Noname manuscript No.
(will be inserted by the editor)

Density matrices and the Standard Model

C. A. Brannen

Received: date / Accepted: date

Abstract We use density matrices to explore the possibility that the various flavors of quarks and leptons are linear superpositions over a single particle whose symmetry follows the finite subgroup S_4 of the simple Lie group SO(3). Density matrices allow modeling of symmetry breaking over temperature, and can incorporate superselection sectors. We obtain three generations each consisting of the quarks and leptons and an SU(2) dark matter doublet. We apply the model to the Koide mass equations and propose extensions of the theory to other parts of the Standard Model and gravitation.

Keywords density matrices, standard model, quarks, leptons, dark matter, Koide mass equations

1 Introduction

We propose that the various flavors of quarks and leptons arise as linear superpositions of an underlying fermion we will call the citronen.[1] The citronen follows a gauge Lie group SO(3) from which we derive the Standard Model gauge Lie group SU(3)×SU(2)×U(1). If we SO(3) rotate the citronen far enough, we get a new flavor citronen in the same way we can SU(3) rotate a red quark to obtain a green quark. The 24 flavors of citronens are defined by the permutation group S_4 which is a finite subgroup of the Lie group SO(3).

We begin by outlining the density matrix theory needed in this paper. Starting with the assumption of a general Lie group G with finite subgroup F, we use the color symmetry of the Standard Model to deduce that $F = S_4$ so that S_4 defines a basis set for the citronens. We show that this implies the citronen gauge Lie group $G = \mathrm{SO}(3)$. Standard Model particle interactions are left and right handed so we convert the S_4 symmetry basis set for the citronens

Department of Physics, Washington State University, Pullman, WA 99164, USA

[1] Citronen means "lemon" in some European languages. Nelogoism due to Forrest LeDuc.

to four handed basis sets $\{L \to L,\ L \to R,\ R \to L,\ R \to R\}$, each with S_3 symmetry. This converts 24 S_4 basis elements to four sets of six. We examine applications of S_3 to the Standard Model, especially the Koide mass equations for the quarks and leptons and propose further work with these ideas including gravitation.

1.1 Contents

Section 2 covers density matrix ideas that we use to model the Standard Model fermions. These include temperature dependence, superselection sectors, symmetry breaking, the cold limit (pure) density matrices, density matrices using general algebras instead of matrix algebras, Schwinger's method of finite quantum field theory using an algebra, and the Standard Model in block diagonal density matrix form. For complex algebras over finite groups, the symmetry and particle content for the density matrix particles can be read from the character table of the finite group, and the internal symmetries of the particles are representations of F. This allows F to be determined.

In Section 3 we determine $F = S_4$ by examining how color SU(3) permutes the colors of the quarks. We find three copies (or generations) of the finite group S_4. Examining the character table of S_4 we deduce that the Standard Model includes a dark matter SU(2) doublet. We write the quarks and leptons, and the Gell-Mann color matrices in the complex S_4 algebra and give the Young's tableaux for the fermions.

In Section 4, we consider possibilities for the gauge Lie group G and conclude that it is SO(3). This makes S_4 a point symmetry, so we write the permutations using gamma matrices. For visualization, we put the symmetries of the elementary particles into crystal (gyroidal or 432) form. The spherical harmonics are SO(3) symmetries and with S_4 a subgroup there is a subtle relationship between the first four spherical harmonics and the quarks, leptons and dark matter.

In Section 5 we propose the quantum state for the citronen. It must have the symmetries of S_4. The result is related to the quantum information idea of "mutually unbiased bases". These 24 citronens give the basis for the quarks and leptons. But the Standard Model particle interactions are between left and right handed particles so we convert from the citronen basis with 24 bases of S_4 symmetry to four basis sets depending on the incoming and outgoing handedness, left to left, etc. These four basis sets each has six elements with S_3 symmetry. We show that the weak quantum numbers are idempotents of the complex algebra over S_3 and hope that a more complete theory can derive them. Finally we derive the mass interaction and give support to the Koide mass equations for the leptons and quarks.

Section 6 discusses the advantages of this theory, the problems with it, the problems with this paper in particular and directions in which this work might be extended. These include quantum statistics and the quasi-normal modes of black holes, compatibility with general relativity using Gauge Theory Gravity,

Gullstrand-Painlevé coordinates and gravitons, and the weak mixing matrices and parameterizations in S_3 circulant form.

1.2 Modified Density Matrices

In regards to the Standard Model, quantum mechanics has been well explored so a new idea will require a modification of some previous assumption. Since this is a paper about the flavors of the quarks and leptons, we will be working in finite dimensional quantum mechanics and we will explain the modifications using the example of SU(2) spin-1/2.

Assuming density matrices are the fundamental quantum state "opens up a much larger variety of ways that the density matrix might respond to various symmetry transformations."[45] But we will go further than Steven Weinberg's 2014 paper by reconsidering the assumption that density matrices use matrix multiplication. Instead, we generalize complex matrices to "algebras". This may be confusing to physicists unfamiliar with algebra (as distinct from "Lie algebra") so we've tried to be careful with our language. We call su(2) a "Lie algebra", SU(2) is a "Lie group", the permutation group S_4 is a "finite group" and it is also a "finite subgroup" of the Lie group SO(3).

If one begins with the assumption that state vectors are the fundamental quantum state then a density matrix ρ is defined by the multiplication of a ket and bra. For spin-up $(+z)$, the multiplication is:

$$\begin{aligned} \rho_{+z} &= |+z\rangle\langle+z|, \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}. \end{aligned} \tag{1}$$

With kets as complex vectors taken from an n-dimensional Hilbert space, the above definition automatically defines density matrices as $n \times n$ complex matrices. Our generalization is that we will allow density matrices to live on more general algebras than the complex matrices. These more general algebras support density matrices that cannot be immediately translated to bra-ket or state vector form.

An "algebra" is a vector space with multiplication. To see how we will be generalizing matrices, we can define matrix multiplication as we would for an algebra. We begin with matrices as a vector space. A vector space has a set of basis vectors. An element of the vector space is a sum of complex multiples of the basis elements. The usual basis set for the 2×2 complex matrices is

$$\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}. \tag{2}$$

Given two vectors, we define addition by adding them term by term. A vector can be multiplied by a complex number, by multiplying each term by the complex number. This scalar multiplication applies to matrices as they are a vector space. To make a vector space into an algebra we need to define how to multiply a vector by a vector.

To define a multiplication on a vector space, we first define multiplication for the basis elements. Then we distribute multiplication over addition and reduce the result by using the basis element multiplication. For example, using the standard matrix basis set,

$$\begin{aligned}
\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} &= [3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-3) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}] \quad [-2i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 2i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}], \\
&= (3)(-2i) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (3)(2i) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&\quad +(-3)(-2i) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-3)(2i) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= (-6i) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 + 0 + (-6i) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -6i \\ -6i & 0 \end{pmatrix}.
\end{aligned} \tag{3}$$

In the above multiplication, the two matrices each have two non-zero entries so there are four products of bases. Of those four products, the middle two were zero because the basis elements multiply to give zero. Of the sixteen possible products of the four basis elements for the 2×2 matrices, half are zero.

One chooses the basis for a vector space according to convenience. An alternative basis for the 2×2 complex matrices is the "Pauli basis":

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

In the Pauli basis, the multiplication shown in Equation 3 is

$$(3\sigma_z)(2\sigma_y) = (6)(\sigma_z\sigma_y) = (6)(-i\sigma_x) = -6i\sigma_x. \tag{5}$$

In the Pauli basis none of products of basis elements are zero. The same is true of the gamma matrix basis which consists of the 16 gamma matrix products. Our first modification of density matrices will be to block diagonal form due to superselection sector rules and is described in Subsection 2.2.

In making a measurement of a spin-1/2 system, we choose an operator, for example σ_z for spin in the z direction, and there will be two possible results from the measurement according to the two eigenvalues of σ_z. This works beautifully for spin-1/2 measurements with our present experimental equipment and we do not propose any changes. The two possible results are a "complete set of basis states", that is, a basis for state vectors. One will find that spin is in the $+z$ or $-z$ direction.

Our generalization is to consider "over complete" sets of basis states. For spin-1/2, the natural over complete set of basis states is already in use in quantum information theory where it is called a "set of mutually unbiased bases." [14] For spin-1/2, these consist of the bases resulting from spin measurements in the x, y and z directions. They are "mutually unbiased" in that the transition probabilities between states from different bases are all equal. For such a system, there are six possible results for a spin-1/2 measurement

corresponding to the six directions $\pm x$, $\pm y$ and $\pm z$. The spin of a classical system could be measured in any direction, so our modification is part of the way to the classical situation.

Rather than the Pauli spin matrices, this paper will be dealing with gamma matrices but the ideas are similar. The (more or less) mutually unbiased bases will be defined by a finite subgroup F of a Lie group G. In a certain way, the finite subgroup F defines how far one must G rotate to change a state enough to make it independent (unbiased) with respect to the unrotated state.

An advantage of using over-complete bases is that they have non-zero transition amplitudes between different bases, such as $\langle +x| + y\rangle$. The transition amplitudes define interactions between density matrices so we get interacting systems without having to specify a Lagrangian or Hamiltonian. In analyzing the mass interaction we will be making density matrices from density matrices. In doing this, the new density matrix has diagonal elements that are real multiples of the pure density matrices and the off diagonal elements are complex (Hermitian) multiples of products of the diagonal density matrices. In terms of quantum field theory, fermion propagators are on the diagonal and interactions, where a fermion is annihilated and another created, are off diagonal.

2 Density Matrix Theory

In this section we discuss the density matrix ideas needed in this paper including temperature modeling, superselection sector rules and block diagonal density matrices, symmetry breaking, cold limit (pure) density matrices, density matrices derived from an algebra with S_3 as an example, Schwinger's Measurement Algebra and the relation between quantum field theory and density matrices, and the algebra which supports the Standard Model fermions.

2.1 Temperature and Density Matrices

Density matrices are particularly useful for exploring the effects of temperature in quantum mechanics. For a Hamiltonian H, there is an associated density matrix that depends on temperature by the familiar formula:

$$\rho(T) \ \propto \ \exp(-H/(k_B T)). \tag{6}$$

In this form, the temperature can be changed by taking powers: $\rho(T/a) \propto [\rho(T)]^a$. In the high temperature limit we have

$$\rho(T=\infty) \ = 1/\mathrm{tr}(1) \ \propto \ 1 \tag{7}$$

The heat bath that one usually associates with a canonical ensemble exchanges photons with the ensemble. The effect of temperature on such an ensemble is described by density matrices. This paper is concerned with heat baths at extreme temperatures where the photons will be supplemented by

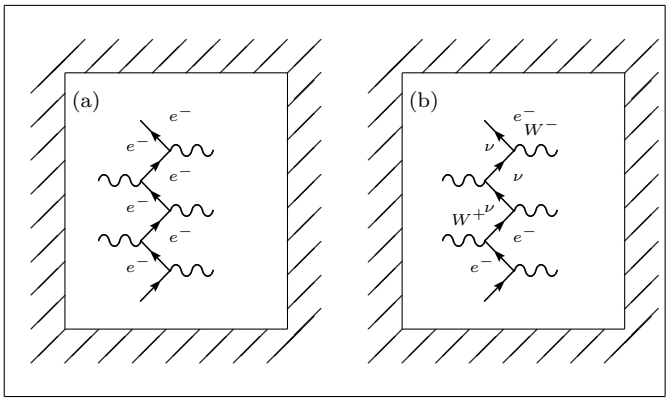

Fig. 1: (a) Heat bath with photons as the gauge bosons, so the fermions do not change flavor. (b) Higher temperature heat bath also includes $W^{\pm}$ in the interactions converting between neutrinos and electrons.

all the other gauge bosons such as the $W^{\pm}$. At these temperatures, the heat bath allows the fermions to change flavor and so particle content changes with temperature. See Figure 1. We suppose that at even higher temperatures the number of gauge bosons will continue to increase to include ones that can convert between leptons and quarks so our density matrices will include all the elementary fermions. And we will assume that some, so far undiscovered, lepto-quark-dark matter gauge boson converts between leptons, quarks and dark matter so that they all approach the same state at high temperatures. A lepto-quark gauge boson has been proposed as an explanation for recently observed 4 standard deviation experimental violations of lepton universality.[10]

2.2 Superselection Sectors and Density Matrices

Given a set of states $\{|x_j\rangle\}$ and probabilities p_j that sum to unity, a density matrix is traditionally given by:

$$\rho = \Sigma_j \; p_j \; |x_j\rangle\langle x_j|. \tag{8}$$

If the $|x_j\rangle$ are all spin-up or spin-down, the density matrix will have the form:

$$\rho = p_+|+z\rangle\langle+z| \;\; + \;\; p_-|-z\rangle\langle-z| = \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}. \tag{9}$$

To get non zero off diagonal elements of the density matrix we have to make the density matrix from states that are linear superpositions of spin-up and spin-down. Otherwise, the density matrix is restricted to have a block diagonal form.

One can represent a beam of spin-up electrons with a state vector $|+z\rangle$. Such a beam can be rotated to have spin in the $+x$ direction by a suitable

arrangement of magnets and the new beam $|+x\rangle$ is given by a linear superposition of spin-up and spin-down:

$$|+x\rangle = (|+z\rangle \ + \ |-z\rangle)/\sqrt{2}. \tag{10}$$

It is thought to be impossible to produce a quantum state that is a linear superposition of an electron and a neutrino. The reason for this is that the particles have different electric charge and so are in different "superselection sectors". This type of restriction also prevents the linear superposition of a fermion with a boson.[46,9] Superselection sectors arise from symmetries and conservation laws; in this paper we assume that the usual superselection sectors arise from the symmetry calculations given here. See [42] for a recent discussion of symmetry and superselection sectors. While it is possible to modify the symmetry so as to eliminate superselection sectors, this does not give one a prescription for designing an experimental apparatus which will violate the superselection rules.[44]

Since the electron and neutrino are in different superselection sectors, our density matrices cannot use states such as $(|e\rangle + |\nu\rangle)/\sqrt{2}$. Instead, we can only combine superselection sectors incoherently and the density matrix for a combination of electrons and neutrinos will appear as:

$$\rho \ = \ p_e|e\rangle\langle e| \ + \ p_\nu|\nu\rangle\langle\nu| = \begin{pmatrix} p_e & \\ & p_\nu \end{pmatrix}. \tag{11}$$

We've left the off diagonal entries in the above matrix blank because they are not just zero but are forbidden by the superselection rule. This is an example of the block diagonal form that arises when density matrices are used for mixtures of particles from different superselection sectors. For the above, the two blocks are 1×1. If we had included spin-1/2, the p_e and p_ν are replaced with 2×2 density matrices and the block diagonals are 2×2. This is an example of the fact that density matrices naturally give rise to particle symmetries of the form SU(j)×SU(k)×...U(1).

The density matrices given in Equation 11 are elements of an algebra of block diagonal matrices of general form:

$$\begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \tag{12}$$

where α_1 and α_2 are complex numbers. These matrices are closed under multiplication and addition and include zero and one. That is, they form an algebra.

An "idempotent" of an algebra is an element that is unchanged by squaring. This is an attribute of pure density matrices: $\rho \ \rho = \rho$. A "primitive" idempotent is (by a simplified physicist appropriate definition) one that has trace 1, also an attribute of density matrices. The above algebra has two primitive idempotents; they correspond to the electron and the neutrino:

$$e = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \ \ \nu = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \tag{13}$$

Primitive idempotents are also used to identify bispinors defined with gamma matrices [22], and are the elements of an algebra to associate with particles [5]. In this section we will be examining an algebra defined by the permutation group S_3. We will put that algebra into block diagonal form and interpret the blocks as particles.

2.3 SU(2) Symmetry Breaking

Suppose that we have two states that are similar in some way, say ρ_e and ρ_ν, for the weak force and the left-handed electron and neutrino states. Since these states are in different superselection sectors, the general form for a density matrix representing a mixture of them at some finite temperature is given by Equation 11 above. The infinite temperature limit is given by Equation 7.

Now consider what the situation would be if the symmetry wasn't broken and the electron and neutrino were a true SU(2) doublet. A general form for a mixed SU(2) density matrix is

$$\begin{aligned} \rho(x,y,z) &= \tfrac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z, \end{pmatrix}, \\ &= (1 + x\sigma_x + y\sigma_y + z\sigma_z)/2. \end{aligned} \tag{14}$$

where x, y, z are real with $x^2 + y^2 + z^2 < 1$. Rotating (x, y, z) to the z direction puts this mixed density matrix into the form of Equation 11 so its high temperature limit is the same Equation 7.

More generally, the high temperature limit of a block diagonal mixed density matrix is the same as the high temperature limit of the usual (single block) mixed density matrix. That is, in the high temperature limit, it is not possible to detect the presence of superselection sectors:

$$\begin{array}{rcc} & \text{finite } T & T \to \infty \\ \text{coherent:} \\ \text{(State vector)} & \tfrac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z, \end{pmatrix} & \to \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{incoherent:} \\ \text{(Density matrix)} & \tfrac{1}{2}\begin{pmatrix} p_e & \\ & p_\nu \end{pmatrix} & \to \tfrac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{array} \tag{15}$$

Thus if a boson makes it possible to convert between two similar particles in different superselection sectors, when described in state vector form it will have the appearance of spontaneous symmetry breaking. But the description in density matrices will simply be the usual block diagonal form.

2.4 Density Matrix Cold Limit Particles

Suppose we begin with a randomly chosen mixed density matrix and repeatedly lower its temperature by squaring and dividing by the trace (to keep the

trace 1). For 2×2 matrices we can either have a single 2×2 block or two 1×1 blocks. These are the cases of the electron and neutrino mixture being either coherent or incoherent. Beginning with arbitrary legal initial conditions, the limit result of repeated cooling is:

$$\begin{array}{lcc} & \text{finite } T & T=0 \\ 2\times 2 \text{ block:} & \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z, \end{pmatrix} & \to \frac{1}{2}\begin{pmatrix} 1+z' & x'-iy' \\ x'+iy' & 1-z', \end{pmatrix} \\ \text{two } 1\times 1 \text{ blocks:} & \frac{1}{2}\begin{pmatrix} p_e & \\ & p_\nu \end{pmatrix} & \to \quad \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \end{array} \tag{16}$$

where $x' = x/\sqrt{x^2+y^2+z^2}$ etc., and the "or" follows as p_e is greater or smaller than p_ν. That is, for the 2×2 case we get an SU(2) doublet state that mixes an electron and neutrino with weak isospin in the (x', y', z') direction, and with the two 1×1 case we get either an electron or a neutrino. This is how density matrices model weak isospin either remaining unbroken or breaking to the electron and neutrino.

Let P_k be a projection operator for the kth block. So if the kth block is a $j\times j$ block, then P_k has j ones on the diagonal in that block and is elsewhere zero. Such a projection operator annihilates anything outside of the k block and it is the identity operator on things that are in the k block. Thus P_k commutes with any element of the algebra. In particular, if ρ is any pure density matrix state so that ρ is a primitive idempotent, then ρP_k is idempotent and so is either zero or is ρ and we have that the pure states of a block diagonal matrix algebra are associated with one block or another. So given an algebra in block diagonal form, we can read off the particle content by looking at the sizes of the blocks.

Cooling a random mixed density matrix will form a limit that approaches a pure density matrix. As a visual example of how density matrix renormalization produces pure states by cooling, see Figure 2. The details of the group theory behind this example will be give in the next subsection.

2.5 Density Matrix Algebras

As algebra elements, density matrices can be added and multiplied. The results are generally not another density matrix, but make sense mathematically. As an example of the usefulness of adding density matrices, consider two density matrices ρ_1 and ρ_2. Given two real numbers p and $q = 1-p$ we can combine the two density matrices by $p\rho_1 + q\rho_2$. This describes a mixture that corresponds to a mixing of the contents of the two ensembles.

The procedure of squaring a density matrix is a way of changing its temperature. So squaring is a mathematical operation that makes sense physically. To enlarge squaring to multiplication, consider what happens when we square

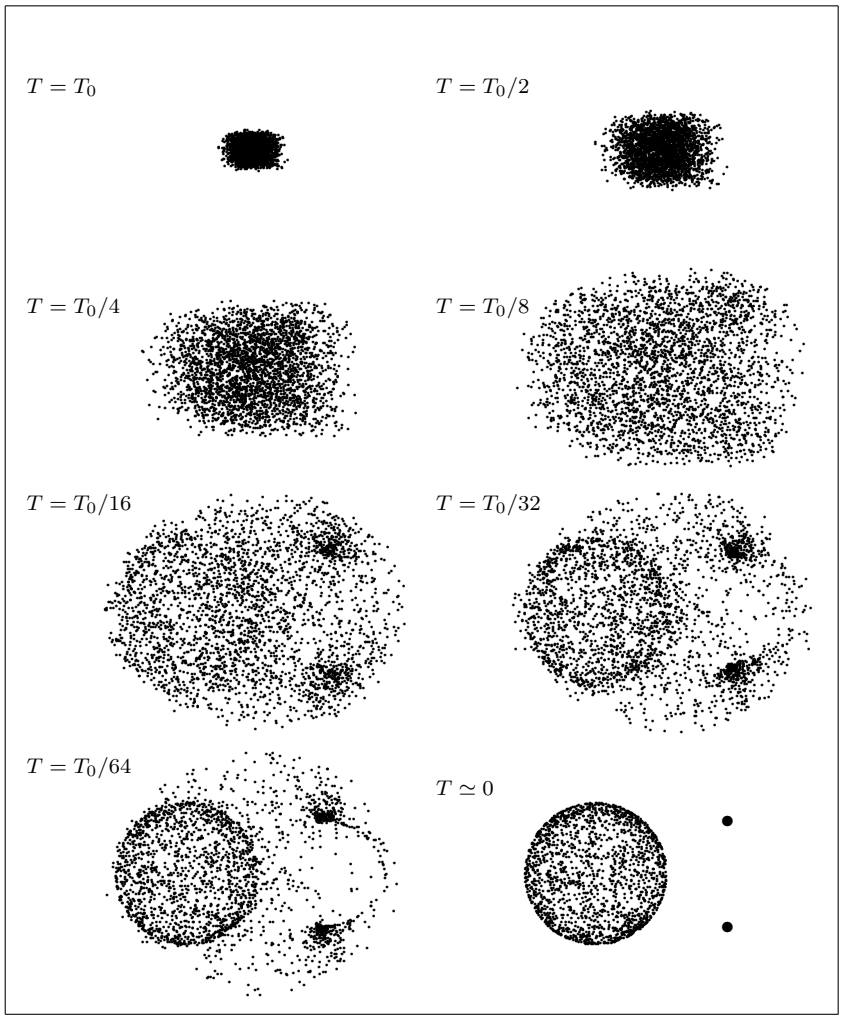

Fig. 2: Density matrix renormalization flow of $\mathcal{C}(S_3)$ with 3,000 states. As the mixed density matrices cool down they approach pure states. The symmetry and representation is $SU(2)\times U(1)\times U(1)$ with one SU(2) doublet and two U(1) singlets. The two axes of the SU(2) Bloch sphere are given by $P_{3z} = -i[(123)-(132)]/\sqrt{3}$ and $P_{3y} = [(23)-(13)]/\sqrt{3}$. The vertical separation between the two U(1) singlets is given by the commutative center element $(12)+(13)+(23)$ and these two states are horizontally separated from the SU(2) doublet by another central element $(123) + (132)$. Thus the graphs are obtained by plotting S_3 quantum states at $(x, y) = (P_{3y} + (123) + (132),\ \ P_{3z} + (12) + (13) + (23))$. Central elements commute with the whole algebra and so cannot be a part of any internal SU(j) symmetry and define the particle charges. Thus they do not distort the SU(j) j-plets and can be used to separate them in graphs like this.

a sum such as $p\rho_1 + q\rho_2$. Then the products of density matrices contribute to the result and so density matrix multiplication also makes sense physically.

This suggests we should analyze "density matrix algebras", that is, the mathematical objects that contain density matrices. We've shown that superselection sectors direct our interest to block diagonal matrices and while these are always algebras, and their particle content can be read off from the block sizes, we will be interested in algebras that have more direct physical content.

This paper considers a finite subgroup F. Finite groups can be used to create algebras called "complex finite group algebras", called "group algebra" in [18]. We will designate the complex group algebra as $\mathcal{C}(F)$. The algebra is defined by using the group elements as a basis for a vector space. To describe the structure of these algebras we will use the finite group S_3, the group of permutations of three objects to create the finite group algebra $\mathcal{C}(S_3)$.

The group S_3 has six elements $\{(\,), (123), (132), (23), (13), (12)\}$with a multiplication table:

S_3	$(\,)$	(123)	(132)	(23)	(13)	(12)
$(\,)$	$(\,)$	(123)	(132)	(23)	(13)	(12)
(123)	(123)	(132)	$(\,)$	(12)	(23)	(13)
(132)	(132)	$(\,)$	(123)	(13)	(12)	(23)
(23)	(23)	(13)	(12)	$(\,)$	(123)	(132)
(13)	(13)	(12)	(23)	(132)	$(\,)$	(123)
(12)	(12)	(23)	(13)	(123)	(132)	$(\,)$

(17)

The six group elements are a basis for the algebra. So an element of the algebra, say $a \in \mathcal{C}(S_3)$ is given by six complex numbers a_χ:

$$a = a_{(\,)}(\,) + a_{(123)}(123) + a_{(132)}(132) + a_{(23)}(23) + a_{(13)}(13) + a_{(12)}(12). \quad (18)$$

Addition is the same as in any other vector space so that, for example:

$$[8(\,) + 2(123) + 3(13)] + [5(123) - 4(12)] = 8(\,) + 7(123) + 3(13) - 4(12). \quad (19)$$

Multiplication is defined by using the group multiplication. Since $(\,)$ is the multiplicative identity and $(12)(13) = (132)$, we have

$$[2(\,) + 3(12)]\,[4(\,) + 5(13)] = 8(\,) + 10(13) + 12(12) + 15(132). \quad (20)$$

The finite group elements have inverses, for example, the inverse of (123) is (132). This falls through to the algebra so the inverse of $0.5(123)$ is $2.0(132)$.

The six basis elements of the algebra follow the multiplication of the finite group and so are a faithful representation of the finite group. And when the algebra is put into block diagonal form, each block can be taken separately as a representation of the finite group.[18] Since $(12)(13)=(132)$ while $(13)(12)=(123)$, the finite group is not commutative. Therefore at least one of the blocks has to be at least as large as 2×2. Such a block would use four degrees of freedom and there are only six degrees of freedom available so there must be three blocks with degrees of freedom given by their sizes:

$$6 = 1^2 + 1^2 + 2^2, \quad (21)$$

that is, there are three blocks, two are 1×1 blocks and one is a 2×2 block.

In the $\mathcal{C}(S_3)$ algebra there will be a projection operator for each of the three blocks. These are easy to write in block diagonal form:

$$P_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}. \tag{22}$$

The other three $\mathcal{C}(S_3)$ degrees of freedom are in the traceless part of the 2×2 block and so can be written using the Pauli spin matrices as a basis.

$$P_{3x} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}, \quad P_{3y} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \begin{matrix} 0 & -i \\ +i & 0 \end{matrix} \end{pmatrix}, \quad P_{3z} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \begin{matrix} +1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix}. \tag{23}$$

These six elements $\{P_1, P_2, P_3, P_{3x}, P_{3y}, P_{3z}\}$ provide a basis for the $\mathcal{C}(S_3)$ algebra that is convenient for block diagonal form. With this basis, an element of the algebra can be written as:

$$a = a_1 P_1 + a_2 P_2 + a_3 P_3 + a_{3x} P_{3x} + a_{3y} P_{3y} + a_{3z} P_{3z}. \tag{24}$$

These six basis elements have the following multiplication table:

S_3	P_1	P_2	P_3	P_{3x}	P_{3y}	P_{3z}
P_1	P_1	0	0	0	0	0
P_2	0	P_2	0	0	0	0
P_3	0	0	P_3	P_{3x}	P_{3y}	P_{3z}
P_{3x}	0	0	P_{3x}	P_3	$+iP_{3z}$	$-iP_{3y}$
P_{3y}	0	0	P_{3y}	$-iP_{3z}$	P_3	$+iP_{3x}$
P_{3z}	0	0	P_{3z}	$+iP_{3y}$	$-iP_{3x}$	P_3

(25)

The density matrices for the 1×1 blocks are simply P_1 and P_2. the 2×2 block has SU(2) symmetry so its density matrices are defined by a Bloch sphere. Letting (x, y, z) be a real vector with length less than or equal to 1, a density matrix is given by

$$\rho_{x,y,z} = (P_3 + xP_{3x} + yP_{3y} + zP_{3z})/2. \tag{26}$$

This gives the particle content of the block diagonal version of the $\mathcal{C}(S_3)$ algebra, at least using the block matrix algebra description of the algebra. This basis is how elementary particle physicists would see these three particles.

It remains to describe the block matrix algebra in terms of the S_3 basis. The three projection operators $\{P_1, P_2, P_3\}$ can be read off of the group's character table:[18]

S_3	( )	(abc)	(ab)
Size:	1	2	3
χ_1	1	1	1
χ_2	1	1	−1
χ_3	2	−1	0

(27)

In the above table, the top row gives the classes. For example, the "(abc)" class consists has two elements: $\{(123), (132)\}$. The number of elements in each class is given in the second row. Below that, each row corresponds to an irreducible representation of the finite group S_3. The three χ_j correspond to the three projection operators P_j. The first two χ_j correspond to 1×1 blocks and are used exactly as shown in the character table, divided by 6 which is the number of elements in S_3. The last χ_j corresponds to a 2×2 block and so is multiplied by 2:

$$\begin{array}{lllllll} P_1 = [(\,) & + & [(123)+(132)] & + & [(23)+(13)+(12)]]/6, \\ P_2 = [(\,) & + & [(123)+(132)] & - & [(23)+(13)+(12)]]/6, \\ P_3 = [4(\,) & - & 2[(123)+(132)]]/6 & & \end{array} \tag{28}$$

These projection operators sum to unity.

The remaining density matrix basis elements $\{P_{3x}, P_{3y}, P_{3z}\}$ are similar to the spin-1/2 matrices so their definition is subject to an SU(2) symmetry. So rather than define them directly, let's first choose a representation of S_3 in the block diagonal algebra. The representations for the elements in the 1×1 blocks are defined by the character table; that is, χ_1 is the trivial representation where all elements are represented by 1 and χ_2 is the alternating representation that gives -1 to the odd permutations and +1 to the even. Beyond that, we can use the diagonal part of the 2×2 block for the representations of even elements and the off diagonal parts for the odd elements. We have:

$$\begin{array}{l} (\,) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}, \quad (123) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \begin{matrix} \omega & 0 \\ 0 & \omega^* \end{matrix} \end{pmatrix}, \quad (132) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \begin{matrix} \omega^* & 0 \\ 0 & \omega \end{matrix} \end{pmatrix}, \\ (23) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \begin{matrix} 0 & \omega^* \\ \omega & 0 \end{matrix} \end{pmatrix}, \quad (13) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \begin{matrix} 0 & \omega \\ \omega^* & 0 \end{matrix} \end{pmatrix}, \quad (12) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}, \end{array} \tag{29}$$

where $\omega = \exp(2i\pi/3)$. From the above definitions of the elements of S_3 in block diagonal form, we obtain the $P_{3\chi}$:

$$\begin{array}{l} P_{3x} = [2(12) - (13) - (23)]/3, \\ P_{3y} = [(23) - (13)]/\sqrt{3}, \\ P_{3z} = -i[(123) - (132)]/\sqrt{3}. \end{array} \tag{30}$$

The reader can verify the above by noting that they each square to P_3 and have the correct commutation relations for su(2). Note that the $P_{3\chi}$ are orthogonal in the S_3 basis just as they were orthogonal in block diagonal form.

The 2×2 block gives a representation of S_3 as:

$$\begin{array}{l} (\,) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (123) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^* \end{pmatrix}, \quad (132) = \begin{pmatrix} \omega^* & 0 \\ 0 & \omega \end{pmatrix}, \\ (23) = \begin{pmatrix} 0 & \omega^* \\ \omega & 0 \end{pmatrix}, \quad (13) = \begin{pmatrix} 0 & \omega \\ \omega^* & 0 \end{pmatrix}, \quad (12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{array} \tag{31}$$

The above is a faithful representation of S_3. It suggests a process for reverse engineering the symmetrization of the Standard Model. We will attempt this in the next section; that is, we will look for F, a finite subgroup of color SU(3), that happens to define a density matrix algebra with quarks and leptons. In our search, we will know that the size of the finite group has to equal the number of degrees of freedom in the block diagonal algebra. And that number we can approximate by counting the Standard Model fermions.

Random mixed density matrices cool down to the pure particles. This can be done with computer calculations. If the simulation is plotted using two of the $P_{3\chi}$ axes one gets a Bloch sphere shell for the su(2) doublet. The 1×1 U(1) singlets show up as single dots. They would be in the center of the Bloch sphere unless we add some center charges that distinguish between the particles. See Figure 2 for the renormalization flow from high temperatures to low in $\mathcal{C}(S_3)$. The three particles found from S_3 are a doublet and two singlets. This is a pattern seen in the Standard Model: the left handed leptons form a broken SU(2) doublet while the right handed leptons are singlets.

2.6 Schwinger's Measurement Algebra and Quantum Field Theory

In 1955, Julian Schwinger "began to write an article on the Quantum Theory of Fields. The introduction contained this description of its plan. 'In part A of this article a general scheme of quantum kinematics and dynamics is developed within the nonrelativistic framework appropriate to systems with a finite number of dynamical variables.' "[37]. This was published as "The Algebra of Microscopic Measurement"[35] in 1959 and it's now known as "Schwinger's Measurement Algebra", and has been rediscovered.[1]

The objective of Schwinger's Measurement Algebra is to put the foundations of quantum mechanics in how measurements relate to one another. It begins with the algebra of Stern-Gerlach experiments. Such an experiment splits a beam of particles into two beams, one with spin-up, the other with spin-down. Then addition means the combining of two beams while multiplication means taking the output of one beam and putting it into another experiment. These ideas are used to introduce quantum mechanics in his introductory textbook, "Quantum Mechanics, Symbolism of Atomic Measurements."[38]

Our work here is an extension of Schwinger's Measurement Algebra. Our first extension is to include temperature. In terms of particle beams, this can be modeled by Stern-Gerlach experiments that are imperfect. For a measurement of spin-1/2 in the (x, y, z) direction that has an efficiency $0 < p \leq 1$, then the algebra element that represents the experiment is a sum of the p scaled pure measurement and the $1 - p$ scaled no measurement. Instead, the $1 - p$ operator splits the beam into two equal parts ignoring the incoming beam spin orientation.

$$\rho = \frac{1-p}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{p}{2}\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \tag{32}$$

This extension of the Measurement Algebra allows for the analysis of imperfect measurement apparatus.

If one wishes to produce (in the limit) a perfect spin-up beam from imperfect Stern Gerlach experiments, one takes the spin-up result of a first experiment and sends it into a second imperfect Stern-Gerlach experiment. In the algebra, this is modeled by squaring the element given in Equation 32. Of course the beam is attenuated by this process. If one is not interested in the attenuation, one can correct for it by dividing by the trace. This gives a physical meaning of the density matrix renormalization process we use in this paper; the squaring corresponds to repeating an experiment and the division by the trace corresponds to removing the attenuation.

More generally, one can take the beam from one imperfect Stern-Gerlach experiment and send it into a different, imperfect Stern-Gerlach experiment and this physical process is modeled by the matrix multiplication of two different mixed density matrices. Thus Schwinger's Measurement Algebra provides physical meaning for the multiplication and addition of mixed density matrices.

A second extension of Schwinger's Measurement Algebra is that we take notice of superselection sectors. A third modification is that we will be keeping track of Berry-Pancharatnam or quantum phases and will avoid splitting measurements into creation and annihilation operators as we discuss next.

Schwinger extended the "Measurement Algebra" to a quantum field theory in an article published in 1960.[36] The paper gives the procedure for converting between finite density matrices and quantum field theories. In the bra ket notation, one writes a density matrix as:

$$\rho_x = |x\rangle\langle x|. \tag{33}$$

Schwinger splits the bra and ket into annihilation and creation operators by introducing what he calls a "fictitious null state" or vacuum. The vacuum is also a density matrix $|0\rangle\langle 0|$ so $\langle 0|0\rangle = 1$ and this can be inserted between the ket and bra to give

$$\rho_x = |x\rangle\langle 0|0\rangle\langle x|. \tag{34}$$

These are split into creation and annihilation operators[2] as:

$$\begin{array}{c} \psi_x^\dagger = |0\rangle\langle x|, \\ \psi_x = |x\rangle\langle 0|. \end{array} \tag{35}$$

Thus the density matrix is an annihilation followed by a creation: $\rho_x = \psi_x\ \psi_x^\dagger$.

As an example of Schwinger's idea, the CKM matrix[34] gives the transition amplitudes for weak force conversion between the three $+2/3$ charged quarks $\{u, c, t\}$ and the three $-1/3$ quarks $\{d, s, b\}$:

$$V_{CKM} = \begin{pmatrix} \langle u|d\rangle & \langle u|s\rangle & \langle u|b\rangle \\ \langle c|d\rangle & \langle c|s\rangle & \langle c|b\rangle \\ \langle t|d\rangle & \langle t|s\rangle & \langle t|b\rangle \end{pmatrix} \tag{36}$$

[2] Schwinger's notation has been translated here into the usual modern form.

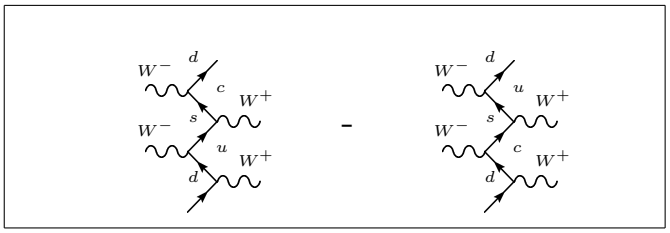

Fig. 3: Jarlskog amplitude $2\mathcal{J} = k_{duscd} - k_{dcsud} = \rho_d\rho_c\rho_s\rho_u\rho_d - \rho_d\rho_u\rho_s\rho_c\rho_d$ written as quantum field theory per Julian Schwinger's 1960 prescription.

The CKM matrix is the source of CP violation in quark weak interactions. Since CPT is conserved, CP violation implies T violation so it appears in a finite field theory as a dependence on the time ordering of a process. That is, reversing the process will convert the transition amplitude to its complex conjugate. For these to differ, we need a transition amplitude with a nonzero imaginary part.

Suppose a quark begins as a d and it is converted by W^+ absorption to a u, then W^- absorption to an s, then W^+ absorption to c and finally a W^- absorption back to d. Each of these transitions takes an amplitude from the V_{CKM} matrix or its inverse. Since V_{CKM} is unitary, the inverse transition amplitudes are the complex conjugates and the overall transition amplitude k_{duscd} for this sequence is:

$$k_{duscd} = \langle d|c\rangle\langle c|s\rangle\langle s|u\rangle\langle u|d\rangle. \tag{37}$$

The imaginary part of k_{duscd} is known as the Jarlskog [24] invariant $\mathcal{J}$; the transition amplitudes for CP violation are proportional to $\mathcal{J}$. The above is almost in density matrix form. Since it is just a complex number, take the trace and use $\langle d|d\rangle = 1$ to get

$$\begin{aligned} k_{duscd} &= \mathrm{tr}(\ |d\rangle\langle d|c\rangle\langle c|s\rangle\langle s|u\rangle\langle u|d\rangle\langle d|\ ), \\ &= \mathrm{tr}(\ \rho_d\ \rho_c\ \rho_s\ \rho_u\ \rho_d\ ). \end{aligned} \tag{38}$$

The above begins and ends with the projection operator (pure density matrix) ρ_d so it is a complex multiple of ρ_d. The density matrices can be translated into creation and annihilation operators by Schwinger's prescription, see Figure 3.

Since reversing the order takes the complex conjugate, twice the imaginary part is:

$$2\ \mathrm{Im}\ (k_{duscd}) = k_{duscd} - k_{dcsud}. \tag{39}$$

To see that the above is invariant, note that $\rho_d + \rho_s + \rho_b = 1$ so $\rho_s = 1 - \rho_d - \rho_b$ can be substituted into Equation 37 eventually giving $k_{duscd} - k_{dcsud} = k_{dcbud} - k_{dubcd}$, etc. For a state vector approach to this idea, see [15].

2.7 The Standard Model Density Matrix

The four Standard Model fermions $\{\nu, e, u, d\}$ are described as having symmetry SU(3)×SU(2)×U(1) in that this is the symmetry of their gauge boson couplings. Since the weak interactions depend on the handedness of the particle, there are eight symmetry conditions. But the left handed leptons $l_L = \{\nu_L, e_L\}$ and the left handed quarks $q_L = \{u_L, d_L\}$ are treated as (broken) SU(2) doublets:

	$SU(3)$	$SU(2)$	$U(1)$
l_L	1	2	$-1/2$
q_L	3	2	$+1/6$
ν_R	1	1	0
e_R	1	1	-1
u_R	3	1	$+2/3$
d_R	3	1	$-1/3$

(40)

The two doublets allows the 8 cases to be described with 6 representations.

The rule for cold density matrices is that a particle has to fall in a single diagonal block. The q_l does not fit in this scheme; in the above it is a triplet in SU(3) and a doublet in SU(2). However, the SU(2) doublet is a broken symmetry and thus doesn't count as a block for density matrix purposes. In addition, the above symmetries are for the right handed and left handed particles. These are limiting cases. The probability that a random hot density matrix spin-1/2 state would be pure left or right is zero so random initial conditions cannot give these states; they're a consequence of a particular choice for the external degrees of freedom.

So far we've not been using the external bispinor symmetry of the quarks and leptons so we've not been able to split the quarks and leptons into their handed parts. In a later section we will use the handedness operator $i\gamma^0\gamma^1\gamma^2\gamma^3$ to perform a basis transformation from S_4 bases to handed bases. To do that we will have to define our symmetry group in terms of gamma matrices. For now, under the rules of the previous section for density matrices, the up and down quarks require different SU(3) blocks. Similarly the electron and neutrino require separate blocks. So the density matrix particle symmetry of the quarks and leptons is SU(3)×SU(3)×U(1)×U(1):

$$\begin{pmatrix} e & & & & & & \\ & \nu & & & & & \\ & & u_{11} & u_{12} & u_{13} & & \\ & & u_{21} & u_{22} & u_{23} & & \\ & & u_{31} & u_{32} & u_{33} & & \\ & & & & & d_{11} & d_{12} & d_{13} \\ & & & & & d_{21} & d_{22} & d_{23} \\ & & & & & d_{31} & d_{32} & d_{33} \end{pmatrix} \tag{41}$$

where u_{jk} (d_{jk}) are the degrees of freedom for the up (down) quark.

The above block diagonal density matrix has a total of 20 complex degrees of freedom:

$$20 = 1^2 + 1^2 + 3^2 + 3^2. \tag{42}$$

These 20 degrees of freedom can be split into four degrees of freedom for the particle projection operators and $20-4 = 16$ degrees of freedom for the internal degrees of freedom of the two quarks. For example, the projection operator for the up quark is:

$$P_u = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 1 & 0 & 0 & & \\ & & 0 & 1 & 0 & & \\ & & 0 & 0 & 1 & & \\ & & & & & 0\,0\,0 & \\ & & & & & 0\,0\,0 & \\ & & & & & 0\,0\,0 \end{pmatrix}. \tag{43}$$

The electron and neutrino have no internal degrees of freedom while the quarks have $16/2 = 8$ each, which we will now explore.

A pure 3×3 density matrix can be derived from a state vector with three elements. Such a state vector has 3 complex degrees of freedom or 6 real degrees of freedom. Of these six, one is lost in the arbitrary complex phase and another in the normalization requirement. Therefore pure 3×3 density matrices have 4 real degrees of freedom. An Hermitian 3×3 matrix has 9 real degrees of freedom but mixed density matrices require that the trace is 1. This leaves 8 real degrees of freedom.

Since the trace of a mixed density matrix is 1 and, for 3×3 matrices, tr $(1/3) = 1$, when we subtract $1/3$ from a mixed 3×3 density matrix what is left is a traceless Hermitian matrix. Such a matrix can be defined using the Gell-Mann matrices as a basis. So we can express any mixed 3×3 density matrix ρ_a as:

$$\rho_a = 1/3 + \Sigma_{k=1}^{8}\, a_k\, \lambda_k \tag{44}$$

where a_k are eight real numbers and the λ_k are the eight Gell-Mann matrices. The Gell-Mann matrices are used to parameterize the su(3) Lie algebra, but in the above we are instead using them to parameterize mixed density matrices. These mixed density matrices transform according to the SU(3) Lie group.

These calculations generalize so we have established that when we rewrite an algebra into block diagonal form, the degrees of freedom of the algebra will appear as projection operators (one for each particle) and real multiples of the symmetry operators for the su(j) Lie algebra defining representation.

3 The Standard Model Finite Group $F = S_4$

Following the method suggested in subsection 2.5, we will now look for F by finding a finite subgroup of color SU(3). The density matrix algebra for the Standard Model given in Equation 41 has 20 degrees of freedom, however the

possible presence of dark matter gives us some latitude so we will look for a finite subgroup with 20 or more elements.

Suppose that ρ is a Standard Model fermion, that is a pure density matrix. Then ρ is an Hermitian primitive idempotent.[12] Now consider the transformations of ρ by an element g of G:

$$\rho \to g(\rho) \;=\; g^{-1}\,\rho\,g. \tag{45}$$

These transformations preserve Hermiticity, idempotency and trace so $g^{-1}\,\rho\,g$ is also a pure density matrix. Lie groups are continuous so $g(\rho)$ can be infinitesimally close to ρ and therefore the difference between ρ and $g(\rho)$ must consist of continuous internal degrees of freedom such as color. Since F is a subgroup of G these observations also apply to rotations by elements of F.

Consider the set of rotations of ρ by elements of F:

$$F(\rho) = \{f(\rho) \mid f \;\in\; F\}. \tag{46}$$

$F(\rho)$ is a finite set of particles. If we choose ρ to be a red quark u_R, then there are only three possibilities for $f(u_R)$, it must be one of $\{u_R, u_G, u_B\}$, that is, rotations by elements of F will preserve the leptons and can only change the colors of the quarks. Accordingly, we now look for elements of color SU(3) that permute color.

3.1 SU(3) Color Permutations

We will now look for a finite group F of elements of SU(3) that permute colors. We want the finite group to have 20 or somewhat more elements. Since there are only 6 elements in the permutations of three objects, we will have more than one representation of a permutation.

P. O. Ludl's PhD thesis[30] lists the finite subgroups of SU(3). There is only one that has around 20 elements and treats the colors equally, the octahedral group S_4 with 24 elements. Given one representation of S_4 we can make another by making an SU(3) transformation. But we are interested in finite groups that permute colors; this eliminates most SU(3) transformations from consideration. The result is that there are exactly three representations of S_4 that permute colors. We now derive them.

Since S_4 has 24 elements while the permutations on 3 objects has only 6, we must have that each permutation will have four representatives. They will differ by phase so we can keep track of them by using a notation of the 3 object permutation with a subscript giving the complex phases. In addition, we will distinguish the permutations of four objects (S_4) from the permutations of three objects (S_3) by putting a bar on top of the permutations on three objects. See Figure 4.

The general SU(3) matrix that swaps green and blue is

$$(\bar{2}\bar{3})_{\alpha,\beta} \;=\; \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\alpha^*\beta^* & 0 \end{pmatrix} \tag{47}$$

where α and β are two phases. For example, the action of this element on ρ_B is indeed ρ_G:

$$\begin{pmatrix} \alpha^* & 0 & 0 \\ 0 & 0 & -\alpha\beta \\ 0 & \beta^* & 0 \end{pmatrix} \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\alpha^*\beta^* & 0 \end{pmatrix} = \begin{pmatrix} 0\,0\,0 \\ 0\,1\,0 \\ 0\,0\,0 \end{pmatrix} \tag{48}$$

Squaring $(\bar{2}\bar{3})_{\alpha,\beta}$ gives an SU(3) matrix that is the identity when acting on ρ_R, ρ_G and ρ_B:

$$(\bar{2}\bar{3})^2_{\alpha,\beta} = (\bar{\ })_{\alpha,\beta} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & -\alpha^* & 0 \\ 0 & 0 & -\alpha^* \end{pmatrix} \tag{49}$$

The above is not the only representation of the identity $(\bar{\ })$. We also have the identity matrix and considering the requirement that all colors are treated equally we have two more representations of $(\bar{\ })$:

$$\begin{pmatrix} -\alpha^* & 0 & 0 \\ 0 & -\alpha^* & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \quad \begin{pmatrix} -\alpha^* & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & -\alpha^* \end{pmatrix}. \tag{50}$$

Altogether this gives four representations for $(\bar{\ })$. A multiplicity of four, applied to all six permutations gives 24 distinct matrices. This implies that F has 24 elements which is compatible with the minimum 20 we were looking for. These four representatives of $(\bar{\ })$ need to be closed under multiplication (and also distinct). This implies $\alpha = 1$ and the four representations of $(\bar{\ })$ are:

$$\begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{51}$$

Putting $\alpha = 1$, and multiplying by the four representations of the identity, the four versions of $(\bar{2}\bar{3})$ are:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta^* & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & -\beta^* & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta^* & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & \beta^* & 0 \end{pmatrix}. \tag{52}$$

Treating the colors equally gives four representations of $(\bar{1}\bar{2})$:

$$\begin{pmatrix} 0 & \beta & 0 \\ -\beta^* & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\beta & 0 \\ -\beta^* & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \beta & 0 \\ \beta^* & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\beta & 0 \\ \beta^* & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{53}$$

and four representations of $(\bar{1}\bar{3})$:

$$\begin{pmatrix} 0 & 0 & \beta^* \\ 0 & 1 & 0 \\ -\beta & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -\beta^* \\ 0 & -1 & 0 \\ -\beta & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -\beta^* \\ 0 & 1 & 0 \\ \beta & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \beta^* \\ 0 & -1 & 0 \\ \beta & 0 & 0 \end{pmatrix}. \tag{54}$$

But we can compute $(\bar{1}\bar{3})$ from $(\bar{1}\bar{2})$ and $(\bar{2}\bar{3})$. First multiply the representatives of $(\bar{1}\bar{2})$ and $(\bar{2}\bar{3})$ together to give four representatives of $(\bar{1}\bar{2})(\bar{2}\bar{3}) = (\bar{1}\bar{2}\bar{3})$:

$$\begin{pmatrix} 0 & 0 & \beta^2 \\ -\beta^* & 0 & 0 \\ 0 & -\beta^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\beta^2 \\ \beta^* & 0 & 0 \\ 0 & -\beta^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\beta^2 \\ -\beta^* & 0 & 0 \\ 0 & \beta^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \beta^2 \\ \beta^* & 0 & 0 \\ 0 & \beta^* & 0 \end{pmatrix}, \tag{55}$$

Multiplying $(\bar{1}\bar{2}\bar{3})$ by $(\bar{1}\bar{2})$ gives $(\bar{1}\bar{2}\bar{3})(\bar{1}\bar{2}) = (\bar{1}\bar{3})$. The four representatives of $(\bar{1}\bar{3})$ are:

$$\begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & 1 & 0 \\ -\beta^{2*} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\beta^2 \\ 0 & -1 & 0 \\ -\beta^{2*} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\beta^2 \\ 0 & 1 & 0 \\ \beta^{2*} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & -1 & 0 \\ \beta^{2*} & 0 & 0 \end{pmatrix}. \tag{56}$$

Comparing with Equation 54 and requiring that the colors be treated equally leads to $\beta^2 = \beta^*$ or $\beta^3 = 1$, so we have three solutions for β:

$$\beta_g = \exp(2ig\pi/3), \tag{57}$$

for $g = 1, 2, 3$. We will associate these three solutions with the three generations of elementary fermions.

Examining the above equations involving β, we see that it is a modification of 3×3 matrices according to

$$\beta\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right) \rightarrow \begin{pmatrix} a_{11} & \beta a_{12} & \beta^* a_{13} \\ \beta^* a_{21} & a_{22} & \beta a_{23} \\ \beta a_{31} & \beta^* a_{32} & a_{33} \end{pmatrix}. \tag{58}$$

This is an SU(3) transformation:

$$\beta\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right) \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta^* & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^* \end{pmatrix}, \tag{59}$$

given $\beta^2 = \beta^*$.

We've shown that the SU(3) color matrices that permute colors are three copies of S_4. Later we will associate the three possibilities for β with the three generations of elementary fermions. For now, we will restrict to the $\beta = 1$ case. For this case, the SU(3) color permuting matrices are all real and can be thought of as elements of SO(3). Choosing coordinates with $x = R$, $y = G$ and $z = B$, then the octahedral group S_4 corresponds to permutations on the paired corners of a cube while S_3 is the permutations on the paired corners of an octahedron. See Figure 4.

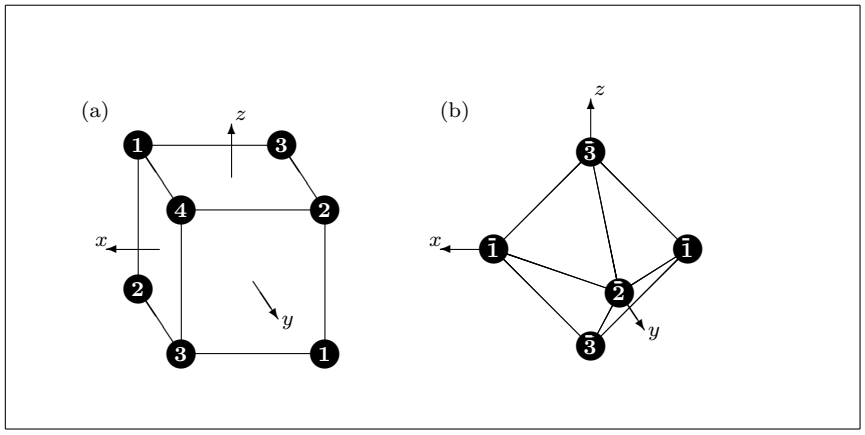

Fig. 4: Orientation of the permutation groups S_4 (a), and S_3 (b), relative to coordinate system. For example, the rotor $(1-\gamma^1\gamma^3)/\sqrt{2}$ gives a rotation by $+90°$ about the y-axis. This rotation is the permutation (1243) in S_4, and is one of the four rotations that give the $(\bar{1}\bar{3})$ swap in S_3. The other three S_4 permutations corresponding to the $(\bar{1}\bar{3})$ swap are (14), (23), and (1342).

3.2 Quarks and Leptons as $F = S_4$ Particles

We saw in Section 2.5 that the particle content of a complex finite group algebra can be read off of its character table. The character table for S_4:

O	E	C_2	C_3	C_2'	C_4	
S_4	( )	$(ab)(cd)$	(abc)	(ab)	$(abcd)$	
Size:	1	3	8	6	6	
A_1	1	+1	+1	+1	+1	leptons
A_2	1	+1	+1	−1	−1	
E	2	+2	−1	0	0	dark matter
T_1	3	−1	0	−1	+1	quarks
T_2	3	−1	0	+1	−1	

(60)

The top line of the above is the chemistry notation for the Octahedral group O. There are five classes in the group $\{E, C_2, C_3, C_2', C_4\}$. The next line gives permutation group S_4 names for these classes, $\{(\,), (ab)(cd), (abc), (ab), (abcd)\}$. The sizes of each class are given in the next line. For example, the $(ab)(cd)$ or C_2 class has three elements: $\{(12)(34), (13)(24), (14)(23)\}$. The first three of the classes, $\{(\,), (ab)(cd), (abc)\}$ are even permutations while the last two, $\{(ab), (abcd)\}$ are odd and are separated by a vertical bar from the even permutations. The relationship between these permutations and the 3 dimensional rotations of SO(3) are given in Figure 4.

The sums over each class are particularly important. The notation we will use for them is:

$$\begin{aligned}
\{(\,)\} &= (\,), \\
\{(ab)(cd)\} &= (12)(34) + (13)(24) + (14)(23), \\
\{(abc)\} &= (123) + (132) + ... + (243), \\
\{(ab)\} &= (12) + (13) + (14) + (23) + (24) + (34), \\
\{(abcd)\} &= (1234) + (1243) + ... + (1432).
\end{aligned} \tag{61}$$

These five form a basis for the center of the algebra, that is, the elements of the algebra that commute with the entire algebra. The projection operators for the five particle types use the sums over classes. These projection operators commute with the entire algebra and so can be written as sums over complex multiples of the class sums.

$$\begin{aligned}
P_{A1} &= (\ \ \{(\,)\} + \ \{(ab)(cd)\} + \ \{(abc)\} + \ \{(ab)\} + \ \{(abcd)\}\ )/24, \\
P_{A2} &= (\ \ \{(\,)\} + \ \{(ab)(cd)\} + \ \{(abc)\} - \ \{(ab)\} - \ \{(abcd)\}\ )/24, \\
P_{E} &= (\ 4\{(\,)\} + 4\{(ab)(cd)\} - 2\{(abc)\} \qquad\qquad\qquad\qquad\ )/24, \\
P_{T1} &= (\ 9\{(\,)\} - 3\{(ab)(cd)\} \qquad\qquad - 3\{(ab)\} + 3\{(abcd)\}\ )/24, \\
P_{T2} &= (\ 9\{(\,)\} - 3\{(ab)(cd)\} \qquad\qquad + 3\{(ab)\} - 3\{(abcd)\}\ )/24.
\end{aligned} \tag{62}$$

These projection operators are idempotent, sum to unity and annihilate each other, and they can be deduced from the character table. They are five of the 24 degrees of freedom of the block diagonal algebra version of S_4. The first two projection operators are primitive and correspond to the leptons. The others are an SU(2) doublet E and two SU(3) triplets T_1, T_2. In terms of defining the S_2 algebra in block diagonal form, it remains to specify the traceless parts of the 2×2 and 3×3 blocks.

The traceless parts of the 2×2 block can be defined by operators that correspond to the Pauli spin matrices:

$$\begin{aligned}
P_{Ex} &= [2(12) - (13) - (14) - (23) - (24) + 2(34) - (1234) \\
&\quad -(1243) + 2(1324) - (1342) + 2(1423) - (1432)\ ]/12, \\
P_{Ey} &= [(13) - (14) - (23) + (24) + (1234) - (1243) \\
&\quad -(1342) + (1432)\ ]/\sqrt{48}, \\
P_{Ez} &= [-(123) + (132) + (124) - (142) - (134) + (143) \\
&\quad +(234) - (243)\ ]i/\sqrt{48}.
\end{aligned} \tag{63}$$

The above square to the 2×2 block projection operator P_E and they satisfy the usual commutation relations for Pauli spin matrices. They are arbitrary in that they can be changed by an SU(2) transformation:

$$\begin{aligned}
P_{Ex} &\to \exp(-\theta\sigma_v) P_{Ex} \exp(\theta\sigma_v), \\
P_{Ey} &\to \exp(-\theta\sigma_v) P_{Ey} \exp(\theta\sigma_v), \\
P_{Ez} &\to \exp(-\theta\sigma_v) P_{Ez} \exp(\theta\sigma_v).
\end{aligned} \tag{64}$$

This rotation is about the real vector $v = (v_x, v_y, v_z)$ for an angle θ The spin matrix σ_v is for spin in the v direction so $\sigma_v = v_x P_{Ex} + v_y P_{Ey} + v_z P_{Ez}$.

Similarly, the Gell-Mann matrices can be used to define operators for traceless parts of the T_1 and T_2 matrices. The two T_χ cases differ only in the sign of the odd permutations so we can combine them into a single set of eight equations with upper signs for T_1 and lower signs for T_2:

$$\begin{aligned}
P_{\pm T1} &= [\mp 2(14) \pm 2(23) + (123) + (132) - (124) \\
&\quad -(142) - (134) - (143) + (234) + (243)]/8, \\
P_{\pm T2} &= [(123) - (132) + (124) - (142) - (134) + (143) \\
&\quad -(234) + (243) \pm 2(1243) \mp 2(1342)]i/8, \\
P_{\pm T3} &= [\pm(12) \mp (13) \mp (24) \pm (34) \pm (1234) \mp (1324) \\
&\quad \mp(1423) \pm (1432) + 2(12)(34) - 2(13)(24)]/8, \\
P_{\pm T4} &= [\pm 2(13) \mp 2(24) + (123) + (132) - (124) \\
&\quad -(142) + (134) + (143) - (234) - (243)]/8, \\
P_{\pm T5} &= [-(123) + (132) - (124) + (142) - (134) \\
&\quad +(143) - (234) + (243) \mp 2(1234) \pm 2(1432)]i/8, \\
P_{\pm T6} &= [\pm 2(12) \mp 2(34) + (123) + (132) + (124) \\
&\quad +(142) - (134) - (143) - (234) - (243)]/8, \\
P_{\pm T7} &= [(123) - (132) - (124) + (142) - (134) + (143) \\
&\quad +(234) - (243) \mp 2(1324) \pm 2(1423)]i/8, \\
P_{\pm T8} &= [\pm(12) \pm (13) \mp 2(14) \mp 2(23) \pm (24) \pm (34) \\
&\quad \mp(1234) \pm 2(1243) \mp (1324) \pm 2(1342) \mp (1423) \\
&\quad \mp(1432) + 2(12)(34) + 2(13)(24) - 4(14)(23)]/\sqrt{192}.
\end{aligned} \tag{65}$$

The above commute with the projection operators P_{T1} or P_{T2} and satisfy the structure constant equations for the Gell-Mann matrices, i.e. $[P_{\pm Tj}, P_{\pm Tk}] = 2if^{jkm}P_{\pm Tm}$. These traceless operators can be rotated by SU(3) transformations similar to the SU(2) transformation given in Equation 64.

The above 16 $P_{\pm T\chi}$ and 3 $P_{E\chi}$ matrices give all the 19 traceless degrees of freedom so together with the 5 projection operators P_χ, they are sufficient to define any of the 24 elements of the block diagonal matrix algebra. Of particular interest in the algebra are the diagonal pure density matrix states ρ_χ. For the leptons, these are P_{A1} and P_{A2}. Using $+$ $(-)$ signs for the $A1$ $(A2)$, the lepton pure states are:

$$\rho_{\pm A} = [(\,) + (ab)(cd) + (abc) \pm (ab) \pm (abcd)]/24 \tag{66}$$

For dark matter, the SU(2) up and down pure states are $(P_E \pm P_{Ez})/2$:

$$\begin{aligned}
\rho_{E\pm} &= [(\,) + (ab)(cd) \\
&\quad +\omega^{\pm 1}((132) + (124) + (143) + (234)) \\
&\quad +\omega^{\mp 1}((123) + (142) + (134) + (243))]/12,
\end{aligned} \tag{67}$$

where $\omega = \exp(2i\pi/3)$. Each of the quarks has three diagonal pure states, the red, green and blue. Like the lepton doublet, the quark doublet differs by the signs of the odd permutations. Using the upper signs for T_1, lower for T_2, the

quark pure density matrices are:

$$\begin{aligned}\rho_{T\pm R} &= [(\,) + (12)(34) - (13)(24) - (14)(23)\\ &\quad \pm(12) \pm (34) \mp (1324) \mp (1423)]/8,\\ \rho_{T\pm G} &= [(\,) - (12)(34) + (13)(24) - (14)(23)\\ &\quad \pm(13) \pm (24) \mp (1234) \mp (1432)]/8,\\ \rho_{T\pm B} &= [(\,) - (12)(34) - (13)(24) + (14)(23)\\ &\quad \pm(14) \pm (23) \mp (1243) \mp (1342)]/8.\end{aligned} \tag{68}$$

This defines the pure diagonal states for the Standard Model fermions in terms of the permutations of S_4.

Young tableaux are often used for state vector symmetries. Linear superposition allows one to describe any SU(j) j-plet state vector with just j states. For example, spin-1/2 state vectors use two states as a basis, spin-up and spin-down. Converted to density matrix language, these j states of SU(j) are all on the diagonal. Accordingly, state vector analysis of quantum symmetry tends to concentrate on just the diagonal states.

The usual Young tableaux uses different diagonal states, by an SU(3) transformation, from the ones used here, but adding all the diagonal states for a particle together, one obtains the same projection operators. For example, adding SU(2) doublet spin-up to spin-down gives the projection operator for SU(2). The projection operators are the same whether one uses Young tableaux or the states given above. The particle projection operators written in Young tableaux are:

$$\begin{aligned}P_{A1} &= \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}\,, \quad \mathrm{P_{A2}} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\,, \quad \mathrm{P_E} = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\,,\\ P_{T1} &= \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array},\ \mathrm{P_{T2}} = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}\,.\end{aligned} \tag{69}$$

The S_4 versions of the dark matter Young's tableaux diagonal pure density matrix states are:

$$\begin{aligned}\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} &= [(\,) + (12)]\,[(\,) + (34)]\,[(\,) - (13)]\,[(\,) - (24)]\,,\\ \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} &= [(\,) - (12)]\,[(\,) - (34)]\,[(\,) + (13)]\,[(\,) + (24)]\,,\end{aligned} \tag{70}$$

the S_4 versions of the T_1 Young's tableaux states are:

$$\begin{aligned}\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array} &= [(\,) + (12)]\,[(\,) + (134) + (143) - (13) - (14) - (34)],\\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array} &= [(\,) + (13)]\,[(\,) + (124) + (142) - (12) - (14) - (24)],\\ \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} &= [(\,) + (14)]\,[(\,) + (123) + (132) - (12) - (13) - (23)]\end{aligned} \tag{71}$$

and the S_4 versions of the T_2 Young's tableaux are:

$$\begin{aligned}
\begin{array}{|c|c|c|}\hline 1&3&4\\\hline\multicolumn{1}{c}{2}\\\cline{1-1}\end{array} &= [(\,) - (12)]\,[(\,) + (134) + (143) + (13) + (14) + (34)].\\
\begin{array}{|c|c|c|}\hline 1&2&4\\\hline\multicolumn{1}{c}{3}\\\cline{1-1}\end{array} &= [(\,) - (13)]\,[(\,) + (124) + (142) + (12) + (14) + (24)], \qquad (72)\\
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\multicolumn{1}{c}{4}\\\cline{1-1}\end{array} &= [(\,) - (14)]\,[(\,) + (123) + (132) + (12) + (13) + (23)],
\end{aligned}$$

The next section discusses the gauge group G.

4 The Standard Model Gauge Group $G =$ SO(3)

In this section we explore the possibilities for the gauge group G. We begin with the conclusion of the previous section: the finite subgroup of G is $F = S_4$. The two obvious candidates for G are SU(3) and SO(3). Our conclusion is that SO(3) is the gauge group. We will write S_4 using rotors made from gamma matrices (we use signature -+++), but first we will look at the possibility that $G = \mathrm{SU}(3)$.

4.1 Why SU(2) or SO(3) instead of SU(3)?

In the previous section we showed that the only candidate for the finite subgroup F is S_4. A typical member of S_4 is the matrix that swaps green and blue:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta^* & 0 \end{pmatrix} \qquad (73)$$

where $\beta^3 = 1$. The above element, and a similar matrix for one of the other two color swaps, generate F so it is sufficient for us to examine this element. For the case $\beta = 1$, we can write this as an exponential of a Lie algebra element by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \exp(\frac{i\pi}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}) \qquad (74)$$

and we see that this copy of S_4 is in triplet SO(3) or SU(2). But generalizing to cover all three generations requires SU(3). Using Equation 59, the generalization to any $\beta = (\exp(2i\pi/3))^n = \omega^n$ is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^n \\ 0 & -\omega^{-n} & 0 \end{pmatrix} = \exp(\frac{i\pi}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\omega^n \\ 0 & i\omega^{-n} & 0 \end{pmatrix}) \,. \qquad (75)$$

The above matrix is in SU(3) and not SO(3).

The gauge group G is a Lie group so before we conclude that G is SU(3) we need to look at the infinitesimal transformations. For the colored quarks, F permutes the colors, so we look for the infinitesimal color transformations. Reading off the infinitesimal color rotation for $\beta = \omega^n$, from Equation 75 gives:

$$S_{xn} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\omega^n \\ 0 & i\omega^{-n} & 0 \end{pmatrix} . \tag{76}$$

The rotation matrices for the other two color rotations are:

$$S_{yn} = \begin{pmatrix} 0 & 0 & i\omega^{-n} \\ 0 & 0 & 0 \\ -i\omega^n & 0 & 0 \end{pmatrix}, \quad S_{zn} = \begin{pmatrix} 0 & -i\omega^n & 0 \\ i\omega^{-n} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{77}$$

We've used S_χ notation as these matrices follow the rules for the su(2) spin-1 Lie algebra. For example:

$$[S_{xn}, S_{yn}] \;=\; i\omega^{3n} S_{zn} \;=\; iS_{zn}. \tag{78}$$

For any n, the three $\{S_{xn}, S_{yn}, S_{zn}\}$ are closed under commutation and form a complete basis for a Lie algebra. By inspection this is the su(2) or so(3) triplet.

So for any generation n, the appropriate gauge group for G is SU(2) or SO(3). The argument for SU(3) requires transformations between generations. But the generation transformation cannot be made infinitesimal. The red up quark and the red charm quark are in the same superselection sector so we can consider mixtures of them such as $\alpha|u_R\rangle + \beta|c_R\rangle$ for α and β complex numbers. But such mixtures are not particles (as the mass is not defined) and so cannot be used as a path of pure density matrices that connects the generations. We will pursue the assumption that G is SO(3) but we understand how another researcher may wish to explore SU(3).

The Hilbert spaces for the SU(3) and SU(2) triplets are both 3-dimensional so a complete basis for state vectors can be taken to be the same for the two. The usual choice for SU(3) are the three colors, red, green and blue. These are:

$$|R\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |G\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |B\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{79}$$

When interpreted as SU(2) triplet states, the above three states are respectively annihilated by S_{xn}, S_{yn}, and S_{zn} and so are the SU(2) triplet states for spin-0 in the x, y and z directions. Note that this is not the usual convention for a complete set of triplet states which would be eigenstates of S_z with eigenvalues $\{-1, 0, +1\}$. Instead, we are taking a triplet su(2) subalgebra of triplet su(3) that amounts to assigning:

$$S_x = \lambda_7, \quad S_y = -\lambda_5, \quad S_z = \lambda_2, \tag{80}$$

where λ_χ are the Gell-Mann matrices with the appropriate generation SU(3) transformation.

4.2 Gauge Symmetry and $G =$ SO(3)

In the Standard Model the fields transform according to their representation of the gauge symmetry SU(3)×SU(2)×U(1). If h is in this gauge symmetry a gauge field A_μ has a global transformation of $A \rightarrow A'_\mu = h\, A_\mu\, h^{-1}$. Making this global symmetry into a local symmetry requires an additional change:[8]

$$A'_\mu = h\, A_\mu\, h^{-1} - \frac{1}{g}(h\, \partial_\mu\, h^{-1}), \tag{81}$$

where g is a coupling constant. But the Dirac operator is $\gamma^\mu \partial_\mu$, so when one rotates the gamma matrices by $\gamma^\mu \rightarrow h^{-1}\, \gamma^\mu\, h$ one expects to see a modification of the above form, $h\, \partial_\mu\, h^{-1}$. This suggests that the gauge symmetry of the Standard Model arises from an SO(3) rotation of the spatial gamma matrices: $\{\gamma^1, \gamma^2, \gamma^3\}$. Such a simple gauge symmetry is not possible for the Standard Model but we saw in the previous section that it is compatible with the assumption that $G =$ SO(3).

With the previous section's conclusion that $F = S_4$, having $G =$ SO(3) becomes attractive in its simplicity. In addition, it provides a means of combining the internal degrees of freedom we've been discussing so far (particle identity and color) with the external degrees of freedom as those external degrees of freedom are defined using gamma matrices. This gives us a method of describing all the features of the elementary fermions in the language of gamma matrices.

4.3 S_4 as a Finite Subgroup of SO(3)

So far, this paper has been treating only the internal degrees of freedom of the quarks and leptons. Since the external degrees of freedom are described using gamma matrices, it is attractive to write the internal symmetries using gamma matrices. To do this one uses rotors, a convenient way of describing the Lie group of rotations with gamma matrices. To simplify calculations, begin with a real unit vector that defines the rotation axis $v = (v_x, v_y, v_z)$ and define a gamma matrix pseudovector in that direction by

$$\gamma^v = (v_x\gamma^1 + v_y\gamma^2 + v_z\gamma^3)\ \ \gamma^1\gamma^2\gamma^3. \tag{82}$$

Then γ^v squares to -1. Multiplying it by a real rotation angle θ gives a set of pseudovectors that are a Lie algebra. Exponentiating gives an element of a Lie group that is called a "rotor". These rotate gamma matrix algebra elements by 2θ around the axis defined by v.

We'll use the notation $r_v(\theta)$ to denote these rotors. The formula for one is:

$$r_v(\theta) = \exp(\theta\gamma^v) = \cos(\theta) + \sin(\theta)\gamma^v. \tag{83}$$

Rotors for a given v are a 1-parameter Lie group so $r_v(\theta)\ r_v(\phi) = r_v(\theta + \phi)$. The inverse rotor is obtained by negating the angle: $r_v^{-1}(\theta) = r_v(-\theta)$.

As a rotation example, let $v = z = (0, 0, 1)$ so $\gamma^z = \gamma^3\ \gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2$. Then

$$r_z(\theta) = \cos(\theta) + \sin(\theta)\gamma^1\gamma^2. \tag{84}$$

For calculations, it is useful to know that γ^0 and γ^3 commute with $r_z(\theta)$. Also, γ^1 and γ^2 when commuted around r_z have the effect of negating the angle: $r_z(-\theta)\ \gamma^1 = \gamma^1\ r_z(\theta)$. The action of $r_z(\theta)$ on the four gamma matrices is:

$$\begin{array}{ll} r_z(\theta)[\gamma^0] = r_z^{-1}(\theta)\ \gamma^0\ r_z(\theta) = \gamma^0\ r_z^{-1}(\theta) r_z(\theta) = & \gamma^0, \\ r_z(\theta)[\gamma^1] = r_z^{-1}(\theta)\ \gamma^1\ r_z(\theta) = \gamma^1\ r_z(\theta)\ r_z(\theta) & = \cos(2\theta)\gamma^1 + \sin(2\theta)\gamma^2, \\ r_z(\theta)[\gamma^2] = r_z^{-1}(\theta)\ \gamma^2\ r_z(\theta) = \gamma^2\ r_z(\theta)\ r_z(\theta) & = \cos(2\theta)\gamma^2 - \sin(2\theta)\gamma^1, \\ r_z(\theta)[\gamma^3] = r_z^{-1}(\theta)\ \gamma^3\ r_z(\theta) = \gamma^3\ r_z^{-1}(\theta) r_z(\theta) = & \gamma^3. \end{array} \tag{85}$$

so $r_z(\theta)$ rotates the γ^1 and γ^2 matrix axes by 2θ. The action of $r_v(\theta)$ is identical to that of $-r_v(\theta)$ so the rotors are an SU(2) triplet double cover of triplet SO(3).

Rotors give us another way of describing the 24 elements of S_4. Each of these rotors performs a distinct rotation to the three axes $\{+x, +y, +z\}$, sending them to $\{\pm x, \pm y, \pm z\}$. We could label the rotations according to this action. We can also consider the rotors as permuting the three objects $\{\pm x, \pm y, \pm z\}$ and this gives an S_3 label to the rotations that is identical to the permutations on the quark colors. The relationship between these descriptions is given by the drawing of the axes in Figure 4. We list the 24 elements of S_4 by these four notations in Figure 5.

4.4 S_4 as a Crystal Point Symmetry

In 1826, Moritz Ludwig Frankenheim[2] showed that crystals could appear in only 32 "crystal classes". Equipment limitations of the day prevented the detection of the microscopic internal details of crystals so his calculations were motivated by observations of the external morphologies of crystals. The crystal classes are also known as "point symmetries" or "point groups".

In an analogy to the days of the early 19th century, present day equipment limitations prevent us from observing the internal details of the Standard Model fermions. And after 200 years we again attempt to describe what is observed at long distances by extrapolating from a model of what we guess happens at short distances. Today we call this concept "renormalization". This paper is concerned with the renormalization of particle identity so it's not too surprising that it reuses Frankenheim's results.

The 32 crystal classes fall into 6 or 7 crystal systems depending on the shape of their unit cell. In space-time we observe no preferred direction so the only system possible is the Isometric or Cubic. Of the five isometric crystal classes, only one is compatible with the observation that elementary fermions are handed. It is the Gyroidal class, international notation 432, the symmetry that Frankenheim called *abwechselnde Hälfte*[3] and that we call S_4.

[3] As of 1977, the existence of gyroidal symmetry minerals was denied in mineralogy textbook used by the present author,[11] but a few rare examples are mentioned in the literature.

$$
\begin{array}{lll}
(\,) & (\bar{\ }) & +x+y+z \quad (1) \\
(12) & (1\bar{2}) & -y-x-z \quad (\gamma^1\gamma^3+\gamma^2\gamma^3)/\sqrt{2} \\
(13) & (2\bar{3}) & -x+z+y \quad (\gamma^1\gamma^2-\gamma^1\gamma^3)/\sqrt{2} \\
(14) & (1\bar{3}) & -z-y-x \quad (\gamma^1\gamma^2-\gamma^2\gamma^3)/\sqrt{2} \\
(23) & (1\bar{3}) & +z-y+x \quad (\gamma^1\gamma^2+\gamma^2\gamma^3)/\sqrt{2} \\
(24) & (2\bar{3}) & -x-z-y \quad (\gamma^1\gamma^2+\gamma^1\gamma^3)/\sqrt{2} \\
(34) & (1\bar{2}) & +y+x-z \quad (\gamma^1\gamma^3-\gamma^2\gamma^3)/\sqrt{2} \\
(123) & (1\bar{2}\bar{3}) & -z+x-y \quad (1+\gamma^1\gamma^2+\gamma^1\gamma^3-\gamma^2\gamma^3)/2 \\
(132) & (1\bar{3}\bar{2}) & +y-z-x \quad (1-\gamma^1\gamma^2-\gamma^1\gamma^3+\gamma^2\gamma^3)/2 \\
(124) & (1\bar{3}\bar{2}) & +y+z+x \quad (1-\gamma^1\gamma^2+\gamma^1\gamma^3-\gamma^2\gamma^3)/2 \\
(142) & (1\bar{2}\bar{3}) & +z+x+y \quad (1+\gamma^1\gamma^2-\gamma^1\gamma^3+\gamma^2\gamma^3)/2 \\
(134) & (1\bar{2}\bar{3}) & +z-x-y \quad (1-\gamma^1\gamma^2-\gamma^1\gamma^3-\gamma^2\gamma^3)/2 \\
(143) & (1\bar{3}\bar{2}) & -y-z+x \quad (1+\gamma^1\gamma^2+\gamma^1\gamma^3+\gamma^2\gamma^3)/2 \\
(234) & (1\bar{3}\bar{2}) & -y+z-x \quad (1+\gamma^1\gamma^2-\gamma^1\gamma^3-\gamma^2\gamma^3)/2 \\
(243) & (1\bar{2}\bar{3}) & -z-x+y \quad (1-\gamma^1\gamma^2+\gamma^1\gamma^3+\gamma^2\gamma^3)/2 \\
(1234) & (2\bar{3}) & +x-z+y \quad (1+\gamma^2\gamma^3)/\sqrt{2} \\
(1243) & (1\bar{3}) & +z+y-x \quad (1-\gamma^1\gamma^3)/\sqrt{2} \\
(1324) & (1\bar{2}) & -y+x+z \quad (1+\gamma^1\gamma^2)/\sqrt{2} \\
(1342) & (1\bar{3}) & -z+y+x \quad (1+\gamma^1\gamma^3)/\sqrt{2} \\
(1423) & (1\bar{2}) & +y-x+z \quad (1-\gamma^1\gamma^2)/\sqrt{2} \\
(1432) & (2\bar{3}) & +x+z-y \quad (1-\gamma^2\gamma^3)/\sqrt{2} \\
(12)(34) & (\bar{\ }) & -x-y+z \quad (\gamma^1\gamma^2) \\
(13)(24) & (\bar{\ }) & +x-y-z \quad (\gamma^2\gamma^3) \\
(14)(23) & (\bar{\ }) & -x+y-z \quad (\gamma^1\gamma^3)
\end{array}
\tag{86}
$$

Fig. 5: The first column lists the 24 permutations of the $F = S_4$ group. The second column gives the corresponding S_3 permutation. The third column describes how the rotation acts on the coordinate axes. The fourth column gives the rotor with a sign convention that the leading term is taken positive. A sign convention is needed as the rotors form a double cover of S_4, for example, the rotor for (12)(34) squares to -1. The swaps (ab) are 180 degree rotations about axes with Miller indices like 110. The (abc) are 120 degree rotations about 111 axes, the $(abcd)$ are 90 degree rotations about 100 axes, and the $(ab)(cd)$ rotations are 180 degrees about 100 axes.

Crystals typically grow at different rates in different directions. A direction where a crystal grows particularly slowly forms a crystal face. These directions are lattice planes for the crystal and are described by "Miller Indices". Miller indices are written as three small integers. Negative integers are written under a bar. For example, $12\bar{3}$ corresponds to the vector $(1, 2, -3)$. These vectors are rotated by the 24 elements of S_4 to give equivalent crystal faces. For the 24 faces to be distinct we must have that the three integers are nonzero with three different magnitudes, for example, $1\bar{5}2$.

Given a Miller index with no zeros or duplicate magnitudes, we can rewrite it as a gamma matrix vector using γ^1, γ^2 and γ^3. for example, $34\bar{5}$ becomes $3\gamma^1 + 4\gamma^2 - 5\gamma^3$. And then we can use the 24 S_4 rotors given in Figure 5 to rotate that gamma matrix vector to get 24 gamma matrix vectors with S_4

Perhaps the nearest locality to the author is the De Lamar mine, Owyhee county, Idaho, USA where a gold silver selenite, Fischesserite Ag_3AuSe_2 appears as a small pink crystal.[4]

symmetry. If we use these vectors to define the perpendicular vectors from the origin to the 24 faces of a crystal we can draw a crystal with gyroidal symmetry as this author did to draw the crystal illustrations in Figure 6.

If we begin with a vector that has two components the same magnitude or zero elements, we will find that more than one rotation will rotate the vector to the same new vector. The resulting crystal will have fewer faces and the gyroidal symmetry will no longer be apparent. For example, beginning with 100 there will only be six different results corresponding to the six faces of a cube.

4.5 Visualizing S_4 States

As was noted above, the S_4 rotors have arbitrary signs so we expect them to be more similar to the SU(2) triplet double cover than SO(3). Pursuing this hand waving argument, we can group the S_4 pure density matrix states into five SU(2) weak isospin doublets. The dark matter doublet has no odd S_4 permutations and is unbroken while the four visible matter doublets are broken by odd S_4 permutations. Then we can associate dark matter with the lowest spherical harmonic Y_0^0 and visible matter with another copy of Y_0^0 for the leptons and the next three real spherical harmonics, $\{p_x, p_y, p_z\}$ for the three colors of quarks:

SU(2):	good	broken			
up:	$\chi^{\uparrow}$	ν	u_R	u_G	u_B
down:	$\chi^{\downarrow}$	e	d_R	d_G	d_B
	$Y_{00} = s$	$Y_{00} = s$	$Y_{1,1} = p_x$	$Y_{1,-1} = p_y$	$Y_{1,0} = p_z$

(87)

Next we illustrate the S_4 density matrix states by crystal drawings. This shows the relation to the spherical harmonics.

In the previous section we associated the 24 Miller indices for a gyroidal crystal with the 24 S_4 rotations. We designate an arbitrary face of the crystal as ( ), and then apply rotations to the crystal to obtain the designations for the other faces. For example, from Equation 68, taking the upper sign to be the up quark, the red up quark is the pure density matrix:

$$\rho_{uR} = [(\,) + (12)(34) - (13)(24) - (14)(23) + (12) + (34) - (1324) - (1423)]/8. \tag{88}$$

We therefore create a gyroidal crystal and put a + on each of the ( ), (12)(34), (12) and (34) faces, and a − on the (13)(24), (14)(23), (1324) and (1423) faces. The remaining 16 faces receive a 0. Then we have the computer write out LaTeX for an illustration. See Figure 6.

5 Fermion States and External Symmetry

The previous sections have been about the symmetry of the citronen. We found that its symmetry is $G = \mathrm{SO}(3)$ with a finite subgroup S_4. In this section we move on to a model of the citronen itself.

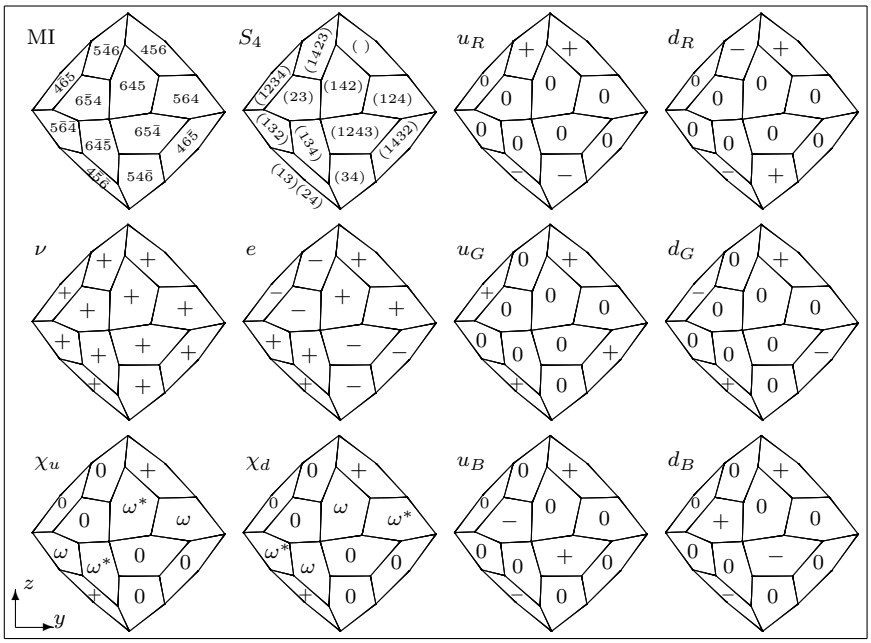

Fig. 6: Visualization of the S_4 quantum states by gyroidal crystal drawings. MI: Miller indices for faces of gyroidal crystal. S_4: Faces labeled with permutations from S_4 symmetry group. ν, e: Neutrino and electron coefficients, $\pm 1/24$. χ_u, χ_d: Dark matter SU(2) up and down coefficients, $\pm 1/12$ or $\omega/12$ where $\omega = \exp(2i\pi/3)$. $u_R, u_G, u_B, d_R, d_G, d_B$: Up and down quarks, red, green and blue coefficients, $\pm 1/8$. Coefficients are from Equations 66 through 68.

Quantum Field Theory (QFT) is required in problems where particles are created or destroyed. This paper has used density matrices (quantum mechanics) as the calculations are easier but this limits the subject to a situation where the particle number does not change: propagators. The simplest generalization we can make of this situation is to consider the mass interaction which relates the left and right handed parts of a single particle. In this situation the external symmetry of the particle changes but the particle number (and identity) does not change. We will attack that problem in this section.

The idea of this paper has been to propose that the quarks and leptons are composed of density matrices made from a citronen, an underlying quantum state. Our first task in this section will be to propose a gamma matrix algebra quantum state to model the citronen. Such a state has to support the S_4 symmetry calculations and will be the subject of our first subsection. In later subsections we will consider how citronens can interact to produce mass. We will be making a complicated density matrix out of simpler density matrices with the interactions defined by the transition amplitudes between the 24 S_4 rotations that define a basis set. The result will be an S_3 model of mass which is compatible with the Koide quark and lepton mass equations.

5.1 The Underlying G Gauge Quantum States

We've written $F = S_4$ in terms of a finite set of gamma matrix rotors in Figure 5. The complete Lie group of rotors includes boosts as well as rotations. These act on the gamma matrices so it's natural to seek the basic quantum states among the gamma matrix algebra. Since the rotors act on the gamma matrix algebra elements on both sides, the quantum states we seek are in density matrix form. So we will look for a quantum state $\rho_{(\,)}$ among the gamma matrix algebra. The symmetry S_4 when applied to that state will give 24 different states. And those 24 states will be the basis for a 24-dimensional Schwinger Measurement Algebra.

Any pure density matrix in a 4×4 matrix can be transformed by an SU(4) transformation to a state $\rho_{(\,)}$ that is diagonal with $(1, 0, 0, 0)$, on the diagonal. Such a transformation defines four diagonal states. We can use sums and differences of these four states to get a set of three commuting matrices:

$$A = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad C = A\,B. \tag{89}$$

The above matrices each square to unity, are different, and any two multiply to give the third. Any two of them form a Complete Set of Commuting Observables (CSCO). The $\rho_{(\,)}$ state can be written as

$$\rho_{(\,)} = (1 + A + B + C)/4 = (1 + A)/2\ (1 + B)/2. \tag{90}$$

The other three diagonal states are obtained by independently changing the signs in $(1 \pm A)/2\ (1 \pm B)/2$. If we are to use $\rho_{(\,)}$ as the original pure density matrix we will be rotating by S_4, we see that it has three parts that can have direction, A, B and C.

To get a faithful algebraic representation of S_4, we can't have $\{A, B, C\}$ all oriented in the same direction. Such an arrangement would not give 24 different bases but instead would have the equivalent of 100 Miller indices and there would be only 6 different elements in the basis. The smaller basis would not give a faithful representation of S_4 and so the character table would not include the quarks.

If we begin with a primitive projection operator among the gamma matrices, it seems plausible that we can use SO(3) to rotate away all the partial angles and so be able to choose A, B and $C = AB$ from the sixteen products of the four gamma matrices $\{\gamma^1, \gamma^2, \gamma^3, \gamma^4\}$. These sixteen products of gamma matrices either have no orientation or are oriented in the x, y, or z directions as follows:

1	x	y	z
1	γ^1	γ^2	γ^3
γ^0	$\gamma^1\gamma^0$	$\gamma^2\gamma^0$	$\gamma^3\gamma^0$
$\gamma^1\gamma^2\gamma^3$	$\gamma^2\gamma^3$	$\gamma^1\gamma^3$	$\gamma^1\gamma^2$
$\gamma^1\gamma^2\gamma^3\gamma^0$	$\gamma^2\gamma^3\gamma^0$	$\gamma^1\gamma^3\gamma^0$	$\gamma^1\gamma^2\gamma^0$

(91)

We can't use 1 which leaves 15 choices for A, B and C. Any of those 15 will commute with eight of the sixteen but those eight include 1 and itself. Therefore there are 15 ways of choosing A and then 6 ways of choosing B. This gives 90 possibilities but order of choosing doesn't matter so there are $90/3! = 15$ ways of picking a CSCO from the products of gamma matrices. For each CSCO, there are four sign choices so there are a total of 60 different pure density matrices derived from CSCOs.

Since 60 is not a multiple of 24, some of these pure density matrices have less than the symmetry needed to be an element of a faithful basis for S_4. In fact, only one set has the right symmetry. The other $60 - 24 = 36$ pure density matrices are oriented in a single direction and so S_4 acting on them puts them into bases of size 6. The seven cases are as follows:

A	B	C	MI	#CSCO	#States
γ^1	$\gamma^2\gamma^0$	$\gamma^1\gamma^2\gamma^0$	321	6	24
γ^1	$+i\gamma^2\gamma^3$	$+i\gamma^1\gamma^2\gamma^3$	100	3	6
γ^1	$-i\gamma^2\gamma^3$	$-i\gamma^1\gamma^2\gamma^3$	100	3	6
$\gamma^2\gamma^3\gamma^0$	$+i\gamma^2\gamma^3$	$-i\gamma^0$	100	3	6
$\gamma^2\gamma^3\gamma^0$	$-i\gamma^2\gamma^3$	$+i\gamma^0$	100	3	6
$\gamma^1\gamma^0$	$+i\gamma^2\gamma^3$	$+i\gamma^1\gamma^2\gamma^3\gamma^0$	100	3	6
$\gamma^1\gamma^0$	$-i\gamma^2\gamma^3$	$-i\gamma^1\gamma^2\gamma^3\gamma^0$	100	3	6

(92)

The first line is the only faithful representative of S_4 symmetry and it is the one that we will use for the citronen. This chosen $\rho_{(\ )}$ has A, B and C oriented in the x, y and z directions, respectively. The remaining six cases have A and B oriented in the x direction and C has no orientation. Since C has no orientation, it cannot be rotated by S_4 to reverse its sign and consequently these three pairs are split into six cases. The final pair are the right $+i\gamma^1\gamma^2\gamma^3\gamma^0$ and left $-i\gamma^1\gamma^2\gamma^3\gamma^0$ handed states. The #CSCO gives the number of complete sets of commuting observables. For example, the line γ^1, $+i\gamma^2\gamma^3$, $+i\gamma^1\gamma^2\gamma^3$ has three CSCOs, they are $\{\gamma^1,\ i\gamma^2\gamma^3,\ i\gamma^1\gamma^2\gamma^3\}$, $\{\gamma^2,\ -i\gamma^1\gamma^3,\ i\gamma^1\gamma^2\gamma^3\}$, and $\{\gamma^3,\ i\gamma^1\gamma^2,\ i\gamma^1\gamma^2\gamma^3\}$. These three sets of CSCOs define three orthogonal bases so the states are equivalent to a complete set of mutually unbiased bases for the Pauli algebra.

The bottom six cases in Equation 92 have their A and B parts oriented in the same direction so rotors act on them as a 3-vector. Thus they have two real degrees of freedom (Bloch sphere) and have the symmetry of SU(2) spin-1/2 doublets. In contrast, the first case has three perpendicular vectors and the rotors will rotate these with three real degrees of freedom. The final pair of states are the left and right handed spin-1/2 states.

5.2 Geometric Phase and Mutually Unbiased Bases

The first several sections of this paper have been concerned only with the symmetries of the propagators. We had stable particles that did not interact so we could analyze them using density matrices rather than quantum field

theory. Now that the previous subsection gives a proposal for the underlying quantum states we can look at how those states interact.

The common method of giving interactions to elementary particles is by creating a Lagrangian that is carefully crafted to have symmetries that are compatible with the observations of experiments. Since we have a theory that purports to explain the origin of the quarks and leptons we hope that the interactions can be derived from the states.

In the usual quantum mechanics, one uses a CSCO to describe a set of basis states. These basis states annihilate each other. For example, with spin-1/2 the usual basis states are spin-up and spin-down. In density matrix form these are:

$$|+z\rangle\langle+z| = \rho_{+z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |-z\rangle\langle-z| = \rho_{-z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{93}$$

The above states have zero/one transition amplitudes. That is, $\langle+z|\,|-z\rangle = 0$. Therefore transition amplitudes cannot provide interesting particle interactions in the usual quantum mechanics; hence the need for Lagrangians to describe particle interactions.

The subgroup S_4 is created by considering 90 degree rotations on SO(3). If we instead applied 90 degree rotations to spin-1/2 SU(2) we would have only six basis states corresponding to spin in the $\{+x, -x, +y, -y, +z, -z\}$ directions. From the point of view of state vector quantum mechanics, this is an "over complete" basis. That is, the basis for this Schwinger Measurement Algebra includes six states:

$$\rho_{\pm x} = (1 \pm \sigma_x)/2, \;\; \rho_{\pm y} = (1 \pm \sigma_y)/2, \;\; \rho_{\pm z} = (1 \pm \sigma_z)/2, \tag{94}$$

instead of the usual two. The six states come in three traditional bases, one each oriented in the x, y and z directions. The transition amplitudes for states from bases with the same orientation are the usual zero or one. What's new are the transitions between states with different orientation. These transitions tend to have the same magnitude so the information content is in their geometric phases which we will now discuss.

For a sequence χ of pure density matrix states that begins and ends with the same state, the transition amplitude α_χ is defined as the ratio of the product state to the state at the beginning and end. For example, we define $\alpha_{+x+y-z-y+x}$ by:

$$\alpha_{+x+y-z-y+x} \;\; \rho_{+x} = \rho_{+x}\, \rho_{+y}\, \rho_{-z}\, \rho_{-y}\, \rho_{+x}. \tag{95}$$

The magnitude can be computed by noting that each 90 degree transition contributes $\sqrt{1/2}$. For the above there are four transitions so the magnitude will be $(\sqrt{1/2})^4 = 1/4$. If the sequence defines a path on the Bloch sphere that carves out an area, the transition amplitude will pick up a complex phase proportional to the area. The above path cuts out two octants, $11\bar{1}$ and $1\bar{1}\bar{1}$, but in the negative direction so the area is $-2/8$ of the sphere. The sphere has a total "geometric phase" area of 2π so that a spin-1/2 particle rotated

through 360 degrees around a vector perpendicular to its initial spin direction will trace a path that splits the Block sphere in two, and so picks up the familiar fermion rotation factor of $\exp(2i\pi\ (1/2)) = -1$. Since the above $+x+y-z-y+x$ path cuts out -2 octants (or $-2/8$), the geometric phase will be $\exp(2i\pi\ (-2/8)) = -i$ and the amplitude will be $(-i)(1/4) = -i/4$ as can be verified by multiplying out the states. Computing the amplitude using state vectors we have

$$-i/4 = \langle +x| + y\rangle\langle +y| - z\rangle\langle -z| - y\rangle\langle -y| + x\rangle. \tag{96}$$

Note that the above product is an observable; it includes a ket for every bra so there is no arbitrary complex phase. This is why the density matrix amplitude (trace) requires a path that begins and ends with the same state. A state vector product such as $\langle +x| + y\rangle$ depends on arbitrary complex phases and so only its magnitude is an observable. The state vector version amounts to dividing both sides of Equation 95 by $\rho_{+x} = |+x\rangle \quad \langle +x|$.

The transition amplitudes between states of different orientation in Equation 94 are all the same $\sqrt{1/2}$ so the transition probabilities are the square, $1/2$. In quantum information theory, these three bases are "mutually unbiased bases" or MUBs. Thus, in a certain sense, the meaning of $F = S_4$ is that, for the underlying quantum states, "unbiased" implies something like "different degrees of freedom". For finite dimensions, mutually unbiased is equivalent to being orthogonal in a metric induced by an inner product in a real Hilbert space.[41,3]

When one makes calculations by products of states chosen from mutually unbiased bases, the only information content is in the phases. George Svetlichny noticed that this is also a feature of the Feynman path integral and wrote an arXiv paper proposing that there is a relationship.[40] Jiri Tolar and Goce Chadzitaskos confirmed the relationship by analyzing the short-time propagator on a finite dimensional analogue of a free quantum particle.[43] The central connection lies in the fact that two complementary observables are mutually unbiased. If one has a state with a precisely known momentum then one can know nothing of its position and vice versa.

The standard quantum mechanical example of complementary observables are position and momentum. The relationship between their bases is the Fourier transform. The Fourier transform works on situations with translational symmetry. The discrete Fourier transform has a similar form and applies to situations where there are a finite number of positions with discrete translational symmetry. The corresponding relationship for rotational symmetry are the spherical harmonics with the related finite symmetries being the point groups such as the S_4 used here.

5.3 Left and Right Handed States

The Standard Model fermions are split into left and right handed parts by the handedness operator $i\gamma^1\gamma^2\gamma^3\gamma^0$. This operator commutes with rotors, both

rotations and boost. What we will be doing is defining an alternative basis for the S_4 algebra. There will still be 24 elements but they will be left or right handed, and we will lose the explicit S_4 symmetry.

The left and right handed projection operators are:

$$\begin{aligned} R &= (1 + i\gamma^1\gamma^2\gamma^3\gamma^0)/2, \\ L &= (1 - i\gamma^1\gamma^2\gamma^3\gamma^0)/2. \end{aligned} \tag{97}$$

These satisfy the usual equations for projection operators:

$$\begin{aligned} RR &= R, \\ LL &= L, \\ RL = LR &= 0. \end{aligned} \tag{98}$$

We can rewrite any pure density matrix (or any gamma matrix algebra element) four different ways with these handed projection operators. Define them as follows:

$$\begin{aligned} \rho_{LL} &= L\,\rho\,L, \\ \rho_{LR} &= L\,\rho\,R, \\ \rho_{RL} &= R\,\rho\,L, \\ \rho_{RR} &= R\,\rho\,R. \end{aligned} \tag{99}$$

Since $L + R = 1$, the four $\rho_{\chi\chi}$ add up to give ρ:

$$\rho = \rho_{LL} + \rho_{LR} + \rho_{RL} + \rho_{RR}. \tag{100}$$

Applying the R and L operators to the 24 S_4 basis states gives 96 states but these can be recombined.

We begin with the primary state:

$$\rho_{(\,)} = (1 + \gamma^1)(1 + \gamma^2\gamma^0)/4. \tag{101}$$

Since $\gamma^2\gamma^0$ commutes with $i\gamma^1\gamma^2\gamma^3\gamma^0$, it also commutes with L and R. On the other hand, γ^1 anticommutes with $i\gamma^1\gamma^2\gamma^3\gamma^0$ so $\gamma^1 L = R\gamma^1$ and $\gamma^1 R = L\gamma^1$. Using the above rules we can compute the four handedness portions of $\rho_{(\,)}$. Since the Standard model particles are described in terms of handedness and spin, we rearrange terms to get:

$$\begin{aligned} \rho_{(\,)RR} &= (1 + i\gamma^1\gamma^3)(1 + i\gamma^1\gamma^2\gamma^3\gamma^0)/4, \\ \rho_{(\,)LL} &= (1 - i\gamma^1\gamma^3)(1 - i\gamma^1\gamma^2\gamma^3\gamma^0)/4, \\ \rho_{(\,)RL} &= (\gamma^1 + i\gamma^3)(1 - i\gamma^1\gamma^2\gamma^3\gamma^0)/4, \\ \rho_{(\,)LR} &= (\gamma^1 - i\gamma^3)(1 + i\gamma^1\gamma^2\gamma^3\gamma^0)/4. \end{aligned} \tag{102}$$

The above equations could be interpreted as a way of rearranging the S_4 basis. For example, the first line ρ_{1RR} is oriented in the y direction so when S_4 is applied to it the 24 results will duplicate to leave only six different cases which can be written as:

$$S_4(\rho_{(\,)RR}) = \{1 \pm i\gamma^k\gamma^1\gamma^2\gamma^3)(1 + i\gamma^1\gamma^2\gamma^3\gamma^0)/4\}, \tag{103}$$

where $k = 1, 2, 3$. Similarly for $\rho_{(\ )LL}$. The $\rho_{(\ )LR}$ and $\rho_{(\ )RL}$ would appear to give 12 cases each but they can be combined in pairs by factoring a $\pm i$ out. For example, $(\gamma^3 - i\gamma^1)$ is one of the transformations of $(\gamma^1 + i\gamma^3)$ but $(\gamma^3 - i\gamma^1) = -i(\gamma^1 + i\gamma^3)$. Thus Equation 102 defines a basis change from S_4 to a basis that is natural for computing handed interactions.

5.4 Weak Couplings and S_3

In the previous subsection we split $\rho_{(\ )}$ into its left and right handed portions and found that they split the 24 basis elements of the S_4 states into four groups $\{RR, LL, LR, RL\}$ each with six bases. The $\rho_{(\ )RR}$ and $\rho_{(\ )LL}$ parts are each closed under multiplication and so form groups. As a 6 dimensional subalgebra of the S_4 complex finite group algebra we expect them to be S_3 complex finite group algebras.

Recall that in Subsection 3.1 we found that the color permuting SU(3) subgroup consisted of S_3 permutations. We used bar notations for these element so that $(\bar{2}\bar{3})$ stands for the swap of the 2nd and 3rd colors. When we converted the S_4 elements into rotors, we kept a column in Figure 5 giving the S_3 permutation corresponding to each rotation. Note that the even (odd) permutations of S_4 correspond to even (odd) S_3 permutations. The particle definitions given in Equations 66 through 68 have the weak isospin doublet elements distinguished by the signs of the odd permutations. This suggests that in looking for a way of describing the weak quantum numbers with S_3 we should use the three odd S_3 permutations to describe weak isospin.

With the standard definitions of weak hypercharge t_0 and weak isospin t_3, electric charge is given by $Q = t_0/2 + t_3$. To make this a bit simpler, we will deal with $t_0/2$ in our equations. With this change, the weak hypercharge and weak isospin quantum numbers for the quarks and leptons particles are:

	$t_0/2$	t_3	$Q = t_0/2 + t_3$
ν_R	0	0	0
d_L	$+1/6$	$-1/2$	$-1/3$
u_L	$+1/6$	$+1/2$	$+2/3$
d_R	$-1/3$	0	$-1/3$
ν_L	$-1/2$	$+1/2$	0
e_L	$-1/2$	$-1/2$	-1
u_R	$+2/3$	0	$+2/3$
e_R	-1	0	-1

(104)

Antiparticles negate the charges and reverse R with L. Accordingly, we've listed the above in increasing $|t_0/2|$.

The natural S_3 element to associate with $t_0/2$ is the identity $(\bar{\ })$. So far in this paper we've required our particles to have trace 1 but the Schwinger Measurement Algebra allows any trace. Since the trace corresponds to the probability of finding a particle, it's necessary that a description of the propagators restrict to trace 1. But here we are looking for an equation for coupling

constants and this restriction disappears. The natural generalization is to lift the requirement that the basis element have trace zero but retain the requirement that it be idempotent.

With these notes, we can associate two of the S_3 central charges with weak hypercharge and weak isospin:

$$\begin{aligned} t_0/2 &= \{(\bar{\ })\} = (\bar{\ }), \\ t_3 &= \{(\bar{1}\bar{2})\} = (\bar{2}\bar{3}) + (\bar{1}\bar{3}) + (\bar{1}\bar{2}). \end{aligned} \tag{105}$$

Recall that the central charges are the basis for the algebra elements that commute with all the algebra and so correspond to conserved quantities. Our assignment uses only two of these central charges; the third appears related to the baryon number.

The idempotency equation for S_3 is obtained by defining an arbitrary element of the complex group algebra of S_3:

$$\alpha = \alpha_{(\bar{\ })}(\bar{\ }) + \alpha_{(\bar{1}\bar{2})}(\bar{1}\bar{2}) + \alpha_{(\bar{1}\bar{3})}(\bar{1}\bar{3}) + \alpha_{(\bar{2}\bar{3})}(\bar{2}\bar{3}) + \alpha_{(\bar{1}\bar{2}\bar{3})}(\bar{1}\bar{2}\bar{3}) + \alpha_{(\bar{1}\bar{3}\bar{2})}(\bar{1}\bar{3}\bar{2}), \tag{106}$$

then simplifying $\alpha = \alpha^2$ by using the S_3 group multiplication rule, then breaking the single equation into six according to the S_3 basis. This gives a set of six coupled quadratic equations in the six unknowns. The α_χ are simply complex numbers and so one can collect terms in the six equations to obtain:

$$\begin{aligned} \alpha_{(\bar{\ })} &= \alpha^2_{(\bar{\ })} + 2\alpha_{(\bar{1}\bar{2}\bar{3})}\alpha_{(\bar{1}\bar{3}\bar{2})} + \alpha^2_{(\bar{1}\bar{2})} + \alpha^2_{(\bar{1}\bar{3})} + \alpha^2_{(\bar{2}\bar{3})}, \\ \alpha_{(\bar{1}\bar{2}\bar{3})} &= 2\alpha_{(\bar{\ })}\alpha_{(\bar{1}\bar{2}\bar{3})} + \alpha^2_{(\bar{1}\bar{3}\bar{2})} + \alpha_{(\bar{1}\bar{2})}\alpha_{(\bar{2}\bar{3})} + \alpha_{(\bar{2}\bar{3})}\alpha_{(\bar{1}\bar{3})} + \alpha_{(\bar{1}\bar{3})}\alpha_{(\bar{1}\bar{2})}, \\ \alpha_{(\bar{1}\bar{3}\bar{2})} &= 2\alpha_{(\bar{\ })}\alpha_{(\bar{1}\bar{3}\bar{2})} + \alpha^2_{(\bar{1}\bar{2}\bar{3})} + \alpha_{(\bar{1}\bar{2})}\alpha_{(\bar{2}\bar{3})} + \alpha_{(\bar{2}\bar{3})}\alpha_{(\bar{1}\bar{3})} + \alpha_{(\bar{1}\bar{3})}\alpha_{(\bar{1}\bar{2})}, \\ \alpha_{(\bar{1}\bar{2})} &= 2\alpha_{(\bar{1}\bar{2})} + \alpha_{(\bar{1}\bar{2}\bar{3})}\alpha_{(\bar{1}\bar{3})} + \alpha_{(\bar{1}\bar{2}\bar{3})}\alpha_{(\bar{2}\bar{3})} + \alpha_{(\bar{1}\bar{3}\bar{2})}\alpha_{(\bar{1}\bar{3})} + \alpha_{(\bar{1}\bar{3}\bar{2})}\alpha_{(\bar{2}\bar{3})}, \\ \alpha_{(\bar{1}\bar{3})} &= 2\alpha_{(\bar{1}\bar{3})} + \alpha_{(\bar{1}\bar{2}\bar{3})}\alpha_{(\bar{1}\bar{2})} + \alpha_{(\bar{1}\bar{2}\bar{3})}\alpha_{(\bar{2}\bar{3})} + \alpha_{(\bar{1}\bar{3}\bar{2})}\alpha_{(\bar{1}\bar{2})} + \alpha_{(\bar{1}\bar{3}\bar{2})}\alpha_{(\bar{2}\bar{3})}, \\ \alpha_{(\bar{2}\bar{3})} &= 2\alpha_{(\bar{2}\bar{3})} + \alpha_{(\bar{1}\bar{2}\bar{3})}\alpha_{(\bar{1}\bar{3})} + \alpha_{(\bar{1}\bar{2}\bar{3})}\alpha_{(\bar{1}\bar{2})} + \alpha_{(\bar{1}\bar{3}\bar{2})}\alpha_{(\bar{1}\bar{3})} + \alpha_{(\bar{1}\bar{3}\bar{2})}\alpha_{(\bar{1}\bar{2})}. \end{aligned} \tag{107}$$

The above six equations are fairly difficult to solve; the solutions include eight 2-manifolds. Fortunately we can simplify by putting $\alpha_{(\bar{1}\bar{2})} = \alpha_{(\bar{2}\bar{3})} = \alpha_{(\bar{1}\bar{3})} = t_3/3$. This reduces to four equation:

$$\begin{aligned} \alpha_{(\bar{\ })} &= \alpha^2_{(\bar{\ })} + 2\alpha_{(\bar{1}\bar{2}\bar{3})}\alpha_{(\bar{1}\bar{3}\bar{2})} + 3t_3^2, \\ \alpha_{(\bar{1}\bar{2}\bar{3})} &= 2\alpha_{(\bar{\ })}\alpha_{(\bar{1}\bar{2}\bar{3})} + \alpha^2_{(\bar{1}\bar{3}\bar{2})} + 3t_3^2, \\ \alpha_{(\bar{1}\bar{3}\bar{2})} &= 2\alpha_{(\bar{\ })}\alpha_{(\bar{1}\bar{3}\bar{2})} + \alpha^2_{(\bar{1}\bar{2}\bar{3})} + 3t_3^2, \\ t_3 &= 2\alpha_{(\bar{\ })}t_3 + 2(\alpha_{(\bar{1}\bar{2}\bar{3})} + \alpha_{(\bar{1}\bar{3}\bar{2})})t_3. \end{aligned} \tag{108}$$

Writing $t_0/2$ for $\alpha_{(\bar{\ })}$, we get solutions for the left and right handed fermions along with a few extra solutions. Six of the extra solutions are different choices for $\alpha_{(\bar{1}\bar{2}\bar{3})}$ and $\alpha_{(\bar{1}\bar{3}\bar{2})}$ which we will ignore. The other two extra solutions are

sums of pairs of other solutions:

	$t_0/2$	t_3	$\alpha_{(\bar{1}\bar{2}\bar{3})}$	$\alpha_{(\bar{1}\bar{3}\bar{2})}$
$\bar{\nu}_L$	0	0	0	0
d_L	1/6	$-1/2$	1/6	1/6
u_L	1/6	$+1/2$	1/6	1/6
$\bar{d}_L$	1/3	0	1/3	1/3
$\bar{\nu}_R$	1/2	$-1/2$	$-i/\sqrt{12}$	$+i/\sqrt{12}$
$\bar{e}_R$	1/2	$+1/2$	$-i/\sqrt{12}$	$+i/\sqrt{12}$
u_R	2/3	0	$-1/3$	$-1/3$
$\bar{e}_L$	1	0	0	0
$u_r + d_L$	5/6	$-1/2$	$-1/6$	$-1/6$
$u_r + u_L$	5/6	$+1/2$	$-1/6$	$-1/6$

(109)

Thus the weak quantum numbers are idempotents of the S_3 symmetry of the handed S_4 states. This provides hope that the weak force could be put into density matrix form where the coupling constants are coefficients for an S_3 symmetry.

5.5 Quark and Lepton Masses

Penrose's Zig-Zag model of the electron has a mass interaction that is particularly simple.[33] The left and right handed portions of the same particle interact with a coupling constant m. The interaction converts a right handed propagator to a left handed one and back: $\cdots \xrightarrow[m]{e_L} \xrightarrow[m]{e_R} \xrightarrow[m]{e_L} \xrightarrow{e_R} \cdots$. In this chain, the left and right projection operators would annihilate each other but the Dirac wave function assigns $\bar{\psi} = \psi^\dagger \gamma^0$ so we get a factor of γ^0 between the left and right handed projection operators. Writing ρ_L and ρ_R for the left and right handed propagators the mass interaction becomes:

$$\cdots \rho_L \; (m\gamma^0) \; \rho_R \; (m\gamma^0) \; \rho_L \; (m\gamma^0) \; \rho_R \; \cdots \tag{110}$$

Since $L \; 1 \; R = R \; 1 \; L = 0$, we can replace $(m\gamma^0)$ in the above with $(2m(1 + \gamma^0)/2)$. This has the advantage that $(1 + \gamma^0)/2$ is a projection operator which we will write P_t:

$$P_t = (1 + \gamma^0)/2. \tag{111}$$

And since $P_t = P_t \; P_t$ we get:

$$(2m(1 + \gamma^0)/2) \; = \; (\sqrt{2m} P_t) \; (\sqrt{2m} P_t). \tag{112}$$

Replacing $m\gamma^0$ with this lets us rewrite the mass chain as:

$$\begin{array}{l} \cdots \; (\sqrt{2m} P_t \rho_L \sqrt{2m} P_t) \; (\sqrt{2m} P_t \rho_R \sqrt{2m} P_t) \; \cdots \quad \text{or} \\ \cdots \; (2m \; P_t \; \rho_L \; P_t) \; (2m \; P_t \; \rho_R \; P_t) \; \cdots \end{array} \tag{113}$$

So each propagator has associated with it a coupling factor of $2m$.

Since γ^0 is an eigenvector of P_t with eigenvalue 1, the operator P_t will take every γ^0 in ρ_L or ρ_R and turn it into unity. So we need consider only the spatial gamma matrices when solving for ρ_L and ρ_R. And each is composed of a projection operator $(1 \pm i\gamma^1\gamma^2\gamma^3\gamma^0)/2$ so they correspond to the 6th or 7th cases listed in Equation 92. The remaining commuting observables are $\pm\gamma^1\gamma^0$ or $\pm i\gamma^2\gamma^3$ and their S_4 rotations. We will use the $\pm i\gamma^2\gamma^3$ as they are a copy of the Pauli matrices according to:

$$\sigma_x \equiv +i\gamma^2\gamma^3, \;\; \sigma_y \equiv -i\gamma^1\gamma^3, \;\; \sigma_z \equiv +i\gamma^1\gamma^2. \tag{114}$$

So the over-complete set of states are:

$$(1 \pm i\gamma^2\gamma^3)/2, \; (1 \pm i\gamma^1\gamma^3)/2, \; (1 \pm i\gamma^1\gamma^2)/2. \tag{115}$$

Since these use only the spatial gamma matrices, we can label them according to Miller indices:

$$\begin{array}{l} \rho_{100} = (1 + i\gamma^2\gamma^3)/2, \;\; \rho_{010} = (1 - i\gamma^1\gamma^3)/2, \;\; \rho_{001} = (1 + i\gamma^1\gamma^2)/2, \\ \rho_{\bar{1}00} = (1 - i\gamma^2\gamma^3)/2, \;\; \rho_{0\bar{1}0} = (1 + i\gamma^1\gamma^3)/2, \;\; \rho_{00\bar{1}} = (1 - i\gamma^1\gamma^2)/2. \end{array} \tag{116}$$

Next we use these six projection operators to create a density matrix model of the electron mass.

The six projection operators of Equation 116 will be the diagonal states of our density matrix. The off diagonal elements of a density matrix act as raising and lowering operators. For example:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{117}$$

With the six states on the diagonal, there will be 30 off diagonal degrees of freedom corresponding to the products of different projection operators. Since we have an over-complete basis, most of our off diagonal matrix elements can be defined by products of the six diagonal states. The six that cannot are annihilating products like $\rho_{100} \; \rho_{\bar{1}00} = 0$. So our matrix will be missing 6 complex degrees of freedom which leaves $36 - 6 = 30$.

The off diagonal matrix positions are products such as $\rho_{100}\rho_{00\bar{1}}$. To save space we will abbreviate these with Miller indices. The left ρ will take a 1 while the right will take a 2 so that $\rho_{100}\rho_{00\bar{1}} = \rho_{10\bar{2}}$. Each of these will be a basis state in a basis for a Schwinger measurement algebra. For an algebra we need a rule for multiplication. What we are doing here is quantum field theory in density matrix form; that means our multiplication is actually concatenation of Feynman diagrams. The diagonal elements are propagators while the off diagonal elements are gauge boson interactions that annihilate a propagator and create a replacement for it. Thus we only consider products where the propagators match. For example, $\rho_{012} \; \rho_{201}$ can be multiplied because the second projection of ρ_{012} and the first projection of ρ_{201} are both $(1+\sigma_z)/2$. This rule turns our algebra into a sort of matrix algebra. The difference is that if two matrix elements are both off diagonal, their product will be multiplied

by a complex coefficient consisting of a transition magnitude and a geometric phase.

The diagonal matrix elements are Schwinger's "elementary selective measurements" that he denotes by $M(a')$, while the off diagonal matrix elements are related to the "most general selective measurement(s)" he denotes by $M(a', b')$. Here a' and b' are possible measurements of the physical quantities A and B respectively.[35] In our case, A and B for example, are σ_z and σ_x. Schwinger avoids dealing with geometric phase by simplified multiplication relations for his $M(a', b')$. His measurement symbols satisfy:

$$M(a'b')\ M(b', a') = M(a', a') = M(a'). \tag{118}$$

In our case we keep the phase and amplitude information in the off diagonal elements so that, with $M(a') \sim \rho_{001}$ and $M(b') \sim \rho_{100}$ we have

$$(\rho_{001}\rho_{100})\ (\rho_{100}\rho_{001})\ =\ (\sqrt{1/2})^2(\rho_{001}) \tag{119}$$

as the product contains two ninety degree transitions or as can be verified by using the Pauli spin matrices. The difference here is the result of our assuming that it is the finite subgroup F that defines a complete set of observables. To get an algebra that fits into Schwinger's assumptions, we would need F to have size 2 and arrange for spin-up to be rotated only to spin-down. Since those states annihilate each other, our method of defining the off diagonal basis elements would fail and we would have to use Schwinger's method.

Products of the ρ_{jkl} can be computed by using the geometric phase method given in Section 5.2. For example:

$$\begin{array}{l}(1+\sigma_y)/2\ (1+\sigma_z)/2\ (1+\sigma_x) = (1+i)/2\ (1+\sigma_y)/2\ (1+\sigma_x)/2, \\ (1+\sigma_y)/2\ (1-\sigma_z)/2\ (1+\sigma_x) = (1-i)/2\ (1+\sigma_y)/2\ (1+\sigma_x)/2.\end{array} \tag{120}$$

Alternatively, one could replace the central σ_z with $-i\sigma_x\sigma_y$ and use $\sigma_x(1+\sigma_x)/2 = (1+\sigma_x)/2$ and similarly for σ_y. This gives the product rules:

$$\begin{array}{l}\rho_{012}\rho_{201} = (1-i)/2\ \rho_{210}, \\ \rho_{01\bar{2}}\rho_{20\bar{1}} = (1+i)/2\ \rho_{210}.\end{array} \tag{121}$$

The calculation can also be done with bras and kets as $\langle +y| + z\rangle\langle +z| + x\rangle = (1-i)/2\ \langle +y| + x\rangle$. Off diagonal products that give a diagonal element will pick up a factor of $1/2$:

$$\rho_{012}\ \rho_{021}\ =\ 1/2\ \rho_{010}. \tag{122}$$

With bras and kets, the $1/2$ factor is given by $\langle +y| + z\rangle\langle +z| + y\rangle = 1/2$.

We can write the collection of $\alpha_{jkl}\rho_{jkl}$ as a matrix:

$$\rho_\alpha = \begin{pmatrix} \alpha_{100}\rho_{100} & 0 & \alpha_{120}\rho_{120} & \alpha_{1\bar{2}0}\rho_{1\bar{2}0} & \alpha_{102}\rho_{102} & \alpha_{10\bar{2}}\rho_{10\bar{2}} \\ 0 & \alpha_{\bar{1}00}\rho_{\bar{1}00} & \alpha_{\bar{1}20}\rho_{\bar{1}20} & \alpha_{\bar{1}\bar{2}0}\rho_{\bar{1}\bar{2}0} & \alpha_{\bar{1}02}\rho_{\bar{1}02} & \alpha_{\bar{1}0\bar{2}}\rho_{\bar{1}0\bar{2}} \\ \alpha_{210}\rho_{210} & \alpha_{\bar{2}10}\rho_{\bar{2}10} & \alpha_{010}\rho_{010} & 0 & \alpha_{012}\rho_{012} & \alpha_{01\bar{2}}\rho_{01\bar{2}} \\ \alpha_{2\bar{1}0}\rho_{2\bar{1}0} & \alpha_{\bar{2}\bar{1}0}\rho_{\bar{2}\bar{1}0} & 0 & \alpha_{0\bar{1}0}\rho_{0\bar{1}0} & \alpha_{0\bar{1}2}\rho_{0\bar{1}2} & \alpha_{0\bar{1}\bar{2}}\rho_{0\bar{1}\bar{2}} \\ \alpha_{201}\rho_{201} & \alpha_{\bar{2}01}\rho_{\bar{2}01} & \alpha_{021}\rho_{021} & \alpha_{0\bar{2}1}\rho_{0\bar{2}1} & \alpha_{001}\rho_{001} & 0 \\ \alpha_{20\bar{1}}\rho_{20\bar{1}} & \alpha_{\bar{2}0\bar{1}}\rho_{\bar{2}0\bar{1}} & \alpha_{02\bar{1}}\rho_{02\bar{1}} & \alpha_{0\bar{2}\bar{1}}\rho_{0\bar{2}\bar{1}} & 0 & \alpha_{00\bar{1}}\rho_{00\bar{1}} \end{pmatrix} \tag{123}$$

and the equation we will be solving is

$$\rho_\alpha \ \rho_\alpha = \rho_\alpha, \tag{124}$$

with the off diagonal products of ρ_{jkl} reducing to complex coefficients. This is 30 quadratic equations, each with 4 or 5 non-zero complex coefficients, written in 30 unknown complex numbers. Symmetry considerations simplify it considerably.

As a start, we write out the quadratic equation for α_{210}. We begin by taking the ρ_{210} part of the equation $\rho_\alpha = \rho_\alpha \ \rho_\alpha$ and obtain:

$$\begin{aligned} \alpha_{210}\rho_{210} &= \alpha_{210}\rho_{210}\alpha_{100}\rho_{100} + \alpha_{010}\rho_{010}\alpha_{210}\rho_{210} \\ &\quad \alpha_{012}\rho_{012}\alpha_{201}\rho_{201} + \alpha_{01\bar{2}}\rho_{01\bar{2}}\alpha_{20\bar{1}}\rho_{20\bar{1}}, \\ &= (\alpha_{210}\alpha_{100})(\rho_{210}\rho_{100}) + (\alpha_{010}\alpha_{210})(\rho_{010}\rho_{210}) \\ &\quad (\alpha_{012}\alpha_{201})(\rho_{012}\rho_{201}) + (\alpha_{01\bar{2}}\alpha_{20\bar{1}})(\rho_{01\bar{2}}\rho_{20\bar{1}}). \end{aligned} \tag{125}$$

Referring to Equation 116 and Equation 121 we reduce the ρ_{jkl} products as:

$$\begin{aligned} \rho_{210}\rho_{100} &= (\rho_{010}\rho_{100})\ (\rho_{100}), & & \\ &= \rho_{010}\rho_{100} & = & \quad \rho_{210}, \\ \rho_{010}\rho_{210} &= (\rho_{010})\ (\rho_{010}\rho_{100}), & & \\ &= \rho_{010}\rho_{100} & = & \quad \rho_{210}, \\ \rho_{012}\rho_{201} &= (\rho_{010}\rho_{001})(\rho_{001}\rho_{100}) & = & e^{-2i\pi/8}\sqrt{1/2}\ \rho_{210}, \\ \rho_{01\bar{2}}\rho_{20\bar{1}} & & = & e^{+2i\pi/8}\sqrt{1/2}\ \rho_{210}. \end{aligned} \tag{126}$$

All the terms in the above are complex multiples of ρ_{210} so we factor them out to obtain the quadratic equation for α_{210}. With the geometric phases $\exp(\pm 2i\pi/8)$ and transition magnitudes $\sqrt{1/2}$, we have:

$$\begin{aligned} \alpha_{210} &= \alpha_{210}\alpha_{100} + \alpha_{210}\alpha_{010} \\ &\quad + e^{-2i\pi/8}\sqrt{1/2}\ \alpha_{012}\alpha_{201} + e^{+2i\pi/8}\sqrt{1/2}\ \alpha_{01\bar{2}}\alpha_{20\bar{1}}. \end{aligned} \tag{127}$$

Repeating the process for ρ_{100} we get the quadratic equation for α_{100} as:

$$\begin{aligned} \alpha_{100} &= (\alpha_{100})^2 + (\sqrt{1/2})^2\alpha_{210}\alpha_{120} + (\sqrt{1/2})^2\alpha_{2\bar{1}0}\alpha_{1\bar{2}0} \\ &\quad + (\sqrt{1/2})^2\alpha_{201}\alpha_{102} + (\sqrt{1/2})^2\alpha_{20\bar{1}}\alpha_{10\bar{2}}. \end{aligned} \tag{128}$$

The off diagonal terms in the above equation are associated with factors of $\sqrt{1/2}$ and if we multiply Equation 127 by $\sqrt{1/2}$, all its off diagonal terms will also have a factor of $\sqrt{1/2}$ and so the $\sqrt{1/2}$s can be removed. Removing the geometric phases $\exp(\pm 2i\pi/8)$ is our next task.

Our objective is to define matrix coefficients β_{jkl} that differ from the α_{jkl} by complex ratios that take the ρ_{jkl} products into account. Then the β_{jkl} can be assembled into matrices that use regular matrix multiplication (except for the entries that have to be zero). The factors of $\sqrt{1/2}$ can be removed by writing $\beta_{jkl} = \sqrt{1/2}\ \alpha_{jkl}$ for the off diagonal elements, but we also want to get rid of the geometric phases so our transformation will have a phase ϕ_{jkl}:

$$\beta_{jkl} = \sqrt{1/2}\exp(i\phi_{jkl})\ \alpha_{jkl} \quad \text{(off diagonal entries)} \tag{129}$$

If we were doing the calculation using bras and kets our problem would be solved by choosing an arbitrary phase for each state. With density matrices, we can choose all the phases in a consistent manner by selecting a "fictitious null state" or vacuum state. We will use:

$$\rho_{111} = (1 + (\sigma_x + \sigma_y + \sigma_z)/\sqrt{3})/2. \tag{130}$$

Now the geometric phase contributed by a segment such as $\rho_{210} = \rho_{100}\ \rho_{010}$ will be the geometric phase of a path that begins at 111 goes to 010 then 100 and finally back to 111. Reading the path right to left, the geometric phase of ρ_{210} can be calculated using bras and kets by:

$$\text{Phase}(\rho_{210}) = \text{Phase}(\langle 111| + y\rangle\langle +y| + x\rangle\langle +x|111\rangle). \tag{131}$$

Each bra appears with a ket so that the arbitrary complex phases cancel.

To compute ϕ_{210} using geometric phases, note that the 111 octant is surrounded by a closed path consisting of three segments: ϕ_{210}, ϕ_{102}, ϕ_{021}. The path goes in the negative direction around the octant so the geometric phase for the path is $\exp(-2i\pi/8)$. By symmetry, each segment contributes equally so they each represent a phase $\exp(-2i\pi/24)$ and we have:

$$\beta_{210} = \sqrt{1/2}\ \exp(-2i\pi/24)\ \alpha_{210} \quad \text{and cyclic permutations.} \tag{132}$$

Similarly, the three segments on the opposite side of the cube, $\phi_{\bar{2}\bar{1}0}$,$\phi_{\bar{1}0\bar{2}}$,$\phi_{0\bar{2}\bar{1}}$, make a closed path, but it is in the positive direction and it encloses 7 octants instead of one so their transformation is:

$$\beta_{\bar{2}\bar{1}0} = \sqrt{1/2}\ \exp(+2i\pi\ 7/24)\ \alpha_{\bar{2}\bar{1}0} \quad \text{and cyclic permutations.} \tag{133}$$

The remaining six segments pass between the 111 and $\bar{1}\bar{1}\bar{1}$ octants. They form a closed path that encloses four octants. In the positive direction, the path segments are $\rho_{10\bar{2}}$,$\rho_{02\bar{1}}$,$\rho_{\bar{2}10}$,$\rho_{\bar{1}02}$, $\rho_{0\bar{2}1}$,$\rho_{2\bar{1}0}$. So the transformation is:

$$\begin{aligned} \beta_{10\bar{2}} &= \sqrt{1/2}\ \exp(+2i\pi\ 4/24)\ \alpha_{10\bar{2}}, \\ \beta_{02\bar{1}} &= \sqrt{1/2}\ \exp(+2i\pi\ 4/24)\ \alpha_{02\bar{1}}, \quad \text{and cyclic permutations,} \end{aligned} \tag{134}$$

This defines the conversions for half of the 24 off diagonal β_{jkl}. The remaining 12 are these reversed so that 1 is swapped with 2. For example $\rho_{10\bar{2}}$ and $\rho_{20\bar{1}}$ are the same segment in reverse. The reversed paths have their geometric phases negated. And the diagonal β are the same as the diagonal α.

With these definitions, the quadratic equations for β_{jkl} correspond to the algebra defined by complex matrix multiplication for a 6×6 matrix with 30 non-zero entries:

$$\rho_\beta = \begin{pmatrix} \beta_{100} & & \beta_{120} & \beta_{1\bar{2}0} & \beta_{102} & \beta_{10\bar{2}} \\ & \beta_{\bar{1}00} & \beta_{\bar{1}20} & \beta_{\bar{1}\bar{2}0} & \beta_{\bar{1}02} & \beta_{\bar{1}0\bar{2}} \\ \beta_{210} & \beta_{\bar{2}10} & \beta_{010} & & \beta_{012} & \beta_{01\bar{2}} \\ \beta_{2\bar{1}0} & \beta_{\bar{2}\bar{1}0} & & \beta_{0\bar{1}0} & \beta_{0\bar{1}2} & \beta_{0\bar{1}\bar{2}} \\ \beta_{201} & \beta_{\bar{2}01} & \beta_{021} & \beta_{0\bar{2}1} & \beta_{001} & \\ \beta_{20\bar{1}} & \beta_{\bar{2}0\bar{1}} & \beta_{02\bar{1}} & \beta_{0\bar{2}\bar{1}} & & \beta_{00\bar{1}} \end{pmatrix}. \tag{135}$$

We're interested in the pure density matrices supported by the algebra defined by β_{jkl} multiplication. This is not an algebra over a finite group so we can't read off the particle content by looking at a character table. Instead, we can use the method described in Subsection 2.4, that is, begin with random mixed density matrices near the high temperature limit (the unit matrix) and repeatedly squaring them and renormalizing them to keep the trace 1 until they are approximately a pure density matrix. This procedure is straightforward, but for a lot of algebras, on completing it one frequently ends up with a dog's dinner that is difficult to organize. Fortunately, in the case of the above algebra, a random matrix quickly converges to a result that has obvious SU(3) symmetry. For example, cooling this high temperature matrix:

$$\rho_T = 1 + \tfrac{1}{100}\begin{pmatrix} .472 & & .644+.460i & -.170+.312i & -.453+.832i & -.147+.462i \\ & -.193 & -.436-.523i & -.758+.059i & -.676+.120i & .769+.070i \\ .644-.460i & -.436+.523i & .926 & & .238+.530i & .690-.599i \\ -.170-.312i & -.758-.059i & & -.878 & .142-.028i & .272+.221i \\ -.453-.832i & -.676-.120i & .238-.530i & .142+.028i & -.494 & \\ -.147-.462i & .769-.070i & .690+.599i & .272-.221i & & .619 \end{pmatrix}, \quad (136)$$

one obtains

$$\rho_{T\approx 0} = \begin{pmatrix} .334 & & .256+.373i & 0 & 0 & .102+.083i \\ & 0 & 0 & 0 & 0 & 0 \\ .256-.373i & 0 & .615 & & 0 & .171-.050i \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \\ .102-.083i & 0 & .171+.050i & 0 & & .052 \end{pmatrix}. \quad (137)$$

The above matrix is nonzero on three rows and columns, those corresponding to $100 = +x$, $010 = +y$ and $00\bar{1} = -z$ so the pure density matrix is restricted to the $11\bar{1}$ octant. The result is generic in that all the results one obtains this way are nonzero on only one of the eight octants. By restricting itself to just three rows and columns, the result is able to avoid any of the six missing matrix spots. This is the largest $n \times n$ matrix one can fit into the 6×6, while avoiding one of the blank entries.

Since the pure density matrices of β_{jkl} all live on a single octant, we will specialize to the 111 octant. The symmetry of the electron and neutrino given in Figure 6 in Subsection 4.5 indicates that we should have all the off diagonal entries identical (subject to hermiticity) so a general solution for the electron or neutrino needs to be an idempotent unit trace 3×3 matrix of form:

$$\rho_a = \begin{pmatrix} \beta_{100} & \beta_{120} & \beta_{102} \\ \beta_{210} & \beta_{010} & \beta_{012} \\ \beta_{201} & \beta_{021} & \beta_{001} \end{pmatrix} = \begin{pmatrix} 1/3 & a & a^* \\ a^* & 1/3 & a \\ a & a^* & 1/3 \end{pmatrix} \quad (138)$$

where a is a complex constant. Solving for $\rho_a^2 = \rho_a$ we have three possibilities for a: $\{1/3,\ \exp(+2i\pi/3)/3,\ \exp(-2i\pi/3)/3\}$. The three cases can be combined as powers of $\exp(2i\pi/3)$ as:

$$\rho_g = \frac{1}{3}\begin{pmatrix} 1 & \exp(+2ig\pi/3) & \exp(-2ig\pi/3) \\ \exp(-2ig\pi/3) & 1 & \exp(+2ig\pi/3) \\ \exp(+2ig\pi/3) & \exp(-2ig\pi/3) & 1 \end{pmatrix} \quad (139)$$

where $g = 1, 2, 3$. The above three solutions are the same three cases for S_4 that we associated with the three generations in Equation 57. Recall that when considering any single generation, F was a finite subgroup of $G = \mathrm{SO}(3)$ but requiring all three generations to appear in the same G meant that we had to have $G = \mathrm{SU}(3)$. The above makes it clear that the generation structure is the same for all three generations except the different generations allow a relative phase between the $+x$, $+y$ and $+z$ axes.

There is another way of obtaining Equation 139. One can begin with the three pure states ρ_{100}, ρ_{010} and ρ_{001}, along with their six products, and compute all possible path integrals over them, looking for what is stable in the long term. This is somewhat more involved than the method shown here but the result is the same and in 2010, the author published the result in this journal.[7] The present paper is about the Standard Model fermions losing the stability of their identity at high temperature, the 2010 paper was about fermions losing their spin-1/2 stability at high temperature. The paper resulted in new equations relating lepton masses. Since then, the subject has been extended by other authors as we describe in the next subsection.

5.6 Koide Mass Equations

In 1982 Yoshio Koide published papers[25,26] with a model of the charged leptons masses that gave the formula:

$$m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2. \tag{140}$$

The charged lepton masses have three real degrees of freedom. The above formula removes one of them. At the time the prediction was off by two standard deviations but later corrections to the τ mass brought the error to near zero. At this time, the Particle Data Group[32] gives the electron, muon and tau masses (MeV) as:

$$\begin{aligned} m_e &= 0.5109989461(31), \\ m_\mu &= 105.6583745(24), \\ m_\tau &= 1776.86(12) \end{aligned} \tag{141}$$

So the latest experimental value for the 2/3 fraction is 0.6666605(70). As far as mass formulas go, the accuracy has attracted some attention. We will stick to papers that are connected to this narrative; the literature is far richer than what we cite.

Koide wrote a 2005 conference proceeding [27] which describes three mysteries of the formula: (a) It's about squared root masses. (b) It's invariant under exchange of masses. (c) It's satisfied at low energy scales but not at high, due to renormalization group equation effects. The usual assumption is that masses come from Yukawa couplings which wouldn't naturally give a square root mass formula. But energy is proportional to the square of momentum and so a formula that gives the square root of mass is natural when that mass comes about from relativity. Koide notes that the invariance under

exchange of masses suggests an S_3 symmetry and this is compatible with our model. Finally, this paper's approach to unification at high temperature is to use the mixed density matrix model. In this model the particles are most distinct in the low temperature limit so this is the only temperature where a mass formula can be exact.

The matrix solution for ρ_g given in Equation 139 is circulant. That is, the top row of the matrix appears two more times in the next two rows, but shifted by 1 and 2 places to the right. So the top row contains all the information. Leaving off the overall factor of 1/3 that row is:

$$(\beta_{100}, \beta_{120}, \beta_{102}) \;=\; (\quad 1 \quad , \exp(+2ig\pi/3) \;, \exp(-2ig\pi/3) \;). \tag{142}$$

These β were the result of eliminating the geometric phases and transition probability information present in the α coefficients. Reversing Equation 132, the values for the α are:

$$\begin{array}{l}(\alpha_{100}, \alpha_{120}, \alpha_{102}) \\ \quad = \; (\; 1 \;,\; \sqrt{2}\exp(+2i\pi(4g-1)/12) \;,\; \sqrt{2}\exp(-2i\pi(4g-1)/12) \;).\end{array} \tag{143}$$

The three α_{jkl} components above are coefficients on the three ρ_{jkl}:

$$(\rho_{100}, \rho_{120}, \rho_{102}) = ((1+\sigma_x)/2, (1+\sigma_x)(1+\sigma_y)/4, (1+\sigma_x)(1+\sigma_z)/4). \tag{144}$$

Mass has no orientation so we're interested in the scalar part s_{jkl} of this vector:

$$(s_{100}, s_{120}, s_{102}) = (1/2, 1/4, 1/4). \tag{145}$$

Combining the above with Equation 143 and summing over the terms we presumably have something that is proportional to the square root of the mass:

$$\begin{array}{rl}\sqrt{m_g} = & (1/2)(1) + (1/4)(\sqrt{2}\exp(+2i\pi(4g-1)/12)) \\ & + \quad (1/4)(\sqrt{2}\exp(-2i\pi(4g-1)/12)), \\ = & \; 1/2 + \sqrt{1/2}\cos(2\pi(4g-1)/12).\end{array} \tag{146}$$

In computing sums of $\sqrt{m_g}$ and m_g, two trigonometry identities are useful:

$$\begin{array}{l}\Sigma_{g=1}^{3} \; \cos(2\pi g/3 + \delta) \;\; = 0, \\ \Sigma_{g=1}^{3} \; \cos^2(2\pi g/3 + \delta) = 3/2,\end{array} \tag{147}$$

where δ is any real number. Summing up the $\sqrt{m_g}$ and their squares m_g we have:

$$\begin{array}{rcl}(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2 & = & ((1/2) \;+\; (1/2) \;+\; (1/2))^2 = 9/4, \\ m_1 + m_2 + m_3 & = & \; 3(1/2)^2 + (3/2)(\sqrt{1/2})^2 \;\; = 3/2.\end{array} \tag{148}$$

The good news is that these three masses satisfy the Koide relation:

$$m_1 + m_2 + m_3 = \frac{2}{3}(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2. \tag{149}$$

The bad news is that the three square root masses are not proportional to the square root masses of the charged leptons. In addition, one of them is negative:

$$\begin{aligned}\sqrt{m_1} &= 1/2 + \sqrt{1/2}\cos(2\pi/3 - 2\pi/12) \approx +0.500,\\ \sqrt{m_2} &= 1/2 + \sqrt{1/2}\cos(4\pi/3 - 2\pi/12) \approx -0.112,\\ \sqrt{m_3} &= 1/2 + \sqrt{1/2}\cos(6\pi/3 - 2\pi/12) \approx +1.112.\end{aligned} \tag{150}$$

Of course if these equations did give the charged lepton masses it would be an improvement on the Koide equation in that it removes a second degree of freedom.

The trig relations in Equation 147 give some hope as they show that "adjusting" the phase δ might move the masses around enough to match the charged leptons. In 2006 this author found that the charged lepton masses were accurately described when one replaces the $-2\pi/12$ of Equation 146 with $2/9$. In 2010 the result was published in this journal [7] as

$$\sqrt{m_g} = 17716\sqrt{\text{eV}}\ (1 + \sqrt{2}\ \cos(2g\pi/3 + 2/9)), \tag{151}$$

accurate to $O(10^{-5})$. The neutrino masses are not well measured but neutrino oscillations give the differences between the squares of their masses. The same 2010 paper found that the neutrino data could be fit to the Koide equation with masses

$$\sqrt{m_{\nu g}} = 0.1000(26)\sqrt{\text{eV}}\ (1 + \sqrt{2}\cos(2g\pi/3 + \pi/12 + 2/9)), \tag{152}$$

one of which has a negative square root. Thus the charged and neutral lepton mass equations both take the curious angle $2/9$ but only the neutrinos need the expected geometric angle $\pi/12$. Perhaps the difference has to do with the neutrinos having a simpler symmetry as illustrated in Figure 6.

The puzzle over the $2/9$ angle was made deeper by two papers[47,48] by Piotr Żenczykowski in 2012 and 2013. He parameterized masses by generation using a formula

$$\sqrt{m_g} = \sqrt{M_g}(1 + \sqrt{2}\ k_g\ \cos(2g\pi/3 + \delta_g)) \tag{153}$$

where M_g, k_g and δ_g are three real numbers. This parameterization is useful in that Koide's equation follows from $k_g = 1$. He applied the parameterization to the up quarks $\{u, c, t\}$ and to the down quarks $\{d, s, b\}$. He found that the up quarks required a δ value very close to $1/3$ of the $2/9$ used by the charged leptons and the down quarks needed $2/3$ of $2/9$. This removes a degree of freedom from the up and down quark masses. Bringing all the fermion mass parameterizations together we have, to close approximation:

$$\begin{aligned}\sqrt{m_{\nu g}} &= \sqrt{M_\nu}(1 + \sqrt{2}\ 1\ \cos(2g\pi/3 + 6/27 + 2\pi/12)),\\ \sqrt{m_{eg}} &= \sqrt{M_e}(1 + \sqrt{2}\ 1\ \cos(2g\pi/3 + 6/27)),\\ \sqrt{m_{dg}} &= \sqrt{M_d}(1 + \sqrt{2}\ k_d\ \cos(2g\pi/3 + 4/27)),\\ \sqrt{m_{ug}} &= \sqrt{M_u}(1 + \sqrt{2}\ k_u\ \cos(2g\pi/3 + 2/27)).\end{aligned} \tag{154}$$

If we can find a derivation for the multiples of 2/27, this will reduce the 12 fermion mass degrees of freedom to six.

In Equation 154, only the neutrino has the geometric phase $\pi/12$. Two explanations for this come to mind. Only the neutrino has a character table entry of all 1s. The other particles are orthogonal to this entry so perhaps that implies that their geometric phases are zero. Alternatively, the mass scale for the neutrinos are much smaller than the others so perhaps all of them have a geometric phase but it is too small to be noticed in the other particles. This could explain some deviations from exact match in the charged lepton masses.

Two unspecified parameters in Equation 154 are k_u and k_d. The corresponding values for the electron and neutrino equations indicate that the Koide equation is satisfied. Instead, the quark values are approximately: [47]

$$\begin{array}{l} k_u = 1.29, \\ k_d = 1.12. \end{array} \tag{155}$$

The present paper finds the k_e and k_ν values from the ratio of the scalar parts of the off diagonal to diagonal values of the α_{jkl}s. The calculation given in this subsection was apparently sufficient for the leptons; perhaps expanding it to the quarks would fix these parameters.

The different values of 6/27, 4/27 and 2/27 are mysterious in that they are not the angles one usually feeds to the cosine function. The cosine came from the sum of two exponentials. Quantum field theory, in dealing with infrared divergences in gauge boson interactions can put charges into exponents. When m virtual photons are considered with charge q and a symmetry factor of $1/m!$, one gets $\Sigma q^m/m! = \exp(q)$. Also the signs are not determined. If it is a charge, one supposes that it is a central charge for an algebra. It is not proportional to electric charge as the up quark takes the 1/3 of the electron value of 2/9 while the down quark takes 2/3.

The last four free parameters in Equation 154 are the masses:

$$\begin{array}{ll} M_\nu = 0.0098, & \text{(eV)} \\ M_e = 3.14 \times 10^8, & \\ M_d = 6.75 \times 10^8, & \\ M_u = 2.05 \times 10^{11}. & \end{array} \tag{156}$$

The presence of factors of 1/27 in the phases suggests that the fundamental charges are 1/3. One can get large ratios by using symmetry to cancel low power contributions to an amplitude calculation so it's interesting that the ratio $\sqrt{M_\nu}/\sqrt{M_e}$ is only a few percent different from $(1/3)^{11}$. Three more powers of 3 separate $\sqrt{M_u}$ from $\sqrt{M_e}$. This paper has been primarily directed to the symmetries of the stable states; to calculate these factors may require working out the Feynman diagrams.

6 Discussion

We discuss the advantages of this theory, the problems with the theory, the problems with this paper in particular, and directions in which this work might be extended.

6.1 Advantages of Theory

Ignoring generations, the Standard Model fermions use six representations of the SU(3)×SU(2)×U(1) symmetry. For example, the left-handed electron and left-handed electron neutrino are an SU(3) singlet, SU(2) doublet and a U(1) singlet. Of course the authors of the Standard Model had an infinite number of alternative choices for the symmetry and for any choice of symmetry there were then an infinite number of choices for the representations. Experiment determined which of these to use. This paper unites the symmetry, representation, generation structure, symmetry breaking and gauge structure of the Standard Model. It provides a hope that many of the experimentally determined parameters of the Standard Model can be calculated from theory.

Grand Unified theories assume that the symmetry of the Standard Model arises as a result of the breaking of another symmetry. To the people who have been working on it for years, this must seem normal but this is not the usual use of symmetry. One normally begins with a model of a system and from that model one derives a set of equations. Solving the equations is eased by noticing that they possess a symmetry. But the underlying object is not assumed to be a symmetry; it's an object that has a symmetry. An advantage of this paper is that it returns to the traditional use of symmetry as a mathematical tool instead of a physical reality.

A problem with symmetry is that it is difficult to use it to calculate the relative parameters of objects in different representations. For example, the spin-1/2 of the electron, along with the spherical harmonics, will give a great deal of information about the excited states of a hydrogen atom. But the symmetry cannot specify the $1/r^2$ potential which is necessary for Schroedinger's equation and emission spectrum. This paper proposes a model for the objects that possess the symmetry and so we can hope that it will allow calculations beyond those of theories limited by symmetry.

Preon models of the Standard Model fermions assume that they are composite. The oldest of these is the 1979 Rishon model independently proposed by Haim Harari[19] and Michael Shupe[39]. The problem with preons is that a composite particle has a size and scattering experiments show the quarks and leptons are smaller than 10^{-18} meters. Preons would have to be smaller yet, but this implies that their binding force must cancel their mass-energy almost exactly. In this theory the elementary particles are linear superpositions and so are naturally point-like. Instead, what we are doing in this paper is proposing a symmetric basis for the Standard Model fermions. In the new basis there are 24 particles with $F = S_4$ symmetry, all subject to the gauge symmetry

$G = \mathrm{SO}(3)$. The difficulty is that the basis is in density matrix theory and cannot easily be put into the usual state vector form.

6.2 Problems with this Theory

This theory is written in a variation of quantum mechanics. A variation like this has consequences all over physics. The author's claim is that the methods used here can also be described using quantum field theory and perhaps that would eliminate the need to show that it is self consistent, but this claim has not been shown.

Quantum mechanics is a big subject that is deeply interwoven so there must be many points where changing its assumptions will create problems. As an example, if measurements of spin-1/2 particles can give six possible results $\{\pm x, \pm y, \pm z\}$ instead of the traditional two $\{\pm z\}$, then the counting of spin-1/2 states has to be modified and this has consequences in quantum statistics. The claim of this paper is that such results only appear at near Planck temperatures and integrate out naturally at low temperatures.[7] We will discuss this again, later in this section.

This paper spends most of its effort on what appears to be a derivation of the symmetry of the Standard Model. It has been designed so that it seems that $F = S_4$ is the natural consequence of assuming a gauge group G with finite subgroup F. This is not at all how the research proceeded. The author spent a lot of effort trying hundreds of density matrix ideas with the idea that "if you dig enough holes eventually you'll find gold", no deep thinking required. So the paper has been made to appear as if it were intelligently thought out as the only possible consequence of a simple assumption. This is not so, it's essentially the result of a random walk through ideas related to density matrices. Once one finds a coincidence it's easy enough to rationalize motivations. For this reason, the reader needs to consider the possibility that the equations here are similar to Standard Model symmetries only as a result of random coincidence. This applies doubly to Subsection 5.6 on mass equations. Try enough equations and it's easy to get 5 or 6 digit coincidences. Reader beware.

6.3 Problems with this Paper

Some of the problems with this paper are simply missing calculations that someone may someday complete. Ideally, a paper proposing an explanation for the Standard Model should provide a calculation for all of its many experimental parameters. The calculations here have been done with a variety of quantum mechanics intended to model the part of quantum field theory that deals with situations where there are no particles created or destroyed. A better paper would use quantum field theory to cover the situations where this does not apply. The primary missing piece to this paper is a model of

the gauge bosons. The creation and annihilation of gauge bosons may require quantum field theory.

Of the parameters of the Standard Model, the only ones discussed are the generation structure of the quark and lepton masses. The biggest improvement from here would be a calculation for the weak quantum numbers. Beyond that derivations of the CKM and MNS matrices would be nice.

6.4 Further Work

On the subject of quantum statistics, there is an interesting paper by Luboš Motl [31] on the ringing of black holes. This is classical general relativity. A stable black hole "has no hair", that is, it is characterized only by mass, charge and angular momentum. If something perturbs the black hole, it relaxes to such a stable state. The deviation from symmetry can be described by the normal modes of the black hole. Unlike the usual normal modes of a vibrating object, these modes decay away so they are called "quasinormal" modes.

A black hole that has hair radiates it off by Hawking radiation. Which particles are radiated presumably depends on the shape of the perturbation. So Motl found the quasinormal modes and determined their spin and the quantum statistics implied by their asymptotic frequencies (or decay rates). The poles and statistics he found were:

Spin	Pole	Naively Implied statistics
$j \in \mathbb{Z} + 1/2$	$\frac{1}{\exp(\beta_H \omega)+1}$	Fermi-Dirac
$j \in \mathbb{Z} + 1$	$\frac{1}{\exp(\beta_H \omega)-1}$	Bose-Einstein
$j \in \mathbb{Z}$	$\frac{1}{\exp(\beta_H \omega)+3}$	Tripled Pauli?

, (157)

where $\beta_H = 1/(k_B T_H)$ is the Hawking thermodynamic beta. The spin-1/2 and spin-1 cases are the expected Fermi-Dirac and Bose-Einstein statistics but spin-2 (graviton) gets a doubly unexpected result. First the sign is positive rather than the negative for Bose-Einstein. Thus the quantum states satisfy the Pauli exclusion principle. Second, it takes a 3 instead of the usual 1. This implies that the quantum state can have three "different forms". It may be just another coincidence but could it be possible to describe gravitons with bases tripled like the ones used here? And does that have anything to do with the mass calculations?

Another question is how to extend this theory to cover gravitation. This is a gauge theory where the symmetries are rotations of objects defined by gamma matrices. Elementary particles have short wavelengths compared to gravitation so a compatible gravitation theory would involve the long wavelength rotations of gamma matrices. Anthony Lasenby, Chris Doran and Stephen Gull found such a gravitation theory called "Gauge Theory Gravity" (GTG) and published it in 1998 [28], but the reader may prefer their improved 2004 version on arXiv. They use the "spacetime algebra" (STA) of David Hestenes which is like gamma matrices but with no matrices ever specified. The reader is also

recommended the 2005 Hestenes introduction to GTG.[21] Unlike this paper, they use the $+---$ sign convention for the gamma matrices. Their theory gives the same results as general relativity restricted to a flat background metric. This amounts to general relativity with a specific choice of metric, subject to a Minkowski transformation. And the flat background prevents science fiction ideas like worm holes but gives identical results to all of the reasonable predictions of general relativity. The GTG gauges local displacements and Lorentz rotations which makes it compatible with this paper.

Since the GTG defines a particular coordinate system for GR problems, one is naturally curious what the simplest GR objects, black holes, look like in GTG form. The idea is to learn something about gravitons. The GTG version of a rotating charged (Kerr-Newman) black hole was found by Doran[13] but the best pedagogical description of the metric, especially in the Cartesian coordinates used in this paper, is the "River Model of Black Holes" by Andrew J. S. Hamilton and Jason P. Lisle.[17]

The non rotating, uncharged black hole is the simplest black hole case. For the GTG, one uses Gullstrand-Painlevé coordinates.[4] If the gravitational force is to be transmitted by a gauge boson (graviton), they must be radially symmetric so this author converted Gullstrand-Painlevé as well as the usual Schwarzschild coordinates into exact Newtonian form, that is, as would be used for a small test mass by Newton's equation $\mathbf{F} = m\mathbf{a}$.[6]

In Gullstrand-Painlevé coordinates, the difference in velocity between incoming and outgoing (radial) light beams is always $2c$. Far from the black hole the two speeds approach c. At the horizon the outgoing speed of light is zero and the incoming speed is $2c$. Closer to the singularity both incoming and outgoing light speeds are above $2c$. So these coordinates do not treat the event horizon as a particularly special case. In analogy with the electric force, we suppose that the gravitational force is due to the differential change in the density of gravitons and from this calculate a graviton density that depends on distance from the singularity. This allows us to conclude that, with either coordinate system, as gravitons move away from the singularity their number increases proportionally to $1/r^4$ so the process that is making more gravitons does it proportionally to the square of the graviton density as a sort of stimulated emission of gravitons.

Eight of the experimental parameters of the Standard Model have to do with how the weak force changes fermion generations. The author worked on this problem with Marni Sheppeard. She pointed out the importance of the discrete Fourier transform, circulant and anti-circulant matrices. Her approach to these problems uses Category theory. Most of her papers are on www.vixra.org; the ones particularly related to this subject are 1004.0083 "Quark Lepton Brainds and Heterotic Supersymmetry" (2010), 0711.0001 " Mass Matrix Transforms in Qubit Field Theory" (2007), 1008.0015 "Arith-

[4] Gullstrand is said to be the physicist who delayed Einstein's Nobel prize partly out of a lack of satisfaction with Schwarzschild coordinates.

metic Information in Particle Mixing" (2010), and 1010.0029 "On Neutral Particle Gravity in Particle Mixing" (2010).

There are two 3×3 unitary matrices, the CKM gives how the quarks are changed by the emission or absorption of a $W^{\pm}$ gauge boson while the MNS matrix does the same for the leptons. Ideally, we should be able to calculate these from first principles. The weak force is handed so perhaps we should try writing the entries of these matrices in terms of the density matrices given in Equation 139. The generation model for mass described in this paper associates the generation g with a phase $\exp(2i\pi\ g/3)$. We expect interactions that depend on generation to depend on the relative phases. If g and g' are the generation numbers for two particles, we have two ways of combining the signs: $\exp(2i\pi(g/3 \pm g'/3))$. The $-$ sign means that the effect depends on the difference in generation so the 3×3 matrix that gives the interaction will be circulant. Similarly the $+$ sign gives an anti-circulant matrix. These sorts of matrices were suggested for the MNS matrix by P. F. Harrison, D. H. Perkins and W. G. Scott in 2002. [20] The form they suggested is called "tri-bimaximal" and is the subject of a great many papers. This gives the form of the matrices we expect, but there also may be coincidences in the values. The one that comes to mind is the Weinberg weak mixing angle that obtains the Z^0 and photon from the weak gauge bosons, the Koide δ angle, and the Cabibo angle in the CKM matrix; all three of these are near $2/9$.

A problem with the CKM and MNS matrices is that experiments can define their entries only subject to arbitrary complex phases multiplying their rows and columns. A unitary 3×3 matrix has 9 real degrees of freedom. When the rows and columns take arbitrary complex phases there are only 4 real degrees of freedom left. The industry parameterizes these matrices with various inelegant methods.

The elimination of arbitrary complex phases has a great deal to do with density matrices so it's natural to look for a parameterization of the weak mixing matrices based on density matrices. The number of parameters for a U(n) matrix is n^2. There are n columns and n rows, but one overall complex phase so in total there are $2n-1$ arbitrary complex phases in the rows and columns. That leaves $n^2 - (2n-1) = (n-1)^2$ parameters for a U(n) mixing matrix. This suggests that the U(n) mixing matrices can be put into a form that is equivalent to the U(n-1) unitary matrices.

For this idea to work, we first must find a Lie subgroup of the U(n) matrices that only needs $(n-1)^2$ parameters. This is most easily done by defining a Lie subalgebra of the u(n) Lie algebra and exponentiating. Such a Lie subalgebra is defined by the u(n) matrices whose rows and columns all sum to zero. This feature reminds one of magic squares so call these matrices magic, or mu(n). Writing down the definition of matrix addition and multiplication is enough to show that mu(n) is closed under addition and multiplication and so they are also closed under commutation and are a Lie subalgebra of u(n). We can parameterize u(n) by the n real elements down the diagonal and the $n(n-1)/2$ complex elements above the diagonal. To make this into a parameterization of mu(n) we adjust the parameters in the bottom row and the rightmost column

so that the rows and columns sum to zero. This removes $2n-1$ parameters and we're left with the expected $(n-1)^2$ parameters. To show that mu(n) indeed is equivalent to $u(n-1)$ we need to provide a transformation that puts these parameters into an Hermitian $(n-1)^2$ block. Such a transformation would be of the form:

$$X' = H\ X\ H^{-1} \tag{158}$$

where H is Hermitian so it preserves Hermiticity. Such a transformation preserves addition and multiplication so it will transform a Lie subalgebra into a Lie subalgebra.

A pure density matrix is Hermitian so things that we make out of it will be Hermitian and will preserve Hermiticity. So let ρ_H be a pure $n \times n$ density matrix. Since ρ_H is idempotent, its 1-parameter Lie subgroups are particularly easy to calculate and since it's Hermitian those Lie subgroups are unitary. For a real number θ,

$$\exp(i\theta\rho_H) = 1 - \rho_H + e^{i\theta}\rho_H, \tag{159}$$

so a unitary matrix that transforms by inverting the ρ_H projection and leaves everything else unchanged is

$$H(\rho_H) = \exp(i\pi\rho_H) = 1 - 2\rho_H. \tag{160}$$

The above is the Householder transformation [23], derived using density matrices. If we transform ρ_H into the $(1,0,0,0...0)$ diagonal matrix, then $H(\rho_H)$ will be the diagonal matrix $(-1,1,1,1...1)$ and the ρ_H projection is indeed negated. Also $H^2 = 1$ and the trace is -1.

We need a transformation that takes a mu(n) matrix into one that is nonzero only for the top left $(n-1) \times (n-1)$ portion. So the ρ we need must be of the form:

$$\rho_H = |H\rangle\langle H| = \begin{pmatrix} a \\ ... \\ a \\ b \end{pmatrix} \begin{pmatrix} a\ ...\ a\ b \end{pmatrix} \tag{161}$$

where a and b are real numbers so that $(n-1)a^2 + b^2 = 1$ to keep the trace 1. And we can test for the right transform by inserting a trivial example into the transformation to obtain $1 - (n-1)(2a^2) - 2ab = 0$. Solving these gives:

$$a = \sqrt{\frac{\sqrt{n}+1}{2\sqrt{n}(n-1)}}, \quad b = -\sqrt{\frac{\sqrt{n}-1}{2\sqrt{n}}}. \tag{162}$$

This proves that mu(n) is equivalent to u(n-1) and so MU(n) is equivalent to U(n-1), and we can parameterize MU(n) by exponentiating and then Householder transforming any convenient parameterization for u(n-1).

The fact that any U(n) matrix can be put into MU(n) form by multiplying rows and columns by phases was discovered by this author by testing random matrices but without proof beyond 2×2 matrices. Philip Gibbs provided the proof for 3×3 matrices in 2009. [16] In 2011, this author posted to the Math

StackExchange the question "Given two basis sets for a finite Hilbert space, does an unbiased vector exist?" which, if true, is sufficient to show that any unitary mixing matrix can be put into magic form. Sam Lisi responded with an unpublished proof using symplectic geometry of Lagrangian intersections.[29]

This method gives a parameterization of U(3) mixing matrices where the four parameters correspond to permutations, either swaps, $\theta_{12}, \theta_{13}, \theta_{23}$ or the rotation θ_{123}. These parameters are correctly labeled in that θ_{12} defines a 1-parameter subgroup that reaches the unitary swap matrix (12) when $\theta_{12} = \pi$. The permutation group S_3 includes a (132) rotation but no such parameter is included. This is because $(123)^2 = (132)$ so (132) is on the 1-parameter subgroup generated by θ_{123}. In exponential form the parameterization is:

$$\exp(i \begin{pmatrix} \theta_{12}+\theta_{13} & -\theta_{12}+i\theta_{123} & -\theta_{13}-i\theta_{123} \\ -\theta_{12}-i\theta_{123} & \theta_{12}+\theta_{23} & -\theta_{23}+i\theta_{123} \\ -\theta_{13}+i\theta_{123} & -\theta_{23}-i\theta_{123} & \theta_{13}+\theta_{23} \end{pmatrix}) \in MU(3). \tag{163}$$

The above can be put into closed (but complicated) form by using a Householder transformation to convert the mu(3) matrix into u(2) form which is easily exponentiated. The result is a symmetric parameterization.

Since the CKM matrix is close to unity, the swap parameters, θ_{jk}, are within 0.0004 of the same labeled parameters of the standard parameterization. The lack of symmetry in the standard parameterization is maximum at the "democratic" mixing matrix, that is, the one with all magnitudes equal. This is the 3×3 mixing matrix with maximum CP violation. To reach that matrix, one puts $\theta_{12} = \theta_{13} = \theta_{23} = 2\pi/9$ and $\theta_{123} = 0$ while the standard parameterization has $\theta_{12} = \theta_{23} = \pi/4$, $\theta_{13} \approx 0.61548$ and $\delta = \pi/2$.

The above parameterization has the swap parameters $\theta_{12}, \theta_{13}, \theta_{23}$ used in anti-circulant mode while the rotation parameters θ_{123} takes an imaginary unit and is used in circulant mode. This is a hint that we can manipulate the algebra and rewrite the parameterization in circulant and anti-circulant fashion. (And this is also a hint on how one can put Equation 163 into closed form.) One finds a parameterization that mirrors the mass density matrices defined in Equations 139 and 132:

$$M(\alpha,\beta,\gamma,\delta) \;=\; \frac{1}{3}\begin{pmatrix} C_1 & C_2 & C_3 \\ C_3 & C_1 & C_2 \\ C_2 & C_3 & C_1 \end{pmatrix} + \frac{i}{3}\begin{pmatrix} D_1 & D_2 & D_3 \\ D_2 & D_3 & D_1 \\ D_3 & D_1 & D_2 \end{pmatrix}, \tag{164}$$

where

$$\begin{aligned} C_n &= \cos(\delta) + 2\cos(\gamma)\cos((\alpha+2n\pi)/3), \\ D_n &= \sin(\delta) + 2\sin(\gamma)\cos((\beta+2n\pi)/3). \end{aligned} \tag{165}$$

The above matrices have rows and columns that sum to $\exp(i\delta)$. They are closed under multiplication so they form a Lie subgroup of U(3) suitable for unitary mixing matrices.

Experiments can only measure magnitudes of the CKM matrix elements. It's up to us to choose the phases. The diagonal entries have magnitudes near 1 while the off diagonal are, to a first approximation defined by the Cabibo angle.

Putting the Cabibo angle to 2/9, we get a real unitary CKM approximation by using $\cos(2/9)$ on the first two diagonal entries and $\pm\sin(2/9)$ on the off diagonals:

$$\text{CKM} \approx \begin{pmatrix} \cos(2/9) & +\sin(2/9) & 0 \\ -\sin(2/9) & \cos(2/9) & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 0.98 & +0.22 & 0 \\ -0.22 & 0.98 & 0 \\ 0 & 0 & 1.00 \end{pmatrix}. \quad (166)$$

When this matrix is written as the sum of a real circulant matrix and an imaginary anti-circulant matrix we get:

$$\text{CKM} \approx \begin{pmatrix} \cos(2/9) & 0 & 0 \\ 0 & \cos(2/9) & 0 \\ 0 & 0 & \cos(2/9) \end{pmatrix} + i \begin{pmatrix} 0 & \sin(2/9) & 0 \\ \sin(2/9) & 0 & 0 \\ 0 & 0 & \sin(2/9) \end{pmatrix} \quad (167)$$

The hope is that the above is the leading contribution in a Feynman diagram series for the CKM matrix.

Acknowledgements I appreciate the direction and assistance of my advisors Sukanta Bose, Fred Gittes and Michael Forbes as well as my fellow grad students. Washington State University has financially supported my research with teaching assistantships. In addition, family and friends have helped support me and I'd like to particularly mention my father, Joseph P. Brannen and my friends Mark Mollo and Forrest LeDuc.

References

1. Anastopoulos, C.: Quantum theory without Hilbert spaces. eprint arXiv:quant-ph/0008126 (2000)
2. Authier, A.: Early Days of X-ray Crystallography. Oxford University Press (2013)
3. Bengtsson, I.: Three Ways to Look at Mutually Unbiased Bases. In: G. Adenier, C. Fuchs, A.Y. Khrennikov (eds.) Foundations of Probability and Physics - 4, *American Institute of Physics Conference Series*, vol. 889, pp. 40–51 (2007). DOI 10.1063/1.2713445
4. Bindi, L., Cipriani, C.: Structural and physical properties of fischesserite, a rare gold-silver selenide from the de lamar mine, owyhee county, idaho, usa. The Canadian Mineralogist **42**, 1733–1737 (2004)
5. Bohm, D.J., Davies, P.G., Hiley, B.J.: Algebraic quantum mechanics and pregeometry. AIP Conference Proceedings **810**(1) (2006). DOI 10.1063/1.2158735
6. Brannen, C.: The Force of Gravity in Schwarzschild and GULLSTRAND-PAINLEVÉ Coordinates. International Journal of Modern Physics D **18**, 2289–2294 (2009). DOI 10.1142/S0218271809015886
7. Brannen, C.A.: Spin path integrals and generations. Found. Phys. **40**, 1681–1699 (2010). DOI 10.1007/s10701-010-9465-8
8. Buchmuller, W., Ludeling, C.: Field Theory and Standard Model. In: High-energy physics. Proceedings, European School, Kitzbuehel, Austria, August 21-September, 2005, hep-ph/0609174 (2006)
9. Chamblin, A.: On the superselection sectors of fermions. ArXiv High Energy Physics - Theory e-prints (1997)
10. Ciezarek, G., Franco Sevilla, M., Hamilton, B., Kowalewski, R., Kuhr, T., Lth, V., Sato, Y.: A Challenge to Lepton Universality in B Meson Decays. Nature **546**, 227–233 (2017). DOI 10.1038/nature22346
11. Cornelius S. Hurlbut, J., Klein, C.: Manual of Mineralogy 19th Ed (after James D. Dana). John Wiley & Sons (1977)

12. Dirac, P.A.M.: Note on the interpretation of the density matrix in the many-electron problem. Mathematical Proceedings of the Cambridge Philosophical Society **27**(2), 240243 (1931). DOI 10.1017/S0305004100010343
13. Doran, C.: New form of the Kerr solution. PRD **61**(6), 067503 (2000). DOI 10.1103/PhysRevD.61.067503
14. Durt, T., Englert, B.G., Bengtsson, I., Życzkowski, K.: On mutually unbiased bases. ArXiv e-prints (2010)
15. Fanchiotti, H., García Canal, C.A., Vento, V.: The Geometric Origin of the CP Phase. ArXiv e-prints (2017)
16. Gibbs, P.: 3x3 unitary to magic matrix transformations (2009). URL http://vixra.org/abs/0907.0002
17. Hamilton, A.J.S., Lisle, J.P.: The river model of black holes. American Journal of Physics **76**, 519–532 (2008). DOI 10.1119/1.2830526
18. Hammermesh, M.: Group Theory and its Application to Physical Problems. Dover (1962)
19. Harari, H.: A schematic model of quarks and leptons. Physics Letters B **86**, 83–86 (1979). DOI 10.1016/0370-2693(79)90626-9
20. Harrison, P.F., Perkins, D.H., Scott, W.G.: Tri-bimaximal mixing and the neutrino oscillation data. Physics Letters B **530**, 167–173 (2002). DOI 10.1016/S0370-2693(02)01336-9
21. Hestenes, D.: Gauge theory gravity with geometric calculus. Foundations of Physics **35**(6), 903–970 (2005). DOI 10.1007/s10701-005-5828-y. URL https://doi.org/10.1007/s10701-005-5828-y
22. Hiley, B.J., Callaghan, R.E.: The Clifford Algebra Approach to Quantum Mechanics B: The Dirac Particle and its relation to the Bohm Approach. arxiv:1011.4033 (2010)
23. Householder, A.S.: Unitary triangularization of a nonsymmetric matrix. J. ACM **5**(4), 339–342 (1958). DOI 10.1145/320941.320947. URL http://doi.acm.org/10.1145/320941.320947
24. Jarlskog, C.: Commutator of the quark mass matrices in the standard electroweak model and a measure of maximal CP nonconservation. Phys. Rev. Lett. **55**, 1039–1042 (1985). DOI 10.1103/PhysRevLett.55.1039. URL https://link.aps.org/doi/10.1103/PhysRevLett.55.1039
25. Koide, Y.: A fermion-boson composite model of quarks and leptons. Physics Letters B **120**(1), 161 – 165 (1983). DOI http://dx.doi.org/10.1016/0370-2693(83)90644-5. URL http://www.sciencedirect.com/science/article/pii/0370269383906445
26. Koide, Y.: New view of quark and lepton mass hierarchy. Phys. Rev. D **28**, 252–254 (1983). DOI 10.1103/PhysRevD.28.252. URL https://link.aps.org/doi/10.1103/PhysRevD.28.252
27. Koide, Y.: Challenge to the mystery of the charged lepton mass formula. In: Lepton and photon interactions at high energies. Proceedings, 22nd International Symposium, LP 2005, Uppsala, Sweden, June 30-July 5, 2005 (2005)
28. Lasenby, A., Doran, C., Gull, S.: Gravity, gauge theories and geometric algebra. Philosophical Transactions of the Royal Society of London Series A **356**, 487 (1998). DOI 10.1098/rsta.1998.0178
29. Lisi, S.: Given two basis sets for a finite hilbert space, does an unbiased vector exist? https://math.stackexchange.com/questions/28413/given-two-basis-sets-for-a-finite-hilbert-space-does-an-unbiased-vector-exist/29819 (2011). Accessed: 2017-08-03
30. Ludl, P.O.: Systematic analysis of finite family symmetry groups and their application to the lepton sector. ArXiv e-prints (2009)
31. Motl, L.: An analytical computation of asymptotic Schwarzschild quasinormal frequencies. Adv. Theor. Math. Phys **6**, 1135–1162 (2002)
32. Patrignani, C., et al.: Review of Particle Physics. Chin. Phys. **C40**(10), 100,001 (2016). DOI 10.1088/1674-1137/40/10/100001
33. Penrose, R.: The road to reality : a complete guide to the laws of the universe. Vintage, London (2005)
34. Roe, B.P.: Particle Physics at the New Millennium. Springer-Verlag New York (1996)
35. Schwinger, J.: The Algebra of Microscopic Measurement. PNAS **45**, 1542–1553 (1959)
36. Schwinger, J.: The Geometry of Quantum States. Proceedings of the National Academy of Science **46**, 257–265 (1960). DOI 10.1073/pnas.46.2.257

37. Schwinger, J.: Quantum Kinematics And Dynamic. Advanced Books Classics. Avalon Publishing (2000). URL https://books.google.com/books?id=hu6t-3Kni-4C
38. Schwinger, J., Englert, B.: Quantum Mechanics: Symbolism of Atomic Measurements. Physics and astronomy online library. Springer (2001)
39. Shupe, M.A.: A composite model of leptons and quarks. Physics Letters B **86**, 87–92 (1979). DOI 10.1016/0370-2693(79)90627-0
40. Svetlichny, G.: Feynman's Integral is About Mutually Unbiased Bases. ArXiv e-prints (2007)
41. Svetlichny, G.: Why Lagrangians? In: P. Kielanowski, A. Odzijewicz, M. Schlichenmaier, T. Voronov (eds.) Geometrical Methods in Physics, *American Institute of Physics Conference Series*, vol. 956, pp. 120–125 (2007). DOI 10.1063/1.2820959
42. Tanimura, S.: Superselection Rules from Measurement Theory. arxiv:1112.5701 (2011)
43. Tolar, J., Chadzitaskos, G.: Feynman's path integral and mutually unbiased bases. Journal of Physics A Mathematical General **42**(24), 245306 (2009). DOI 10.1088/1751-8113/42/24/245306
44. Weinberg, S.: The Quantum Theory of Fields. No. v. 2 in The Quantum Theory of Fields 3 Volume Hardback Set. Cambridge University Press (1995). URL https://books.google.com/books?id=sn9QvU5dmBQC
45. Weinberg, S.: Quantum mechanics without state vectors. PRA **90**(4), 042102 (2014). DOI 10.1103/PhysRevA.90.042102
46. Wick, G.C., Wightman, A.S., Wigner, E.P.: The intrinsic parity of elementary particles. Phys. Rev. **88**, 101–105 (1952). DOI 10.1103/PhysRev.88.101. URL https://link.aps.org/doi/10.1103/PhysRev.88.101
47. Żenczykowski, P.: Remark on Koide's Z_3-symmetric parametrization of quark masses. PRD **86**(11), 117303 (2012). DOI 10.1103/PhysRevD.86.117303
48. Żenczykowski, P.: Koide's Z_3-symmetric parametrization, quark masses, and mixings. PRD **87**(7), 077302 (2013). DOI 10.1103/PhysRevD.87.077302