P}_{\mathcal{D}}$ is on the ellipsoid iff $\boldsymbol{P}_{\mathcal{D}} \wedge \mathbf{E}^{*\mathcal{D}} = 0$. If time is always fixed as $t = 0$, then the DCSTA GIPNS bivector entities $\boldsymbol{\Omega}$ formed from the T_s correspond to entities of $\mathcal{G}_{8,2}$ DCGA [6][10], up to some sign differences in some scalar expressions, due to the different choice of signature.

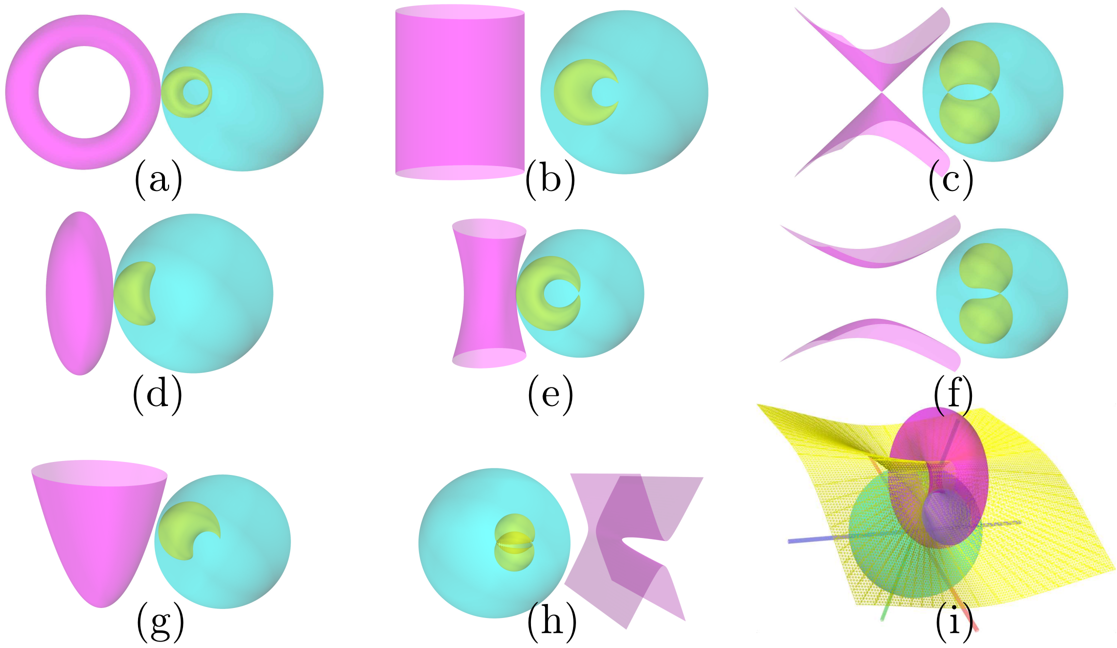

Figure 2. DCSA 2-vector quadrics $\mathbf{Q}$ and their inversions $\mathbf{\Omega}=\mathbf{SQS}^{\sim}$ in sphere $\mathbf{S}=\mathbf{S}_{\mathcal{DS}}$.

Figure 2 shows various DCSA 2-vector quadrics (and tori) $\mathbf{Q}$ and their inversions in a DCSA 2-blade sphere $\mathbf{S}=\mathbf{S}_{\mathcal{DS}}$ ($\mathbf{S}=\mathbf{\Sigma}_{\mathcal{D}}$ holding $t=0$). Figure 2(a) shows the inversion of a quartic torus, which is a quartic Dupin ring cyclide. Figure 2(b) shows the inversion of a quadric cylinder, which is a quartic Dupin needle cyclide. Figure 2(c) shows the inversion of a quadric cone, which is a Dupin horned cyclide. Figures 2(d,e,f,g,h) show the inversions of various other quadrics (ellipsoid, one sheet hyperboloid, two sheets hyperboloid, paraboloid, and hyperbolic paraboloid, resp.), which are various quartic Darboux cyclides. Figure 2(i) shows the inversion of a torus in a sphere that is centered on a surface point of the torus, which is a cubic parabolic cyclide. The DCSTA point at infinity $\mathbf{e}_{\infty}=\mathbf{e}_{\infty1}\mathbf{e}_{\infty2}$ is an *exterior* (outlier) *surface point* of the quadric ellipsoid entity $\mathbf{Q}=\mathbf{E}$ of Figure 2(d), where $\mathbf{S}\mathbf{e}_{\infty}\mathbf{S}^{-1}=\hat{\mathbf{P}}_{\mathcal{DS}}=\mathcal{D}(\mathbf{p}_{\mathcal{S}})$ is the center point of sphere $\mathbf{S}$ and an exterior surface point (not visible in the figure) of the Darboux cyclide $\mathbf{\Omega}=\mathbf{SES}^{\sim}$. An exterior (outlier) point may also be called a handle point.

Any DCSTA GIPNS bivector spatial quadric surface entity $\mathbf{Q}$, formed as a linear combination of the T_s from the first three rows in Table 1, has no time t dependency and appears to have zero velocity in space-time. The entity $\mathbf{Q}$ is also a purely spatial entity in the $\mathcal{G}_{2,8}$ DCSA subalgebra. The spatial quadric entity $\mathbf{Q}$ can be actively boosted into a spatial velocity $\mathbf{v}=\beta c\hat{\mathbf{v}}$ using the DCSTA 4-versor boost operator $B_{\mathcal{D}}=B_{\mathcal{C}^1}B_{\mathcal{C}^2}$ (158). In physics, the speed $0\leq v<c$ of massive bodies can only approach light speed c, and the natural speed $\beta=v/c$ is then limited to $0\leq\beta<1$. If $\mathbf{Q}$ has been translated by $\mathbf{d}$ from the origin (perhaps by using a DCSTA translator $T_{\mathcal{D}}=T_{\mathcal{C}^1}T_{\mathcal{C}^2}$) and is centered at spatial position $\mathbf{p}_0=\mathbf{d}$, then the translated boost operator $B_{\mathcal{D}}^{\mathbf{d}}=B_{\mathcal{C}^1}^{\mathbf{d}}B_{\mathcal{C}^2}^{\mathbf{d}}$ (161) can be used on $\mathbf{Q}$. The boosted quadric entity $\boldsymbol{Q}=B_{\mathcal{D}}^{\mathbf{d}}\mathbf{Q}B_{\mathcal{D}}^{\mathbf{d}\sim}$ has center position $\mathbf{p}_t=\mathbf{p}_0+\mathbf{v}t$ at time t, and has a geometrical length contraction (directed scaling) of the surface in the direction $\hat{\mathbf{v}}$ by factor $d=\sqrt{1-\beta^2}$, which is consistent with special relativity length contraction (66). While $\mathbf{Q}$ (in **bold**) is a spatial entity with no time t dependency, the boosted entity $\boldsymbol{Q}$ (in ***bold italic***) is a space-time entity with time dependent position according to a constant velocity $\mathbf{v}$ of the boost, and also a contraction effect at all times. At $t=0$, the contraction effect, which is a geometrical dilation, is present, and projecting $\boldsymbol{Q}$ on the DCSA subalgebra, effectively setting $t=0$, produces a purely spatial entity $\mathbf{Q}'=\mathcal{P}(\boldsymbol{Q})$ (178) at $t=0$, centered at $\mathbf{p}_0=\mathbf{d}$, that retains the geometrical dilation or length contraction. The result $\mathbf{Q}'$ is a directed scaling operation, but (so far) limited to a scaling factor $0<d\leq 1$.

The boost natural speed β for a length contraction factor d is $\beta=\sqrt{1-d^2}$, by solving for β in the Lorentz factor (65). Admitting the imaginary scalar $i=\sqrt{-1}$ (defined by $i^2=-1$), the boost of a quadric by an imaginary β dilates by $d>1$, and then the result can be projected to the spatial subalgebra $\mathcal{G}_{2,8}$ DCSA to discard time components and achieve directed spatial scaling in the direction $\hat{\mathbf{v}}$ of the boost velocity $\mathbf{v}$.

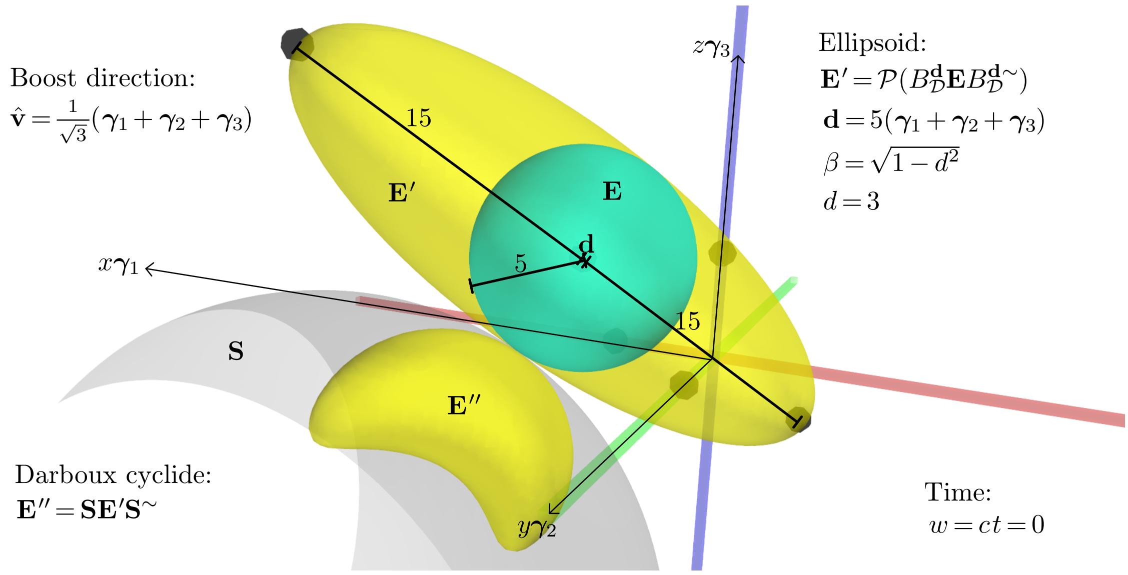

Figure 3. Spherical ellipsoid $\mathbf{E}(r=5)$ dilated by factor $d=3$ in direction $\hat{\mathbf{v}}$ as ellipsoid $\mathbf{E}'$ and then reflected in sphere $\mathbf{S}$ as Darboux cyclide $\mathbf{E}''$.

Figure 3 visualizes [19] a DCSTA GIPNS bivector *spherical ellipsoid* $\mathbf{E}$ dilated in situ by factor $d=3$ in the direction $\hat{\mathbf{v}}$ as $\mathbf{E}'$ using a *translated-boost* operator $B_{\mathcal{D}}^{\mathbf{d}}$ centered on the center position $\mathbf{p}_0 = \mathbf{d}$ of $\mathbf{E}$ and $\mathbf{E}'$. The $\mathcal{G}_{2,8}$ DCSA projection is

$$\mathcal{P}(A) = (A \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1}, \tag{178}$$

where the DCSA unit pseudoscalar is

$$\mathbf{I}_{\mathcal{DS}} = \mathbf{I}_{\mathcal{S}^1}\mathbf{e}_5\mathbf{e}_6\mathbf{I}_{\mathcal{S}^2}\mathbf{e}_{11}\mathbf{e}_{12}. \tag{179}$$

$\mathbf{E}'$ is reflected in a DCSTA GIPNS 2-blade *(hyperpseudo)sphere* $\mathbf{S} = \boldsymbol{\Sigma}(t=0, r_0=15) = \boldsymbol{\Sigma}_{\mathcal{C}^1}\boldsymbol{\Sigma}_{\mathcal{C}^2}$ as $\mathbf{E}''$, which is a Darboux cyclide. The sphere $\mathbf{S}$, initially centered on the origin, was translated using a DCSTA 4-versor translator $T_{\mathcal{D}}$ by a displacement vector $\mathbf{d} + (5+15)R\hat{\mathbf{d}}R^{\sim}$, using $R = \exp\left(\frac{1}{2}\frac{\pi}{2}\frac{1}{\sqrt{2}}(\gamma_2 - \gamma_1)^{*\mathcal{S}}\right)$, to bring the sphere into a tangential position to $\mathbf{E}'$. All are at time $t=0$.

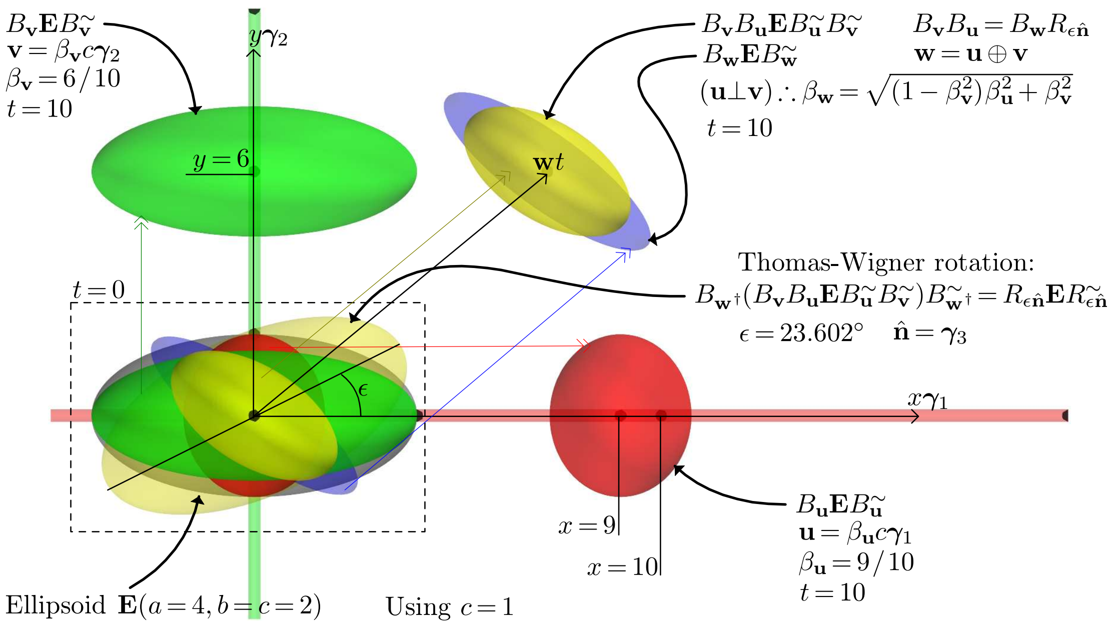

Figure 4. Boosts of ellipsoid $\mathbf{E}$, showing length contractions and Thomas-Wigner rotation.

Figure 4 shows the DCSA 2-vector ellipsoid entity $\mathbf{E}$ (176) with $a=4$ (x-diameter $2a=8$) and $b=c=2$ (y-diameter $2b=4$), centered on the origin with zero initial velocity, which is then boosted by various velocities $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}=\mathbf{u}\oplus\mathbf{v}$. In the figure, natural speeds are used with $c=1$, and all of the boost B and rotation R versors are assumed to be their unimodular DCSTA doubled forms (e.g., $B_{\mathbf{u}}=\hat{B}_{\mathcal{C}^1\mathbf{u}}\hat{B}_{\mathcal{C}^2\mathbf{u}}$). The boosted ellipsoid $B_{\mathbf{u}}\mathbf{E}\widetilde{B_{\mathbf{u}}}=\mathbf{E}\oplus\boldsymbol{u}$ (extending the notation of (70)), by $\mathbf{u}=\beta_{\mathbf{u}}c\gamma_1=(9/10)\gamma_1$ and with center position $\mathbf{u}t=9\gamma_1$ at $t=10$, has length contraction factor $\gamma_{\mathbf{u}}^{-1}=\sqrt{1-\beta_{\mathbf{u}}^2}\approx 0.436$ and the contracted x-diameter is approximately $\gamma_{\mathbf{u}}^{-1}2a\approx 3.487$. The boosted ellipsoid $B_{\mathbf{v}}\mathbf{E}\widetilde{B_{\mathbf{v}}}=\mathbf{E}\oplus\boldsymbol{v}$, by $\mathbf{v}=\beta_{\mathbf{v}}c\gamma_2=(6/10)\gamma_2$ with center position $\mathbf{v}t=6\gamma_2$ at $t=10$, has length contraction factor $\gamma_{\mathbf{v}}^{-1}=\sqrt{1-\beta_{\mathbf{v}}^2}=0.8$ and the contracted y-diameter is $\gamma_{\mathbf{v}}^{-1}2b=3.2$. The boosted ellipsoid $B_{\mathbf{v}}B_{\mathbf{u}}\mathbf{E}\widetilde{B_{\mathbf{u}}}\widetilde{B_{\mathbf{v}}}=B_{\mathbf{w}}R_{\epsilon\hat{\mathbf{n}}}\mathbf{E}\widetilde{R_{\epsilon\hat{\mathbf{n}}}}\widetilde{B_{\mathbf{w}}}$ (by (108)), with a resulting velocity $\mathbf{w}=\beta_{\mathbf{w}}c\hat{\mathbf{w}}\approx 0.9372c(0.7682\gamma_1+0.6402\gamma_2)$ (by (112) and (113)) and center position $\mathbf{w}t=7.2\gamma_1+6\gamma_2$ at $t=10$, has a more complicated contraction due to the composition of boosts; however, when boosted back to zero velocity as $B_{\mathbf{w}^\dagger}(B_{\mathbf{w}}R_{\epsilon\hat{\mathbf{n}}}\mathbf{E}\widetilde{R_{\epsilon\hat{\mathbf{n}}}}\widetilde{B_{\mathbf{w}}})\widetilde{B_{\mathbf{w}^\dagger}}=R_{\epsilon\hat{\mathbf{n}}}\mathbf{E}\widetilde{R_{\epsilon\hat{\mathbf{n}}}}$, then it is only the Thomas-Wigner rotation $R_{\epsilon\hat{\mathbf{n}}}$ (96) of the ellipsoid $\mathbf{E}$. In this example, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular ($\mathbf{u}\perp\mathbf{v}$), so we can also obtain $\beta_{\mathbf{w}}$ by (91) as $\beta_{\mathbf{w}}=\beta_{\mathbf{u}\oplus\mathbf{v}}=\sqrt{(1-\beta_{\mathbf{v}}^2)\beta_{\mathbf{u}}^2+\beta_{\mathbf{v}}^2}$. The boosted ellipsoid $B_{\mathbf{w}}\mathbf{E}\widetilde{B_{\mathbf{w}}}$ has the same velocity $\mathbf{w}$ as $B_{\mathbf{v}}B_{\mathbf{u}}\mathbf{E}\widetilde{B_{\mathbf{u}}}\widetilde{B_{\mathbf{v}}}$, but is a much different result: it is boosted into the frame of $\boldsymbol{w}=\boldsymbol{o}+\mathbf{w}$, not into the frame of $\boldsymbol{u}$ then into the frame of $\boldsymbol{v}$; it does not include the Thomas-Wigner rotation $R_{\epsilon\hat{\mathbf{n}}}$; and, it has a simple contraction by the factor $\sqrt{1-\beta_{\mathbf{w}}^2}\approx 0.3487$ in only the direction $\hat{\mathbf{w}}$.

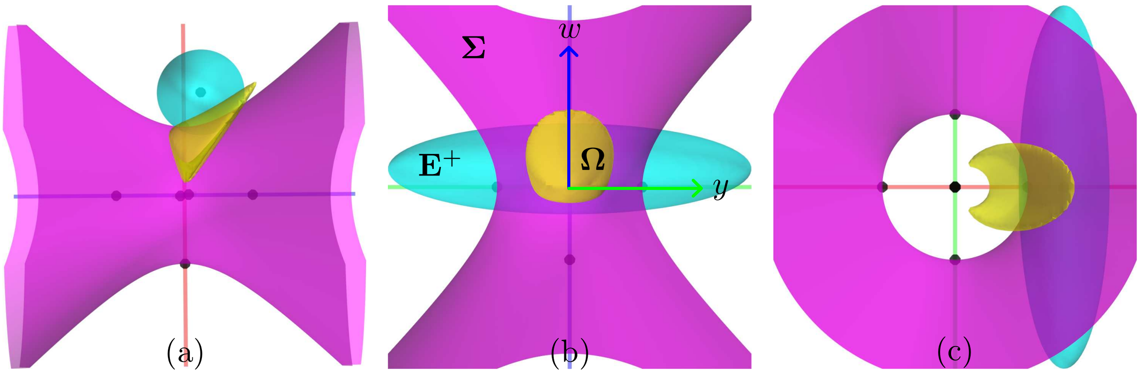

Figure 5. Inversion $\boldsymbol{\Omega}=\boldsymbol{\Sigma}_{\mathcal{D}}\mathbf{E}^+\widetilde{\boldsymbol{\Sigma}_{\mathcal{D}}}$ of pseudoquadric $\mathbf{E}^+$ in pseudosphere $\boldsymbol{\Sigma}_{\mathcal{D}}$.

Figure 5 shows the inversion $\boldsymbol{\Omega}=\boldsymbol{\Sigma}_{\mathcal{D}}\mathbf{E}^+\widetilde{\boldsymbol{\Sigma}_{\mathcal{D}}}$ of a DCSTA 2-vector *pseudoquadric* ellipsoid

$$\mathbf{E}^+=T_{x^2}/a^2+T_{y^2}/b^2+T_{w^2}/c^2-T_1 \tag{180}$$

in a DCSTA 2-blade hyperpseudosphere $\boldsymbol{\Sigma}_{\mathcal{D}}$, which is a pseudosphere (circular space-time hyperboloid) in the three dimensional space-time of the two spatial dimensions x and y with the pseudospatial dimension w, holding $z=0$. A DCSTA 2-vector space-time pseudoquadric (pseudospatial quadric) $\mathbf{Q}^+$ is formed from a DCSA 2-vector spatial quadric $\mathbf{Q}$ by replacing one of the coordinates x, y, or z with the pseudospatial coordinate w. The inversion of the corresponding spatial quadric ellipsoid $\boldsymbol{\Omega}=\boldsymbol{\Sigma}_{\mathcal{D}}\mathbf{E}\widetilde{\boldsymbol{\Sigma}_{\mathcal{D}}}$ viewed in the same three dimensions x, y, and w sees the spatial ellipsoid as a circle (or ellipse) in the xy-plane and as a cylinder in xyw-spacetime (i.e., the same xy-plane circle for all time $w=ct$), and therefore its inversion appears quite different than the inversion of the corresponding pseudoquadric.

The DCSTA *differential elements* are

$$\begin{aligned} D_w &= 2T_wT_{w^2}^{-1} && (181)\\ D_x &= 2T_xT_{x^2}^{-1} && (182)\\ D_y &= 2T_yT_{y^2}^{-1} && (183)\\ D_z &= 2T_zT_{z^2}^{-1} && (184)\\ D_t &= 2T_tT_{t^2}^{-1} && (185)\end{aligned}$$

and the commutator product $\times$ of multivectors A and B is

$$A\times B = (AB-BA)/2=-B\times A. \tag{186}$$

Using the commutator product, the DCSTA differential elements are *differential operators* on any bivector surface entity $\mathbf{\Omega}$ that is formed as a linear combination of the DCSTA extraction elements T_s. The time t derivative of $\mathbf{\Omega}$ is

$$\dot{\mathbf{\Omega}} = \partial_t \mathbf{\Omega} = \frac{\partial \mathbf{\Omega}}{\partial t} = D_t \times \mathbf{\Omega}. \tag{187}$$

For direction $\boldsymbol{n}$ with unit magnitude $\|\boldsymbol{n}\|_2 = \sqrt{\boldsymbol{n} \cdot \boldsymbol{n}^{\dagger}} = 1$, the $\boldsymbol{n}$-directional derivative operator is

$$\partial_{\boldsymbol{n}} = \frac{\partial}{\partial \boldsymbol{n}} = D_{\boldsymbol{n}} \times = (n_w D_w + n_x D_x + n_y D_y + n_z D_z) \times \tag{188}$$

and the $\boldsymbol{n}$-directional derivative of any bivector entity $\mathbf{\Omega}$ is

$$\partial_{\boldsymbol{n}} \mathbf{\Omega} = D_{\boldsymbol{n}} \times \mathbf{\Omega}. \tag{189}$$

The entity $\mathbf{\Omega}$ represents an implicit surface function $F(w, x, y, z)$, and its $\boldsymbol{n}$-directional derivative $\partial_{\boldsymbol{n}} \mathbf{\Omega}$ represents the derivative implicit surface function $\partial_{\boldsymbol{n}} F$. Mixed partial derivatives are obtained by taking successive derivatives in any order.

In Extended CGA (k-CGA), the differential operators have the general form $D_x = k T_{x^{k-1}} T_{x^k}^{-1}$ in (189).

5 Conclusion

$\mathcal{G}_{4,8}$ DCSTA [7][11] extends $\mathcal{G}_{2,8}$ Double Conformal Space Algebra (DCSA) [7], which is different in space signature from the DCGA $\mathcal{G}_{8,2}$ of [4][5][6][8][9][10], into a high-dimensional 12D embedding of Space-Time Algebra $\mathcal{G}_{1,3}$ [14] that has general quadric surface entities with a complete set of space-time transformation operations as versors and projections.

The DCSTA 2-vector general quadric surface entities provide an accurate representation of quadric surfaces in space-time. As discussed, boosts (see Figure 4) of the DCSTA quadric surface entities are moving surfaces that include the special relativity effects of length contraction (66) and Thomas-Wigner rotation (96). In geometry and physics, the DCSTA 2-vector surface entities, including general quadrics and their inversions in hyperpseudospheres, may find uses in education and applications for modeling inversive geometry and surfaces in the space-time of special relativity.

DCSTA is a Geometric Algebra [2][3][15][18] for computing with general quadric surfaces and their inversions in hyperpseudospheres in space-time. For applications, testing, or education, DCSTA $\mathcal{G}_{4,8}$ can be computed using various software packages. During the research and writing of this paper, the author used the free symbolic computer algebra system *Sympy* [22] with the *$\mathcal{G}$Algebra* [1] module. All figures were rendered using *Mayavi* [19] and annotated with mathematical text using TeXmacs [23].

As discussed in the paper, but not elaborated in full detail, not only is it possible to construct a doubling of CGAs as for DCSTA, but it is also possible (in theory) to extend to any number k of orthogonal CGAs $\mathcal{G}_{k(p+1),k(q+1)}$. This CGA extension theory is to be called Extended CGA or k-CGA. In k-CGA, there are k-vector entities (linear combinations of k-vector extraction operators) that represent general degree k hypersurfaces and also certain other hypersurfaces of degrees l, $k < l \leq 2k$, representing all possible inversions (and compositions of inversions) of the general degree k hypersurfaces in hyperpseudospheres.

As another example of k-CGA, the paper [12] presents the Extended CGA (3-CGA) $\mathcal{G}_{3(2+1),3} = \mathcal{G}_{9,3}$ over $\mathcal{G}_{2,0}$, called the Triple Conformal Geometric Algebra (TCGA), as an algebra for general cubic plane curve entities and their inversions. TCGA could be further extended to 3D, as $\mathcal{G}_{3(3+1)3} = \mathcal{G}_{12,3}$ over $\mathcal{G}_{3,0}$, for general cubic surface entities and their inversions.

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