E8 Root Vectors from 8D to 3D

Frank Dodd (Tony) Smith, Jr. - viXra 1708.0369

Abstract:

This paper is an elementary-level attempt at discussing 8D E8 Physics
based on the 240 Root Vectors of an E8 lattice
and how it compares with physics models based on 4D and 3D structures
such as Glotzer Dimer packings in 3D, Elser-Sloane Quasicrystals in 4D,
and various 3D Quasicrystals based on slices of 600-cells
and
a natural progression from 600-cell to Superposition of 8 E8 Lattices.

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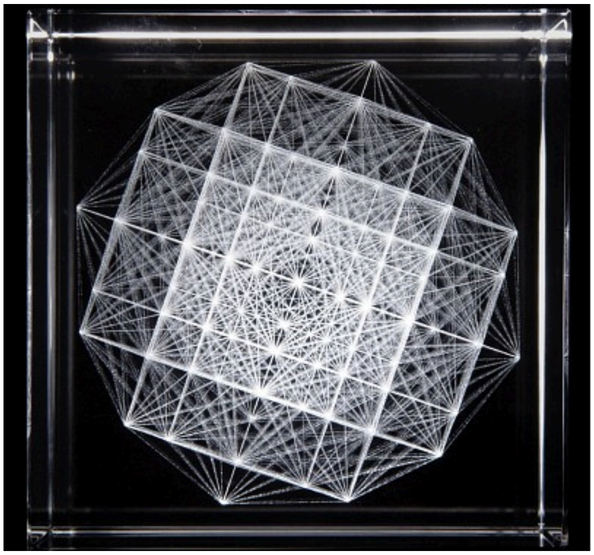
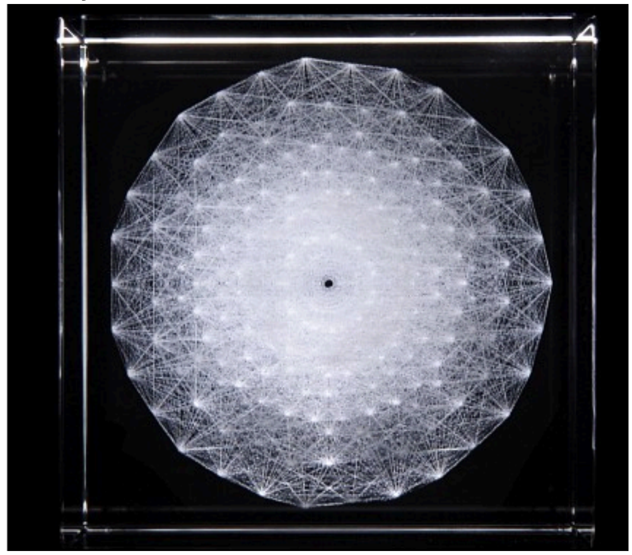
My E8 Physics model described in viXra 1602.0319 is based on physical interpretation of each of the 240 Root Vectors of E8. The E8 Root Vectors live in 8D but it is hard for me to visualize 8 dimensional space so I like to use projections to 3D and 2D. Bathsheba Grossman makes laser sculptures in 3D glass cubes, including a scupture of the 240 E8 Root Vectors. In this E8 sculpture by her



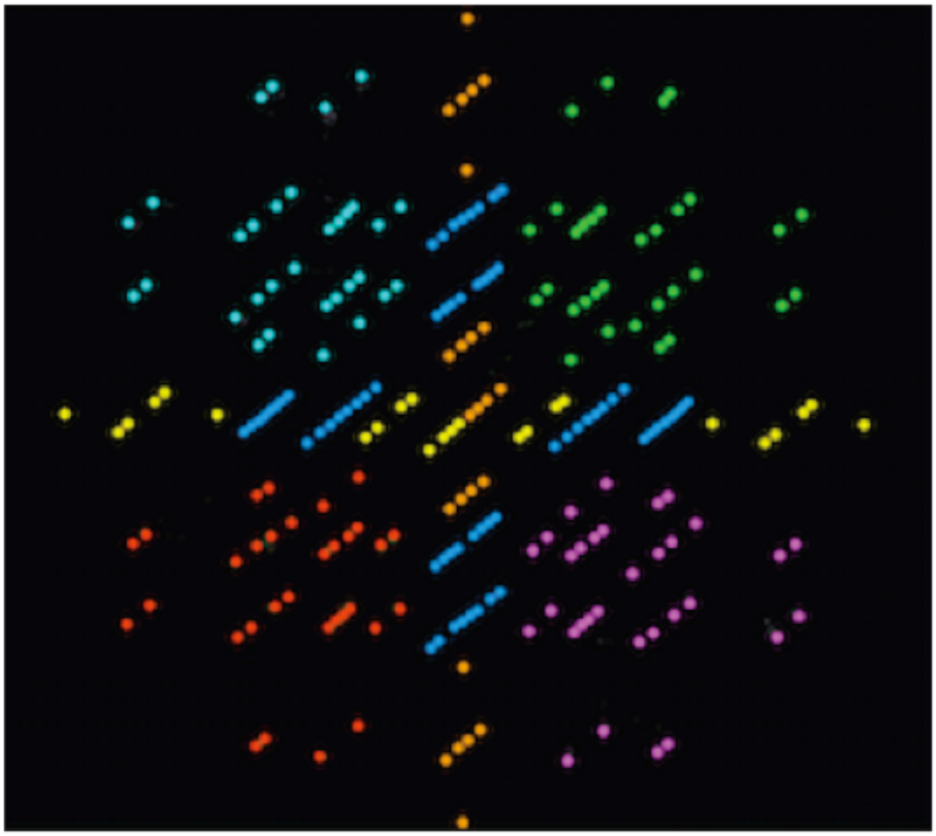


where different 2D face projections of her 3D cube projection from full 8D Root Vectors look quite different, although they obviously represent the same 240 of E8.

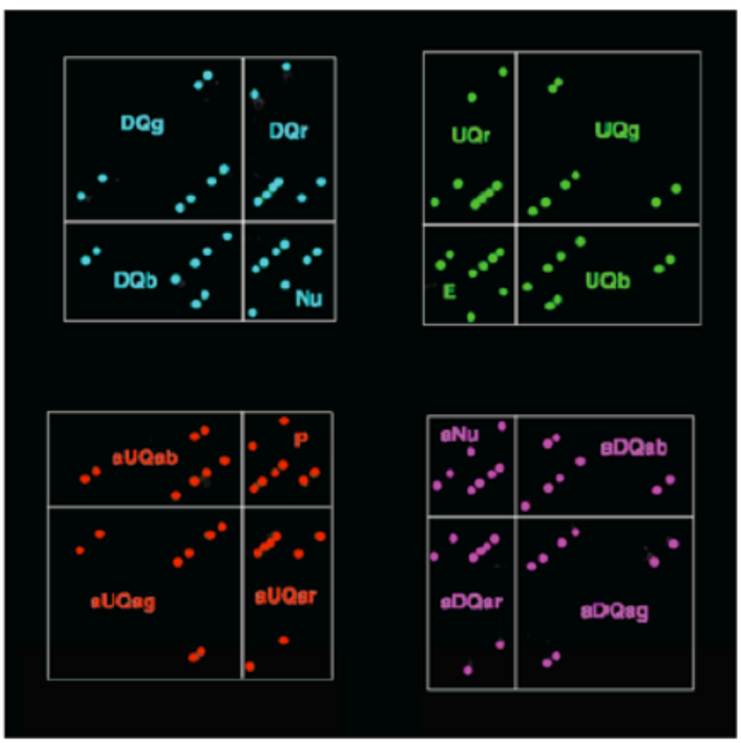
The 2D projection above on the left I call the square-cube projection.
In it there are 112 Root Vector Vertices on the two axes of the square
and there are in each of the 4 off-axis quadrants there are 32 vertices for 4x32 = 128
so that the square-cube projection corresponds to E8 / D8 = (OxO)P2
where E8 has 240 Root Vectors and D8 has 112 Root Vectors and
(OxO)P2 is Rosenfeld's Octo-Octonionic Projective Plane with 64+64 = 128 dimensions
of half-spinors for 8 components of 8 fermion particles and 8 fermion antiparticles.
The D8 axes have structure D8 / D4 x D4 = 64-dim real 4-Grassmannian of R8 which
represents 8 spacetime position x 8 spacetime momentum dimensions
and one D4 represents gauge bosons of gravity and ghosts of standard model
and the other D4 represents gauge bosons of the standard model and ghosts of gravity.

The 2D projection above on the right I call the circle-ball projection.
It has 8 concentric circles each with 30 vertices.
4 circles represent E8 Physics of gravity and the M4 part of M4 x CP2 Kaluza-Klein and
4 represent E8 Physics of standard model and CP2 part of M4 x CP2 Kaluza-Klein.

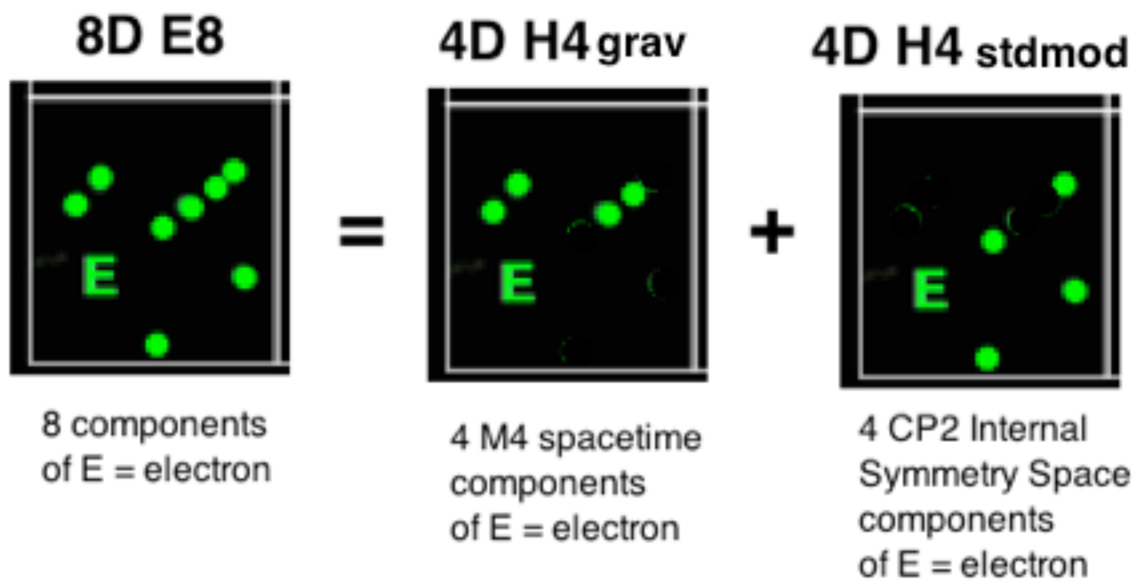
First, look at the 240 E8 Root Vectors in the square-cube projection:



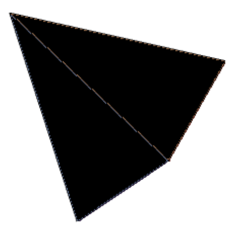
Here is the physical identification of the 128 E8 / D8 fermionic root vector subset:



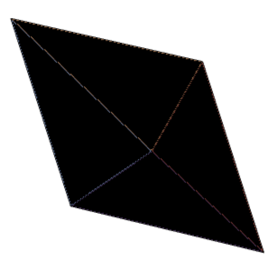
Here are some more details using the electron as example:



I conjecture that
the 4 vertices of M4 components for 4D H4grav 600-cell form a tetrahedron
with N = 1 tetrahedra (imagea from Wolfram CDF file by Ed Pegg Jr)



and
the 4 vertices of CP2 components for 4D H4stdmod 600-cell form another tetrahedron which
when combined with the H4grav tetrahedron forms an 8-vertex dimer
as described by Chen, Engel, and Glotzer in arXiv 1001.0586
with **N = 2 tetrahedra**



representing all 8 components of the electron.

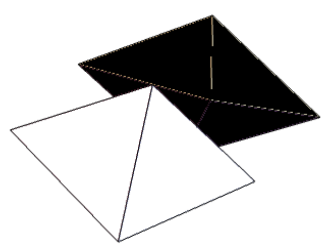
The propagation path of each of the two tetrahedra of the electron dimer remains within its own 4D H4 600-cell inside the E8 Lattice.

Packing densities in 3D for tetrahedral dimer structures are described by Chen, Engel, and Glotzer in arXiv 1001.0586:

#Tetra	Maximum Density		Success	Motifs,
N	Numerical, $\hat{\phi}$	Analytical, ϕ	Rate	Structural Description
1	0.367346	18/49	100%	1 monomer [11]
2	0.719486	ϕ_2	100%	2 monomers, transitive [22]
3	0.666665	2/3	21%	3 monomers, three-fold symmetric
4	0.856347	4000/4671	80%	2 dimers (positive + negative)
5	0.748096	ϕ_5	22%	1 pentamer, asymmetric
6	0.764058	ϕ_6	11%	2 dimers + 2 monomers
7	0.749304	3500/4671	15%	2×2 dimers minus 1 monomer
8	0.856347	4000/4671	44%	2×2 dimers, identical to $N = 4$
9	0.766081		—	1 pentagonal dipyramid + 2 dimers
10	0.829282	ϕ_{10}	2%	2 pentagonal dipyramids
11	0.794604		—	1 nonamer + 2 monomers
12	0.856347	4000/4671	3%	3×2 dimers, identical to $N = 4$
13	0.788728		4%	1 pentagonal dipyramid + 4 dimers
14	0.816834		3%	2 pentagonal dipyramids + 2 dimers
15	0.788693		—	Disordered, non-optimal
16	0.856342	4000/4671	< 1%	4×2 dimers, identical to $N = 4$
⋮	⋮			⋮
8×82	0.850267			Quasicrystal approximant [21]

TABLE I: Maximum numerical densities $\hat{\phi}$ for packings with small cells, obtained with numerical compression via Monte Carlo compression starting from a random configuration. the quasicrystal approximant result with $N = 8 \times 82$ is included. Details about the analytical results $\phi_2 = 9/(139 - 40\sqrt{10})$, $\phi_5 = 0.74809657\ldots$, $\phi_6 = 11228544/(97802181 - 132043\sqrt{396129})$, and $\phi_{10} = 29611698560/(23657426736 + 4919428689\sqrt{6})$

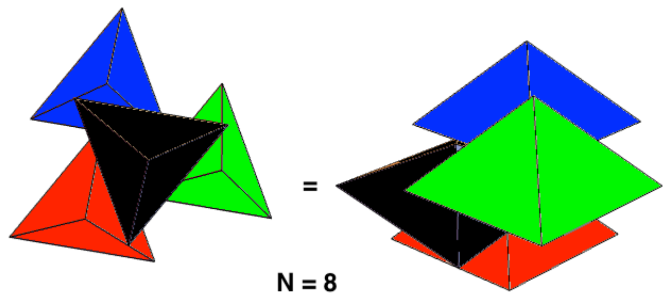
As you increase the number N of tetrahedra you first encounter the maxiumum at
N = 4 which represents two dimers = particle-antiparticle pair (electron-positron)



The maximum is encountered at N = 4, 8, 12, 16 ... for dimer tetrahedra periodicity 4.
A tetrahedron can be seen as a pair of binary binars

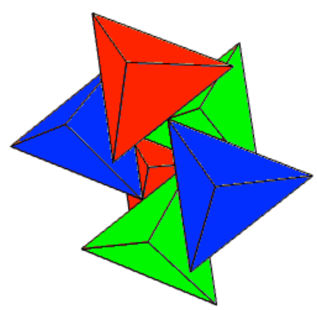
so that the dimer binary periodicity is 2 x 4 = 8
which is the same 8-periodicity as Real Clifford Algebras with binary structure.

The tetrahedral N = 8 is for 4 dimers corresponding to a lepton and G R B quarks (electron and green, red, and blue up quarks)

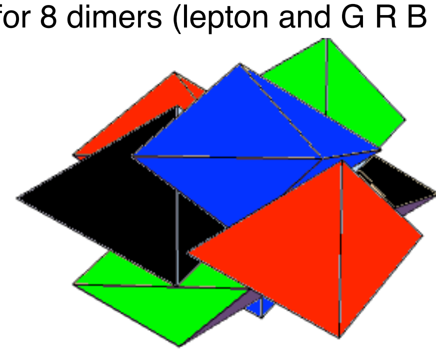


N = 8

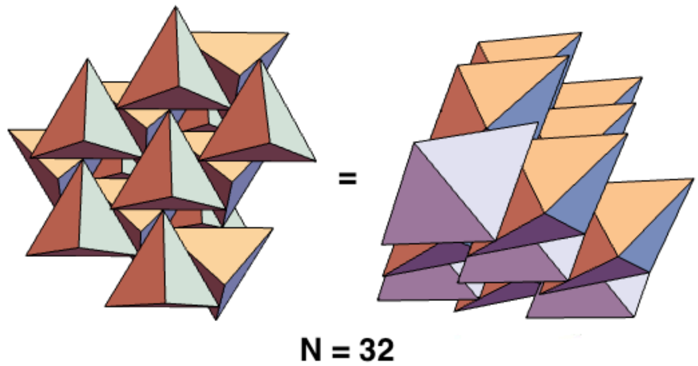
The tetrahedral N = 12 is for 6 dimers corresponding to 3 quark-antiquark pairs (green, red, and blue up quarks and green, red, and blue up antiquarks)



The tetrahedral N = 16 is for 8 dimers (lepton and G R B quarks and their antiparticles)

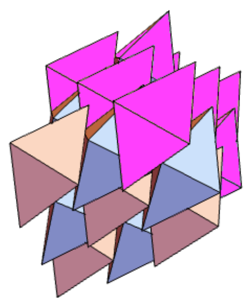


The tetrahedral N = 32 is for 16 dimers that represent E8 / D8 = (OxO)P2
= all 16 fermions x 8 components = 128 Fermionic E8 Root Vectors

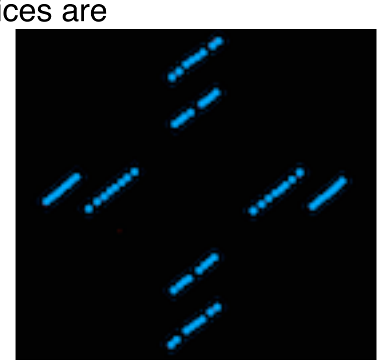


The 128 Fermionic E8 Root Vectors are also consistent with Geoffrey Dixon's fundamental tensor T^2 where T = RxCxHxO = real x complex x quaternion x octonion.

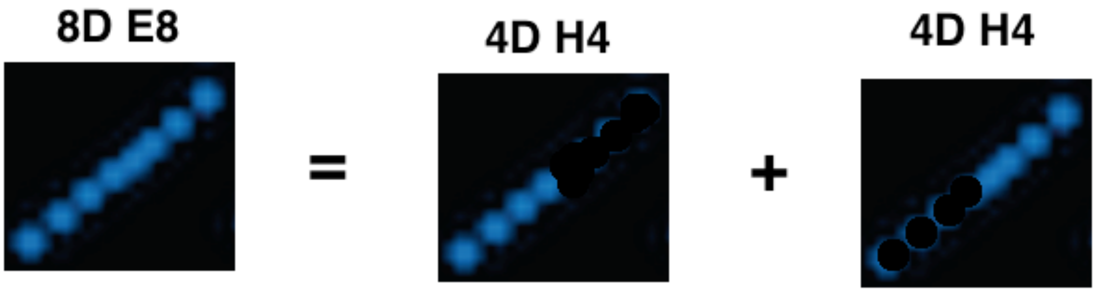
The tetrahedral N = 48 adds 16 / 2 = 8 dimers (magenta) representing (4+4 = 8) dimensions of spacetime and 8x8 = 64 E8 Root Vectors for a total of 128 + 64 = 192 Root Vectors or 96 binars or 24 dimers.



There are two tetrahedra = one Glotzer 8-vertex dimer for each dimension of 8D spacetime. The 8x8 = 64 vertices are



For each dimension of 8D spacetime,
two tetrahedra represent momentum in 4D M4 and in 4D CP2
each propagating in its own H4 600-cell subspace



8 components of momentum for each position dimension of spacetime

4 M4 spacetime components of momentum for each position dimension of spacetime

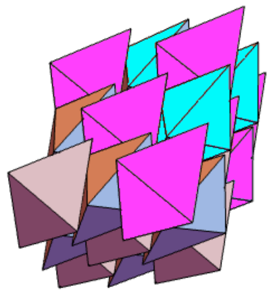
4 CP2 Internal Symmetry Space components of momentum for each position dimension of spacetime

Therefore, 128 + 64 = 192 of the 240 representing fermions and spacetime
can be represented as tetrahedra.

The spacetime 64 are isomorphic by Triality
to the N = 8 lepton and G R B quark particle components (8x8 = 64)
and to their N = 8 lepton and G R B quark antiparticle components (8x8 = 64)

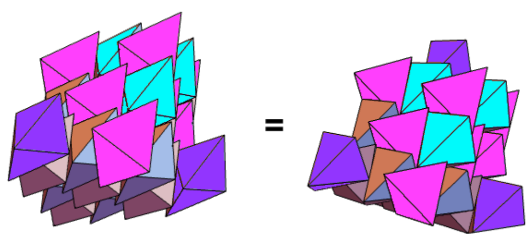
Consistently with Clifford Periodicity (tetrahedral N = 48 is divisible by 4)
Fermions + Spacetime give a packing of the maximum density 4000 / 4671 = 0.8656347
which is more dense than a dodecagonal quasicrystal (0.8324)
and
more dense than a compressed QC approximant at 0.8503
(see Haji-Akbari1, Engel, Keys, Zheng, Petschek, Palffy-Muhoray, and Glotzer
in arXiv 1012.5138)

The tetrahedral N = 54 adds 3 dimers representing 24 gauge bosons and ghosts
(12 gauge bosons for Gravity+Dark Energy and 12 ghosts for Standard Model
or
12 gauge bosons for Standard Model and 12 ghosts for Gravity+Dark Energy)



BUT as tetrahedral N = 54, equivalent to binary 108, is NOT consistent with periodicity because when you add
EITHER 24 vertices of Gravity+Dark Energy OR 24 vertices of Standard Model
to 128 + 64 = 192 Fermion Particles and Antiparticles and Spacetime
then you get 216 vertices or 54 tetrahedra or 108 binars
and 54 is not a multiple of 4 and 108 is not a multiple of 8.

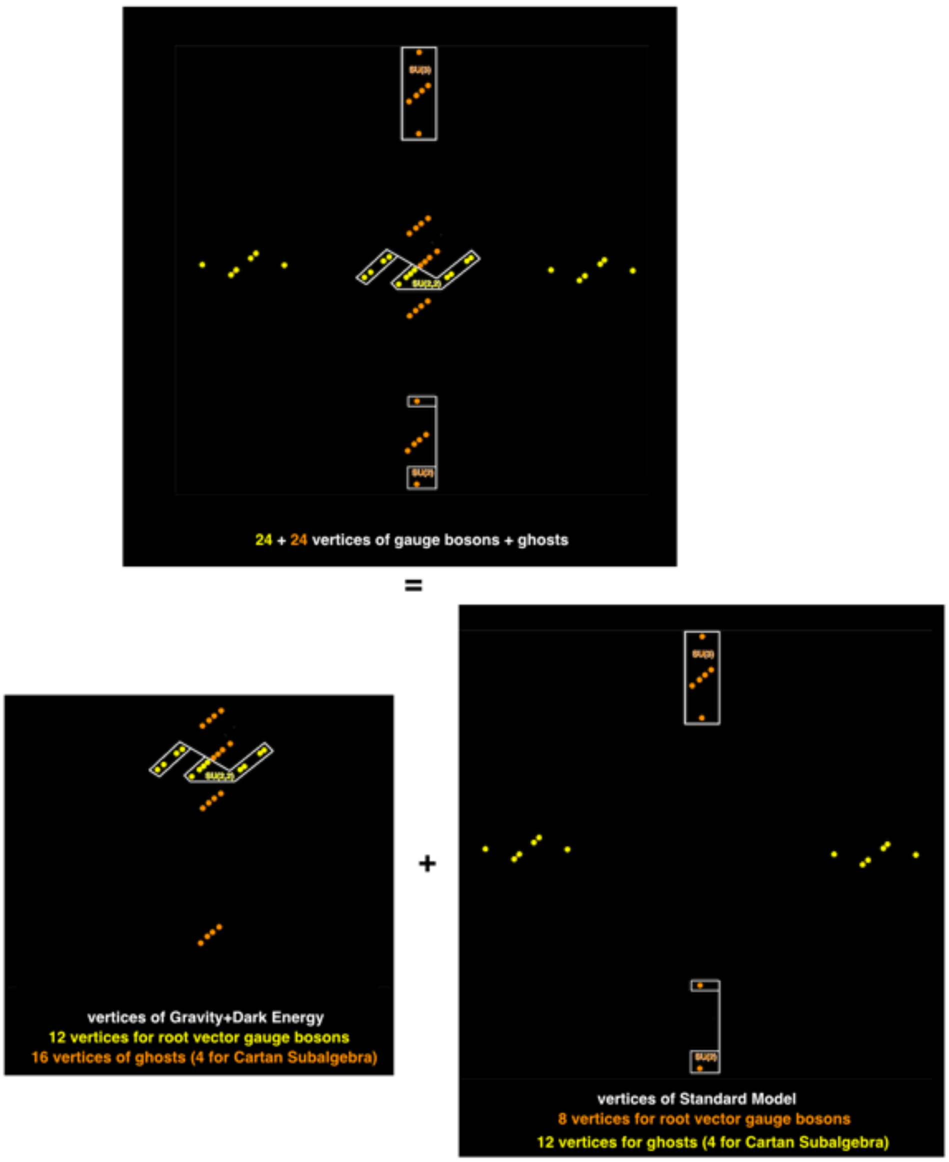
However, when you add
BOTH 24 vertices of Gravity+Dark Energy AND 24 vertices of Standard Model
to 128 + 64 = 192 Fermion Particles and Antiparticles and Spacetime



then you get 240 vertices or 30 dimers or 60 tetrahedra or 120 binars
(30 8-vertex dimers give the circle-ball 2D projection)
so
for N = 60 the totality of all 240 E8 Root Vectors is consistent with periodicity.

What is the physical reason that you cannot add only one of
24-vertex Gravity-Dark Energy and 24-vertex Standard Model
to the 192 vertices of Fermions and Spacetime
but
must add both ?

A non-physical answer is that 192 + 24 vertices = 216 / 4 = 54 tetrahedra
and 54 is not divisible by 4
whereas
192 + 24 + 24 vertices = 240 /4 = 60 tetrahedra is divisble by 4 of periodicity.



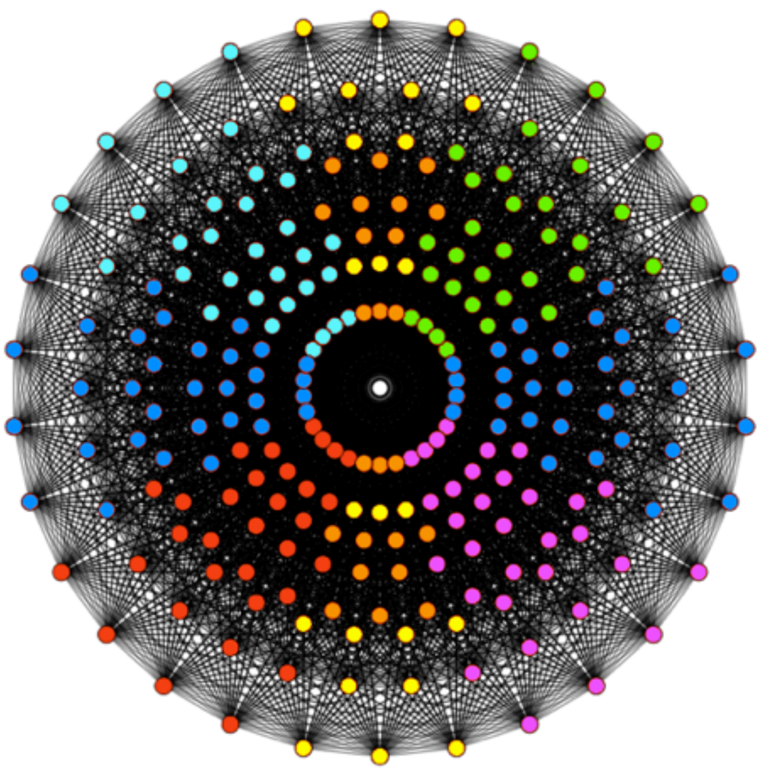
Physically,
the gauge bosons of Gravity+Dark Energy are in M4 (horizontal axis)
and their ghosts are in CP2 (vertical axis) so both axes must be used
and Standard Model similarly requires both axes to be used.

Now, look at the 240 E8 Root Vectors in the circle-ball projection:

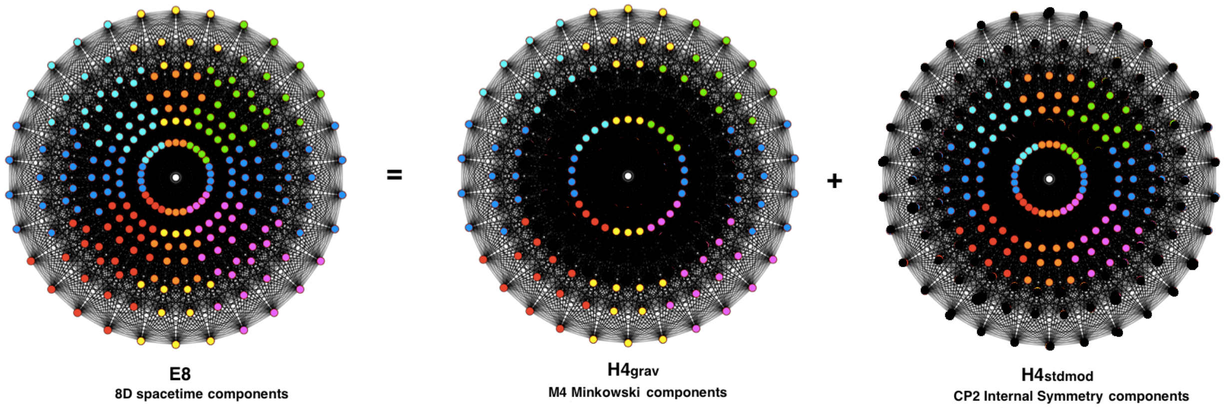
My E8 Physics model Physical Interpretation of the 240 E8 Root Vectors which break down into two sets of 120 each with H4 symmetry that correspond to the M4 gravity and CP2 standard model sectors of M4 x CP2 Kaluza-Klein is:

64 blue = Spacetime
64 green and cyan = Fermion Particles
64 red and magenta = Fermion AntiParticles
24 yellow = D4g Root Vectors = 12 Root Vectors of SU(2,2) Conformal Gravity
+ 12 Ghosts of Standard Model SU(3)xSU(2)xU(1)
24 orange = D4sm Root Vectors = 8 Root Vectors of Standard Model SU(3)xSU(2)xU(1)
+ 16 Ghosts of U(2,2) of Conformal Gravity

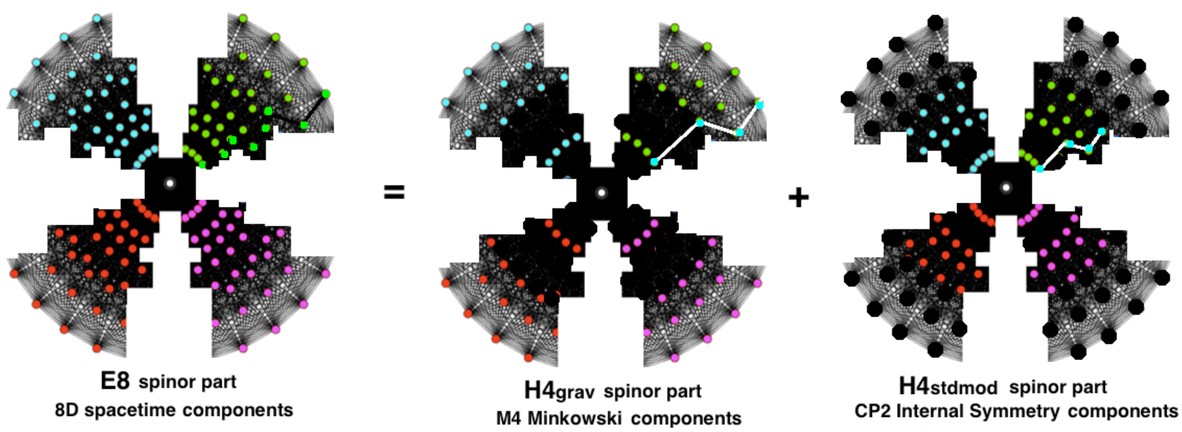
Here they are shown in the circle-ball 2-dim projection with 8 circles of 30 vertices each:



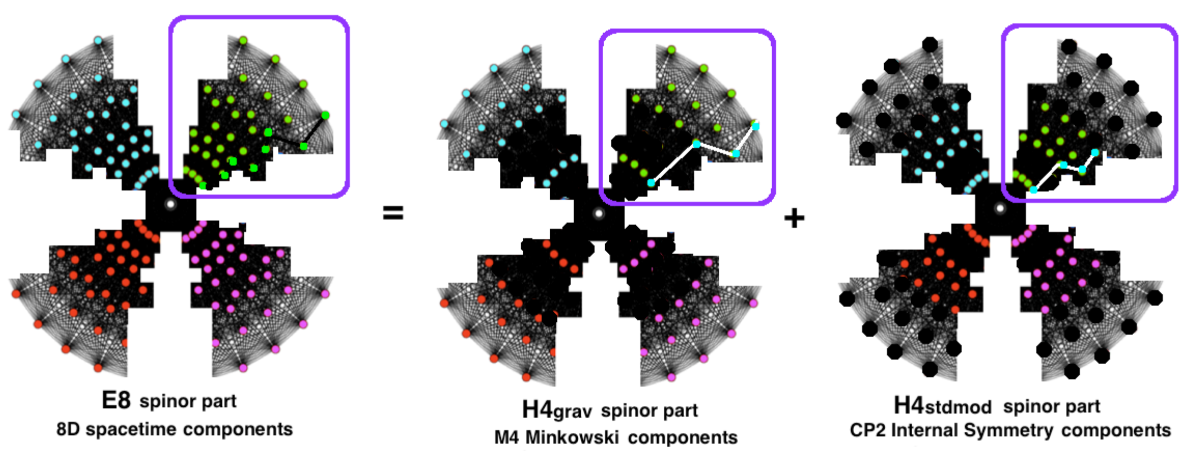
Here is how the 240 break down into 120 + 120 of H4grav and H4stdmod



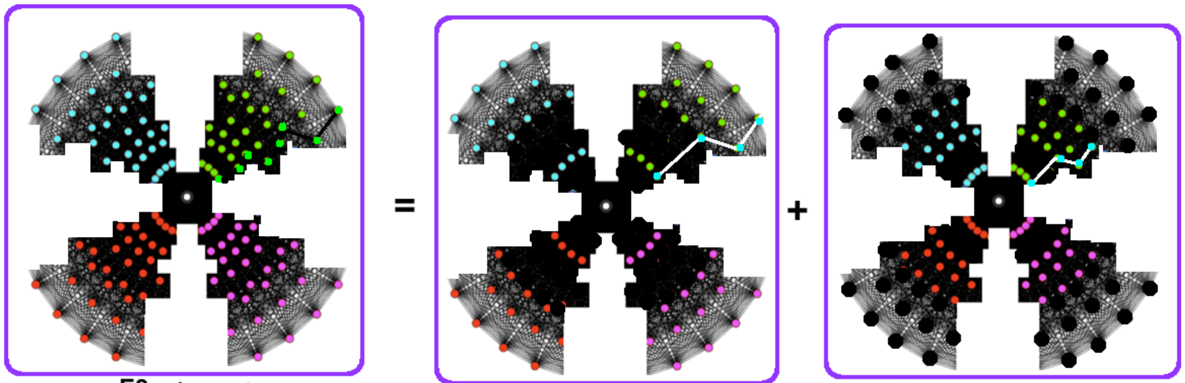
Here are 128 Fermionic Root Vectors with the 8 components for the electron dimer that break into two (M4 and CP2) tetrahedra with 4 vertices shown connected by white lines.



If you combine the dimers for the green, red, and blue up quarks with the electron dimer as shown in purple boxes then you get 4 dimers with maximum packing density



If you then take all 4 Fermion Quadrants

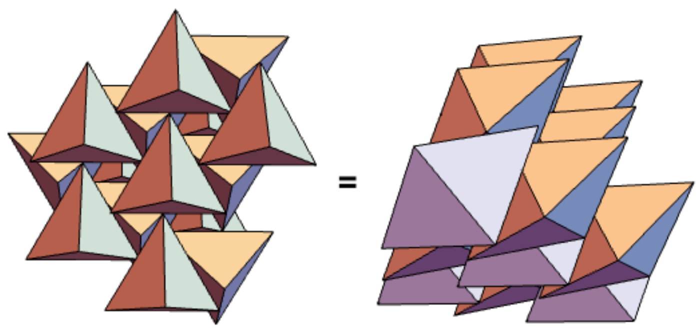


E8 spinor part
8D spacetime components

H4grav spinor part
M4 Minkowski components

H4stdmod spinor part
CP2 Internal Symmetry components

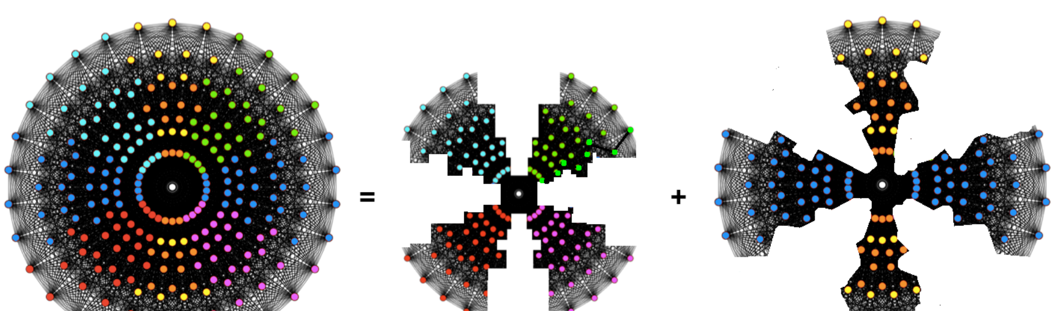
then you get the **tetrahedral N = 32** for 16 dimers that represent E8 / D8 = (OxO)P2
= all 16 fermions x 8 components = 128 Fermionic E8 Root Vectors



N = 32

The 128 Fermionic E8 Root Vectors are also consistent with Geoffrey Dixon's fundamental tensor T^2 where T = RxCxHxO = real x complex x quaternion x octonion.

The 240 of E8 = (128 spinor fermionic E8 / D8) + 112 of D8

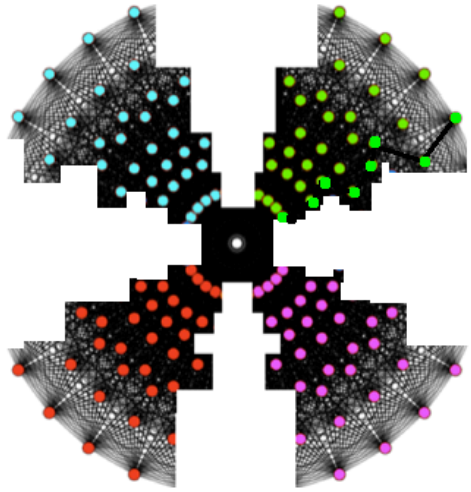


E8

E8 spinor part
8D fermion spacetime components

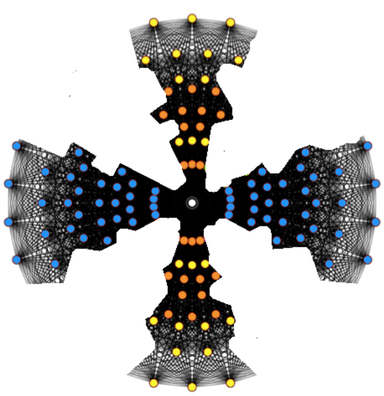
E8 D8 part = 64 + D4 x D4
8D spacetime and
gauge bosons + ghosts

The Spinor Fermion part = E8 / D8 contains 128 vertices = 64 binars = 16 dimers = = 32 tetrahedra so it **has tetrahedral N = 32**



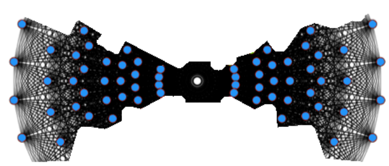
E8 spinor part
8D spacetime components

Since D8 / D4xD4 = 64-dim (OxO)P2
the 112 of D8 = (8x8 = 64 spacetime) + (24+24 = 48 D4xD4)



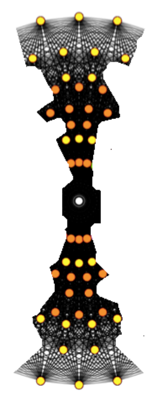
E8 D8 part = 64 + D4 x D4
8D spacetime and
gauge bosons + ghosts

=



E8
8D spacetime components

+



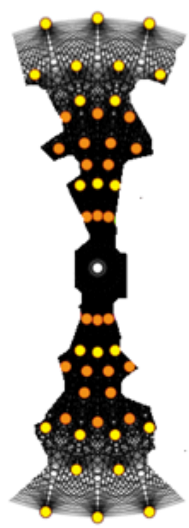
E8
8D gauge bosons + ghosts

The Spacetime part = D8 / D4xD4 contains 64 vertices = 32 binars = 8 dimers =
= 16 tetrahedra so it **has tetrahedral N = 16**

E8
8D spacetime components

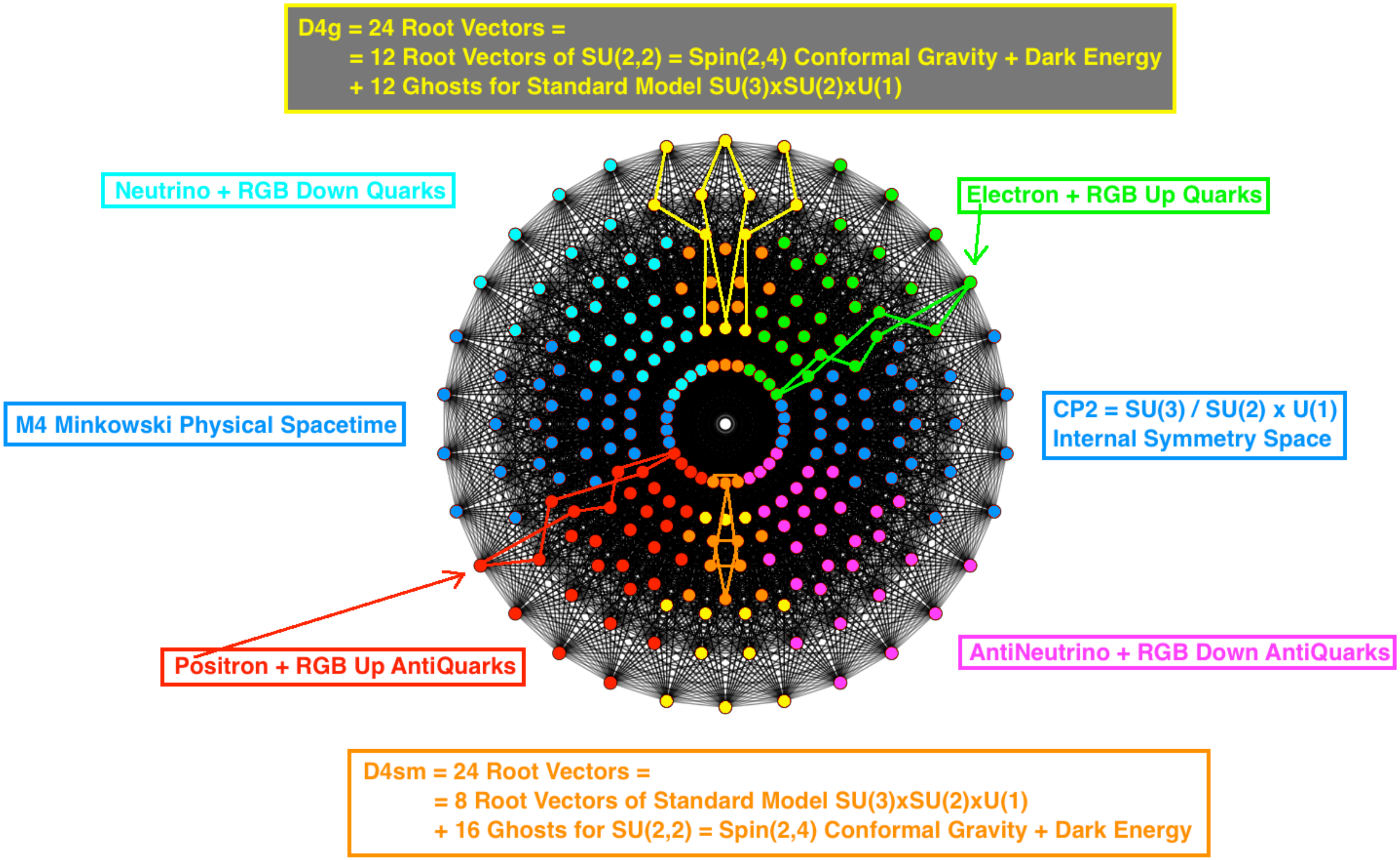
and **the total Spinors + Spacetime** has 192 vertices = 96 binars = 24 dimers =
= 48 tetrahedra so it **has tetrahedral N = 48**

The Gauge Boson + Ghosts part = D4xD4 contains 48 vertices = 24 binars = 6 dimers
= 12 tetrahedra so it **has tetrahedral N = 12**



E8
8D gauge bosons + ghosts

and **the total Spinors + Spacetime + Gauge Bosons + Ghosts has 240 vertices =**
= 120 binars = 30 dimers = 60 tetrahedra so the total E8 tetrahedral N = 60


D4g = 24 Root Vectors =
= 12 Root Vectors of SU(2,2) = Spin(2,4) Conformal Gravity + Dark Energy
+ 12 Ghosts for Standard Model SU(3)xSU(2)xU(1)
Neutrino + RGB Down Quarks
Electron + RGB Up Quarks
M4 Minkowski Physical Spacetime
CP2 = SU(3) / SU(2) x U(1)
Internal Symmetry Space
Positron + RGB Up AntiQuarks
AntiNeutrino + RGB Down AntiQuarks
D4sm = 24 Root Vectors =
= 8 Root Vectors of Standard Model SU(3)xSU(2)xU(1)
+ 16 Ghosts for SU(2,2) = Spin(2,4) Conformal Gravity + Dark Energy

Dimer Packing and QuasiCrystals

In arXiv 1106.4765 Haji-Akbari, Engel, and Glotzer said:

"... Phase Diagram of Hard Tetrahedra ...

Two dense phases of regular tetrahedra have been reported recently.
The densest known tetrahedron packing is achieved in a crystal of triangular bipyramids (dimers) ... phase DIII ... triclinic ... with packing density 4000 / 4671 = 85.63%.
In simulation a dodecagonal quasicrystal is observed;
its approximant, with periodic tiling (3,4,3^2,4),
can be compressed to a packing fraction of 85.03%. ...

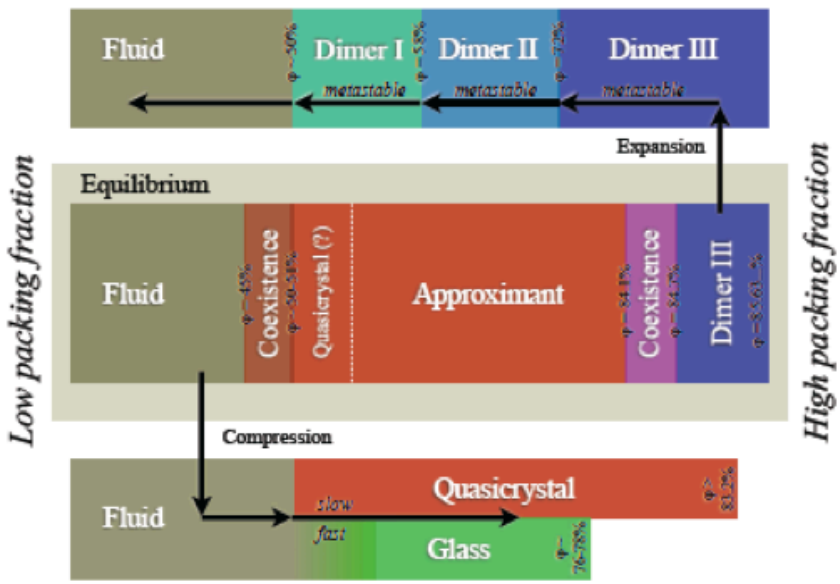
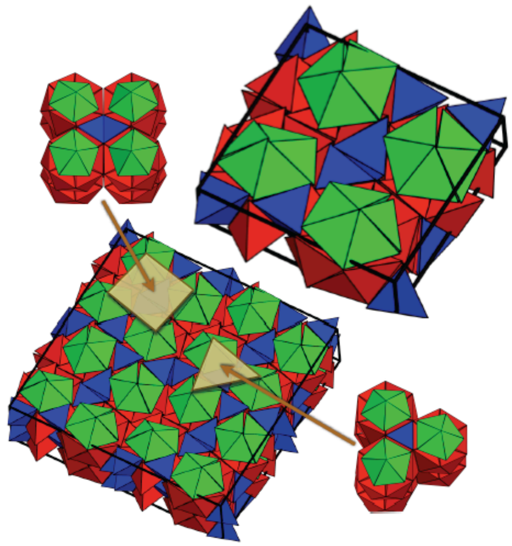


FIG. 11: Schematic phase diagram of hard tetrahedra summarizing our findings. In thermodynamic equilibrium the Dimer III crystal and the approximant are stable (Middle panel). In compression simulations the approximant is never observed, and only the quasicrystal forms. If crystallization is suppressed, then a jammed packing with local tetrahedral order forms [29, 36] (Lower panel). The transformation of the approximant or quasicrystal directly to and from the Dimer III crystal is not observed in simulation. Instead, during expansion the Dimer III crystal transforms into the Dimer II crystal, and then the Dimer I phase prior to melting to the fluid (Upper Panel).

... The phase DIII ... triclinic ... is thermodynamically stable,
DII ... monoclinic ... and Di ... rhombohedral ... are metastable ...
The transformation of the approximant or quasicrystal directly to and from the Dimer III crystal is not observed ... Instead, during expansion the Dimer III crystal transforms into the Dimer II crystal, and then the Dimer I phase prior to melting ...

... **Structurally,**
the quasicrystal is significantly more complicated than the dimer phase;
tetrahedra are arranged into rings that are further capped with pentagonal dipyramids (PDs). The rings and PDs are stacked in logs parallel to the ring axis, which in projection form the vertices of a planar tiling of squares and triangles ...



... Additional particles - referred to as interstitials -
appear in the space between the neighboring logs.
It is noteworthy that the entire structure is a network of interpenetrating PDs spanning all particles in the system.
A periodic approximant of the quasicrystal, i.e. a crystal approximating the structure of the quasicrystal on a local level, with the (3**,** 4**,** 3^2, 4) Archimedean tiling
and 82 tetrahedra per unit cell compresses up to **...** 85**.**03%,
only slightly less dense than the dimer crystal ...
In this paper we demonstrate that the approximant is more stable than the dimer crystal up to very high pressures and that the system prefers the dimer crystal thermodynamically only at packing densities exceeding 84%. ...".

The quasicrystal QC is a cut-and-projection from a full E8 lattice
and so any QC loses by projection some of the full E8 information,
and the lost part of the E8 information corresponds
to complicated empire-phason structure of the QC,
so

the **complexity of the QC phase is due to**
its failure to connect with full E8 information.

For example, consider the Elser-Sloane 4D QuasiCrystal described by them
in J. Phys. A: Math. Gen. 20 (1987) 6161-6168 where they say:
"... Let V be 8D Euclidean space with orthonormal basis e1, ... , e8
...
The unit icosians consist of ... 120 quaternions ...
the icosians ... with the Euclidean ... rational number ... norm lie in a real 8D space
and form a lattice isomorphic to the E8 lattice ...
the Weyl group W(E8) ... is ... [t]he point group G0 of this lattice
...
There are 240 icosians of Euclidean norm unity, consisting of the unit icosians and
sigma ... = (1/2)(1 - sqrt(5)) times the unit icosians,
and these correspond to the 240 minimal vectors of the E8 lattice
...
the group G1 = [3,3,5] ... consist[s] ... of all transformations of the icosians ...
G1 has order 14,400 ...[and]... acts on V as a subgroup of G0 ...
There are two 4D subspaces X and Xbar of V that are invariant under the action of G1
...
We note that E8 has only the origin in common with either of the spaces X or Xbar
...
The Voronoi cell W of E8 is defined by W = { Q in V : II Q II ≤ II Q - P II for all P in E8 }
... The Voronoi cell W is a convex 8D polytope ...[with]... 19,400 vertices ...
The ... [Elser-Sloane] quasicrystal involves the 4D polytope S ... obtained by
projecting W onto the subspace Xbar ...
... The polytope S is the convex hull of the projection of these ... W ... vertices onto Xbar
... to project onto Xbar ...multiply... by

$$\Phi = \begin{bmatrix} c(I+\sigma H) & \tilde{c}(I+\tau H) \\ c(I-\sigma H) & \tilde{c}(I-\tau H) \end{bmatrix}$$

where $I = I_4 = \mathrm{diag}\{1, 1, 1, 1\}$,

$$c = (4+2\sigma)^{-1/2} = 0.602\ldots \qquad H = \tfrac{1}{2}\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

and take the last four coordinates ... S has 720 vertices ...
120 vertices of a copy of the polytope {3,3,5} ...
600 vertices of a copy of the reciprocal polytope {5,3,3} (the 120-cell) ...
S is the convex hull of reciprocal (and concentric) polytopes {3,3,5} and {5,3,3} ,
arranged so that the midpoints of the edges of the {5,3,3} pass through the
centres of the triangular faces of the {3,3,5} ...
S is a 4D analogue of the triacontahedron ... convex hull of ... {3,5} ... and {5,3} ...
arranged so that the midpoints of their edges coincide
...

The **4D quasicrystal C is obtained by projecting the lattice E8**

onto the subspace X ,

subject to the requirement that the projection onto Xbar lies in the polytope S

...

(i) C is invariant under a point group (fixing the origin) isomorphic to G1 = [3,3,5] ...

(ii) C is closed under multiplication by tau ... = (1/2)(1 + sqrt(5)) [Golden Ratio] ...

(iii) C is a discrete set of points ...

(iv) ... 120 of the 240 minimal vectors of E8 project into C ... forming a copy of {3,3,5}
Similarly ... 120 of the 2160 vectors in E8 of length 2 project into C ... forming a ...
larger {3,3,5} concentric with the first ...

(v) **C has a cross section which is a 3D quasicrystal with icosahedral symmetry**. ...".

Boyle and Steinhardt in arXiv1608.08215 and arXiv1604.06426 say:

"... H4 root QL ... corresponds to the icosians ...

then the maximally-symmetric 4D orthogonal projection of the E_8 roots may be achieved by taking the eight columns of the 8×8 matrix

$$\left(\mathbf{v}_1^+ \ \mathbf{v}_2^+ \ \mathbf{v}_3^+ \ \mathbf{v}_4^+ \ \mathbf{v}_1^- \ \mathbf{v}_2^- \ \mathbf{v}_3^- \ \mathbf{v}_4^- \right) = \begin{bmatrix} (I+\sigma H) & (I+\tau H) \\ (I-\sigma H) & (I-\tau H) \end{bmatrix} \qquad (5.11)$$

as an orthogonal basis in eight dimensions, and choosing $\{\mathbf{v}_1^+, \mathbf{v}_2^+, \mathbf{v}_3^+, \mathbf{v}_4^+\}$ as a basis for the $\|$ space, while $\{\mathbf{v}_1^-, \mathbf{v}_2^-, \mathbf{v}_3^-, \mathbf{v}_4^-\}$ are a basis for the $\perp$ space. With this choice, the 240 E_8 roots project onto the parallel space to yield two copies of the 120 H_4 roots (an inner copy and an outer copy that is longer by τ).

...".

Physically, 8D E8 gives two 4D 600-cells,
one in X = vll and the other in Xbar = v_l_

which in E8 physics represent M4 and CP2 of 8-dim Kaluza-Klein spacetime M4 x CP2
Therefore, in terms of E8 Physics based on physical interpretation of Root Vectors,
each of the two 600-cells contains one D4 of D4xD4 in D8 / D4xD4 of E8
and
the 600-cell with D4grav represents M4 spacetime and Gravity+DarkEnergy
and
the 600-cell with D4stdmod represents CP2 symmetry space and Standard Model.

An Elser-Sloane 4D QC is based on either one or the other of those two 600-cells
each of which has 120 vertices corresponding to 120 of the 240 E8 Root Vectors
so an

Elser-Sloane 4D QC cannot describe more than 120 / 240 = = half of E8 Physics.

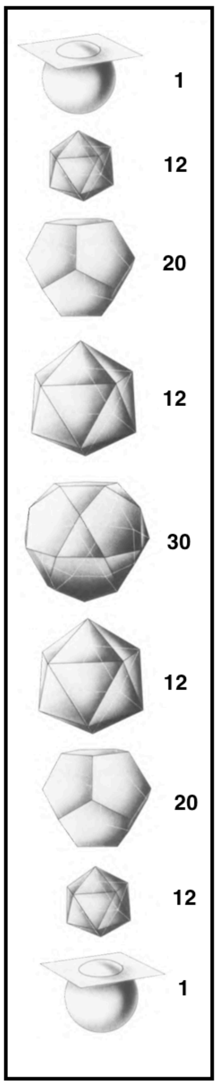
A 3D QC based on 4D 600-cells is even more limited

in the parts of E8 Physics that it can describe,
being based on a cross section of the 600-cell of Elser-Sloane 4D QC
which cross sections have only a subset of the 120 vertices of the 600-cell.

Here are some cross section slices of a 600-cell

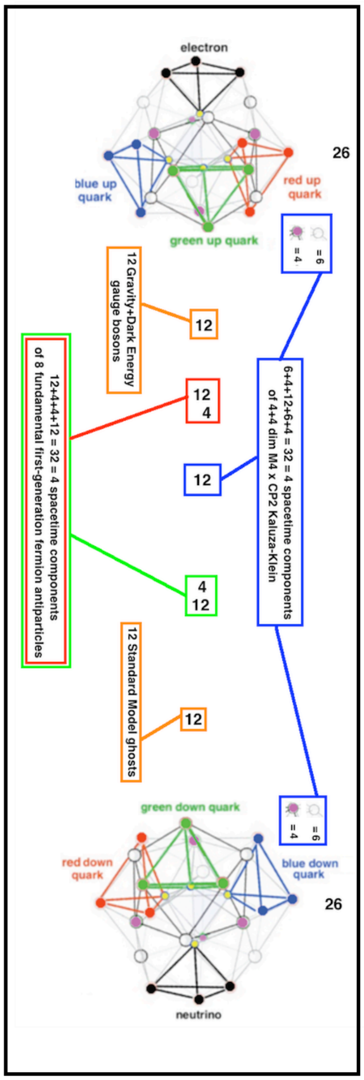
(see “Geometrical Frustration” (Cambridge 1999, 2006) by Sadoc and Mosseri)

vertex first

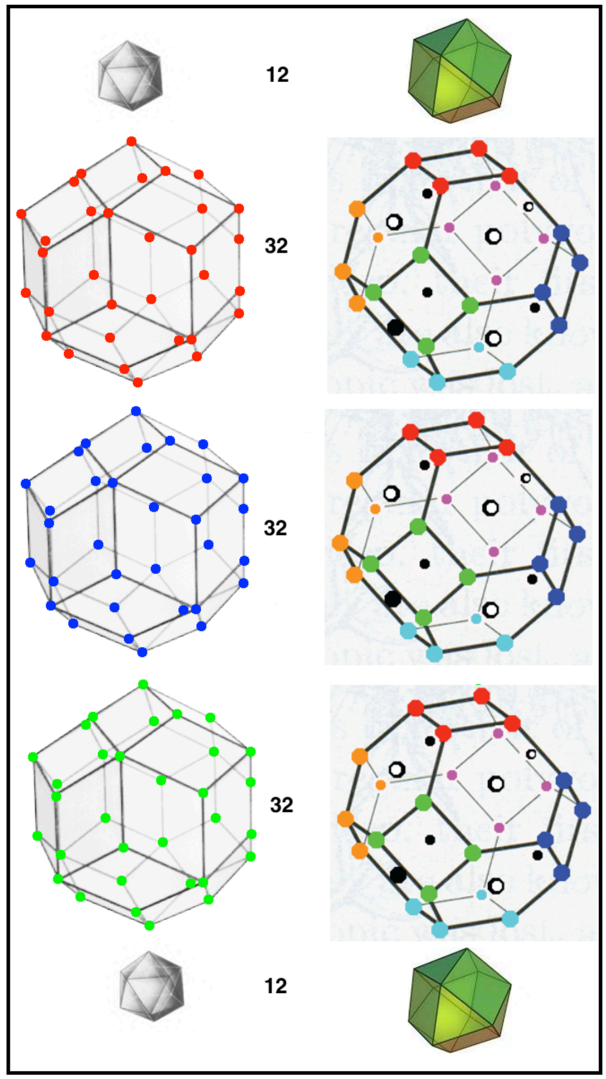


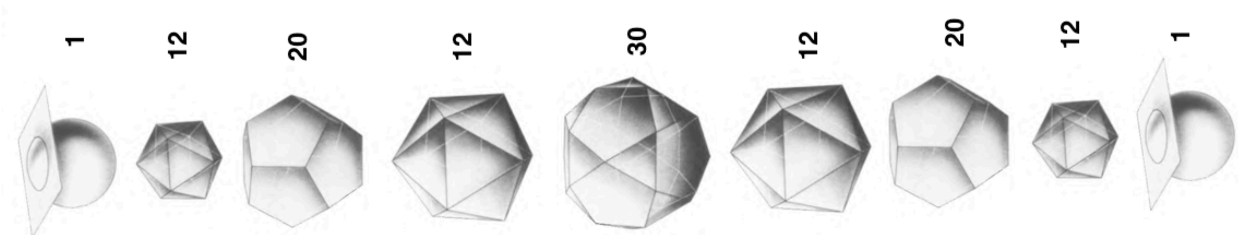
cell first

57G contact neighbors



rhombic triacontahedra
jitterbugs with
truncated octahedra





Vertex-first Tetrahedral Slice Structure:

At Equator is the 30-vertex icosidodecahedron + top and bottom vertices = 32 vertices corresponding to 4 momentum dimensions of 4-dim physical spacetime M4 time 4+4 = 8 dimensions of a M4 x CP2 Kaluza-Klein spacetime where CP2 = SU(3) / SU(2) x U(1) is a compact internal symmetry space carrying the symmetry groups of the Standard Model.

Adjacent to the icosadodecahedron on either side are 20+12 vertices of dodecahedron +icosahedron whose convex hull is the 32-vertex Rhombic Triacontahedron (RTH).
The upper 20+12 = 32 vertices represent 4 covariant components of 4-dim M4 Physical Spacetime of 8 first-generation fermion fundamental particles
(electron, RGB up quarks; neutrino, RGBdown quarks)
and the lower 12+20 = 32 vertices represent 4 covariant components of 4-dim M4 Physical Spacetime of 8 first-generation fermion fundamental particles
(positron, RGB up antiquarks; antineutrino, RGBdown antiquarks).

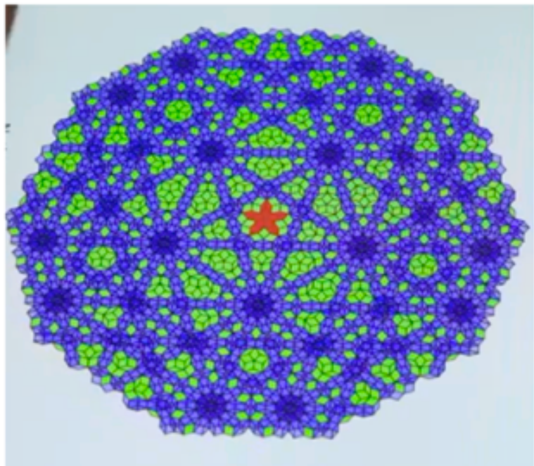
The upper and lower 12-vertex icosahedra represent
the 12 Root Vectors of the SU(2,2) = Spin(2,4) Conformal Group
that gives, by a MacDowell-Mansouri mechanism, Gravity+ Dark Energy
and
the ghosts of the 12 gauge bosons of the SU(3)xSU(2)xU(1) Standard Model.

The Vertex-first structure has H3 icosahedral symmetry
that is inherited from the H4 symmetry of the 600-cell.

The 3D QC quasicrystal does not contain directly in its vertices
all the physics information of all 240 E8 Root Vector vertices
so, due to the missing information, it has a complicated empire - phason structure.

Given a star-like central configuration of a 3D QC such as an icosahedron, its Empire is that part of the 3D QC that is an accurate copy of part of the E8 parent lattice
and
its Phasons are ribbon-like areas of the 3D QC for which projection did not give full information about the E8 parent lattice,
which ignorance allows flips between possible alternative configurations.

Empires and Phasons are described by Fang in a 2D example: "...



... the green area …[has] only one way to tile legally …
these tiles must be forced by the red patch [star]…
The green tiles are called the Empire Field of the red patch [star] ...
the blue area there are multiple ways of tiling …
the blue ribbons are superpositions of left and right [Phason] flip …".

In arXiv 1511.07786 Fang and Klee Irwin describe how QC with Phason Ribbons may be related to Fibonacci Chains:

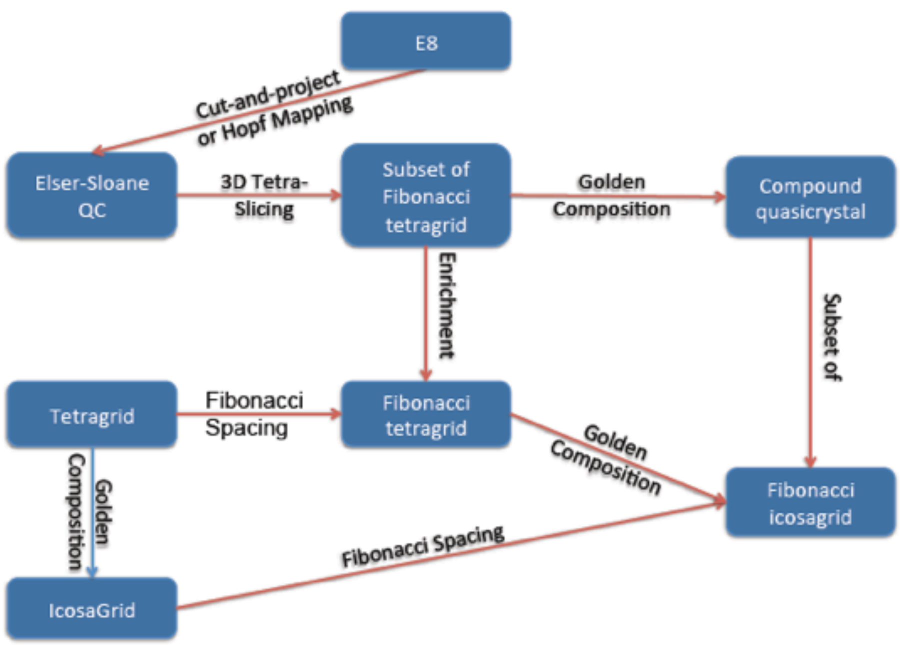


Figure 19: *The relationships between FIG and CQC and how they are generated.*

Klee Irwin, in Toward the Unification of Physics and Number Theory, said:
"... the simplest quasicrystal possible is the two length ... 1D ... Fibonacci chain ...
It possesses two lengths related as the golden ratio. In order for a quasicrystal greater than 1D to have only two letters, the letters must be 1 and the inverse of the golden ratio. ... When a slice of E8 is projected to 4D according to a non-arbitrary golden ratio based irrational angle, the resulting quasicrystal is made entirely of 3-simplexes and is the only way to project that lattice to 4D and retain H4 symmetry. ...
This quasicrystal ... can be described as a network of Fibonacci chains ...
Changing a single point to be on or off in a Fibonacci chain 1D quasicrystal forces an infinite number of additional points throughout the possibility space of the 1D chain to also change state. When a network of Fibonacci chains is formed in 2D, 3D or 4D, a single binary state change at one node in the possibility space changes Fibonacci chains throughout the entire 1+n dimensional network of chains ...
the special dimensions for Fibonacci chain related quasicrystals are 1D, 2D, 3D and 4D. And of these dimensions, 4D can host the quasicrystal with the densest network of Fibonacci chains,
where 60 Fibonacci chains share a single point at the center of the 600-cells in the E8 to 4D quasicrystal discovered by Elser and Sloane ...[they] appear to be the maximum possible density of Fibonacci chains in a network of any dimension ...
3D quasicrystals ordinarily have a maximum of degree 12 vertices with six shared Fibonacci chains. Fang Fang of Quantum Gravity Research discovered how to create a 3D network of Fibonacci chains with degree 60 vertices ...".

Boyle and Steinhardt in arXiv 1608.08220 , arXiv 1608.08215 ,arXiv 1604.06426 say:
"… Unlike an ordinary lattice, which has no scale invariance,
each reflection QL has discrete scale invariance …
WE CANNOT ENUMERATE ALL THE REFLECTION QLs IN 2D,
WE CANNOT ENUMERATE ALL … FACTORS IN 2D …
THE … UNIQUE … MAXIMAL REFLECTION QL … EXISTS IN 4D ...
there is a unique reflection QL … quasilattice … /\ … in 4D …
Every vector … in /\ can … be written as an integer combination of the 120 H4 roots
…
/\ must contain all the golden integers times each H4 root,
and all integer linear combinations of such vectors …
and … it cannot contain anything else
…
it is unique … H4 root QL … corresponds to the … icosians,
which may be obtained by orthogonally projecting the E8 root lattice
on a maximally symmetric 4D subspace …
…
the 240 E8 roots project onto the parallel space
to yield two copies of the 120 H4 roots
(an inner copy and an outer copy that is longer by T [Golden Ratio]) ...
the scaling group of the QL must be a subgroup of the scaling group of its 1D sublattice
... H3 and H4 contain Z(sqrt(5))
...

choose … the minimal star … a 120-pointed star pointing towards the vertices of the 600-cell
… and …

$\lambda_\pm$	τ	$m_2^\pm/m_1^\pm$	S'	L'
$\frac{1}{2}(1 \pm \sqrt{5})$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\frac{1}{2}(1 \pm \sqrt{5})$	$\frac{L}{2}\frac{L}{2}$	$\frac{L}{2}S\frac{L}{2}$

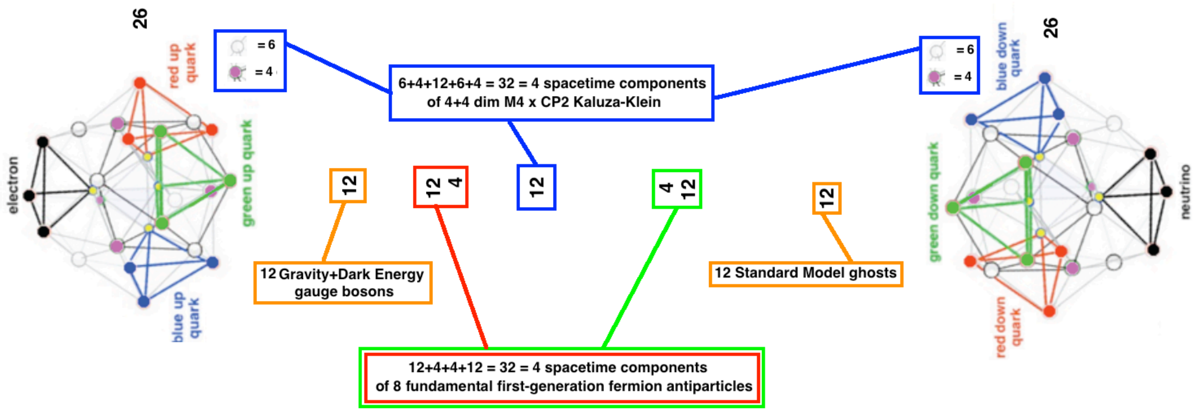
Table 1. Catalog of self-similar quasilattices relevant to constructing higher-dimensional Ammann patterns and Penrose-like tilings in [12]. In this table, we use the convenient notation $\lambda_\pm$ and $m_i^\pm$ where here the superscript/subscript "+" stands for the former subscript/superscript "$\|$", while the "−" stands for "$\perp$". the scaling factor is the "golden ratio", $\lambda_\| = \phi = (1+\sqrt{5})/2$, which is the relevant case for describing systems with 5-fold or 10-fold order in 2D, some systems with icosahedral (H_3) order in 3D, and systems with "hyper-icosahedral" (H_4) order in 4D

… the unique reflection quasilattice corresponding to H4 is the H4 root quasilattice (i.e. the set of all integer linear combinations of the H4 roots) …
H4 reflection QL contain[s] a 1D sublattice corresponding to a ring … Z(sqrt(5) ...
the fundamental unit … = T = (1/2)(1+5) (the golden ratio) …".

In other words, an E8 lattice is made up of two H4 quasilattice 4D QC
(one scaled by integers and the other by Golden Ratio) and
each H4 4D QC is a 120-point star plus Fibonacci Chains based on the 60 Fibonacci Chains through pairs of antipodal points of the 120-point star.

Compare
Fibonacci Chains / Phason Ribbons of Vertex-first Icosahedral Structures
with
Cellular Automata of Truncated Octahedra / Cuboctahedra derived from Rhombic Triacontahedra / Icosahedra by Jitterbug Transformation.

Therefore the
Vertex-first Tetrahedral Slice Structure allows construction of a Realistic Physics Model
IF you can generate
the Standard Model gauge bosons from their ghosts
and
the Gravity+Dark Enery ghosts from their gauge bosons
and
the 4D CP2 components of fermions and spacetime from the existence of M4 x CP2 Kaluza-Klein



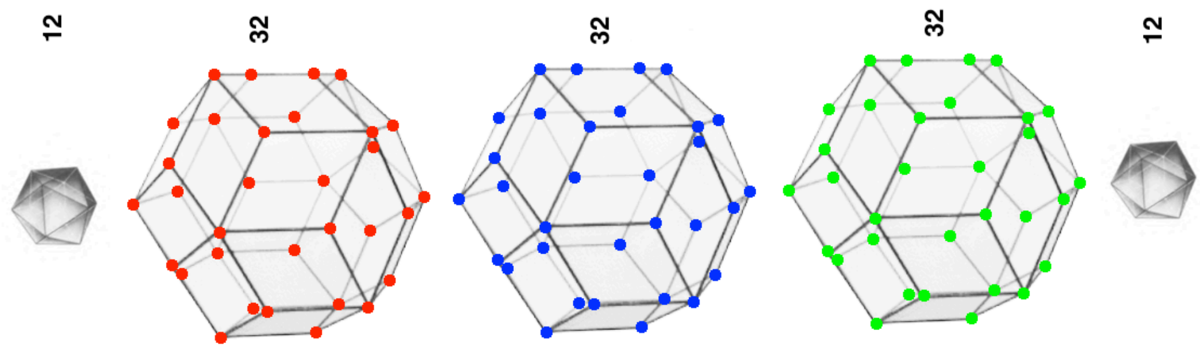
Cell-first Tetrahedral Slice Structure with 57G:

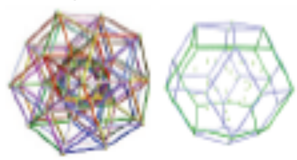
The top and bottom structures are **26-vertex groups of 57 tetrahedra (57G)**
which are the maximal number of tetrahedra in a group
all in contact with each other within the 600-cell.

This configuration most clearly shows how individual tetrahedra represent individual fermions

but

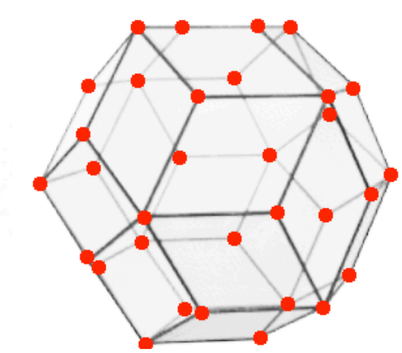
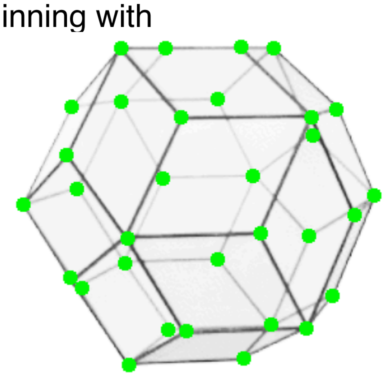
vertices with similar physical interpretation are not grouped together as nicely as in Vertex-first Slicing or as with Rhombic Triacontahedra.



The 32-vertex **Rhombic Triacontahedron (RTH)**, is a combination of the 12-vertex Icosahedron and the 20-vertex Dodecahedron. It "forms the convex hull of ... orthographic projection ... using the Golden ratio in the basis vectors ... of a 6-cube to 3 dimensions." (Wikipedia).  Sharp  and Flat  golden rhombohedra are the basis for constructing Rhombic Triacontahedra.

The 32-vertex Rhombic Triacontahedron does not itself tile 3-dim space but it is important in 3-dim QuasiCrystal tiling. Mackay (J. Mic. 146 (1987) 233-243) said "... the basic cluster, to be observed everywhere in the three-dimensional ... Penrose ...tiling ...[is]... a rhombic triacontahedron (RTH) ... The 3-D tiling can be regarded as an assembly of such RTH, party overlapping ...".

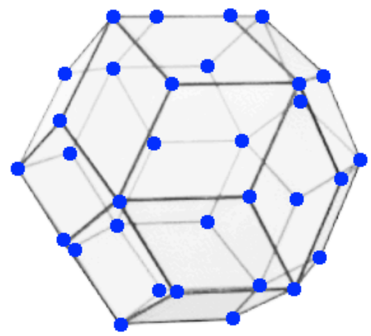
To look at tiling 3-dim space by Rhombic Triacontahedra, the first step is to describe the physical interpretation of the Rhombic Triacontahedra, beginning with

 and 

which are interpreted as 4x8 = 32 vertices representing
4 covariant components of 4-dim M4 Physical Spacetime of 8 first-generation fermion fundamental particles (electron, RGB up quarks; neutrino, RGBdown quarks)
and 4x8 = 32 vertices representing
4 covariant components of 4-dim M4 Physical Spacetime of 8 first-generation fermion fundamental particles (positron, RGB up antiquarks; antineutrino, RGBdown antiquarks)

Since fermion particles are inherently Left-Handed, their RTH is Left-Handed
and
since fermion antiparticles are inherently Right-Handed, their RTH is Right-Handed.

The third RTH with no handedness describes Spacetime as 4x8 = 32 vertices



representing 4 momentum dimensions of 4-dim physical spacetime M4
time 4+4 = 8 dimensions of a M4 x CP2 Kaluza-Klein spacetime
where CP2 = SU(3) / SU(2) x U(1) is a compact internal symmetry space
carrying the symmetry groups of the Standard Model.

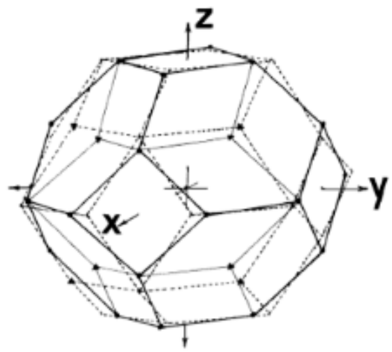
Note that the central RTH of spacetime as a Rhombic Triacontahedron
is dual to the equatorial icosadodecahedron of the vertex-first slices of a 600-cell.

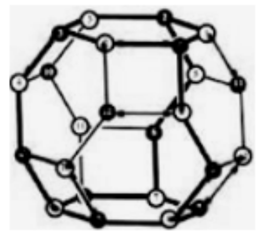
The two 12-vertex icosahedra (top and bottom slices of the 600-cell) represent

the 12 Root Vectors of the SU(2,2) = Spin(2,4) Conformal Group
that gives Gravity+ Dark Energy by a MacDowell-Mansouri mechanism
and
the ghosts of the 12 gauge bosons of the SU(3)xSU(2)xU(1) Standard Model.

Note that the cuboctahedron transforms by Jitterbug to an icosahedron
which is the top and bottom configuration for Vertex-first projection.

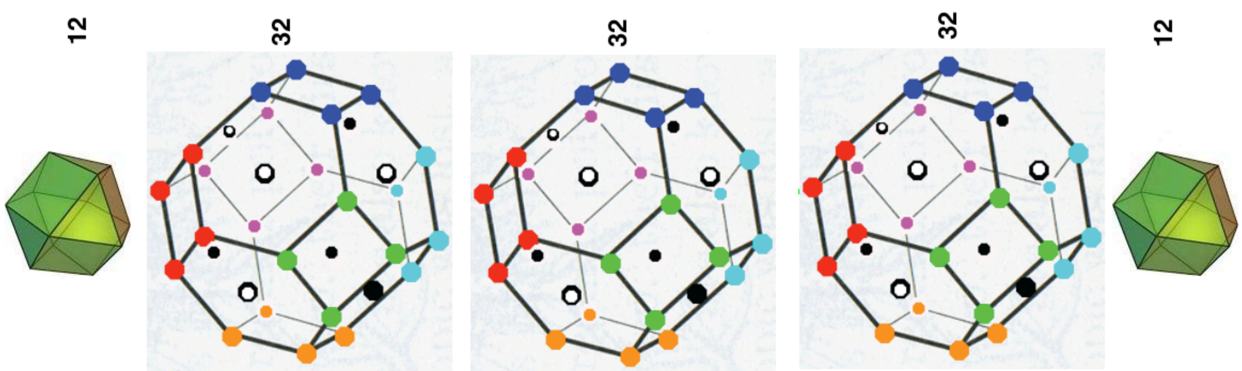
Mackay (J. Mic. 146 (1987) 233-243) said "... **a rhombic triacontahedron (RTH) ... can be deformed to ... a truncated octahedron** ... [which is] the space-filling polyhedron for body-centered cubic close packing ...





... By a similar process ... a cuboctahedr[on]... can be deformed to an icosahedron ...".

Using those Jitterbug transformations the icosahedral / rhombic triacontahedral slicing of the 24-cell goes to cuboctahedral / truncated octahedral structure



The 4x8 = 32 M4 spacetime components of 8 fermion particles
and 4x8 = 32 M4 spacetime components of 8 fermion antiparticles
are indicated by color codes

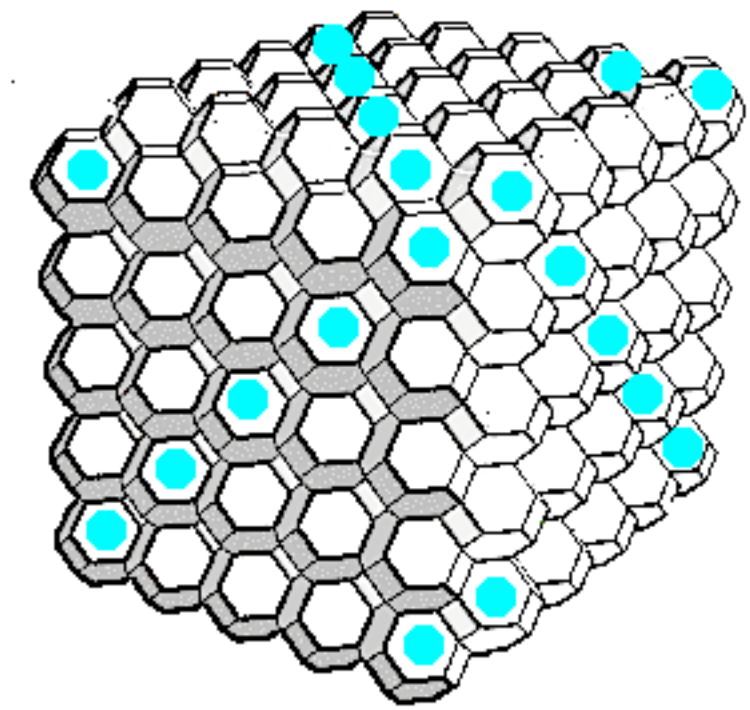
neutrino, red down quark, green down quark, blue down quark;
blue up quark, greem up quark , red up quark, electron

with the quarks at corner vertices of square faces and the leptons at centers of hexagon faces.

For the central configuration representing spacetime
the 8 dimensions of spacetime correspond to the 8 fundamental fermions.

Truncated Octahedra tile 3D space
and
the cuboctahedron has a 6-square configuration that is compatible
with the 6-square space-filling configuration of the truncated octahedron
so

the Rhombic Triacontahedra slicing can, by Jitterbug transformation
tile 3D space with transformed Truncated Octahedra
EXCEPT that some of the Truncated Octahedra (marked in cyan in the following image)
must be replaced by Cuboctahedra:
(image from apgoucher at cp4space (25 Aug 2013))



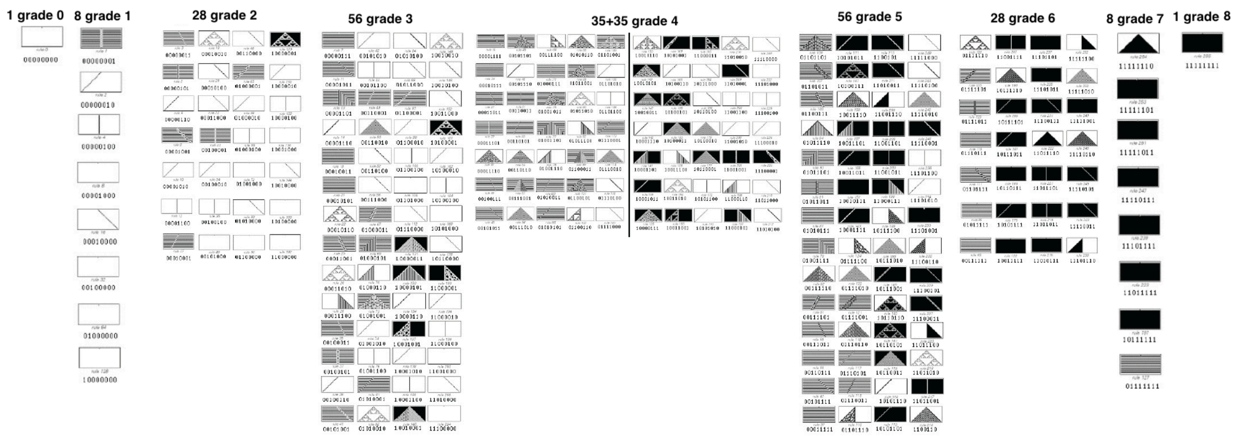
The 3D QC Quasicrystal structure of Rhombic Triacontahedra with Icosahedra
is transformed by Jitterbug into
a 3D almost-space-filling structure of Truncated Octahedra with Cuboctahedra.

Instead of the empire - phason structure of vertex-first 600-cell slicing 3D QC
with points, icosahedra, dodedahedra, and an icosidodecahedron
you have
the pattern of cuboctahedra replacements in the overall truncated octahedral 3D tiling.

From 3D Rhombic Triacontahedra to 24D Leech Lattice

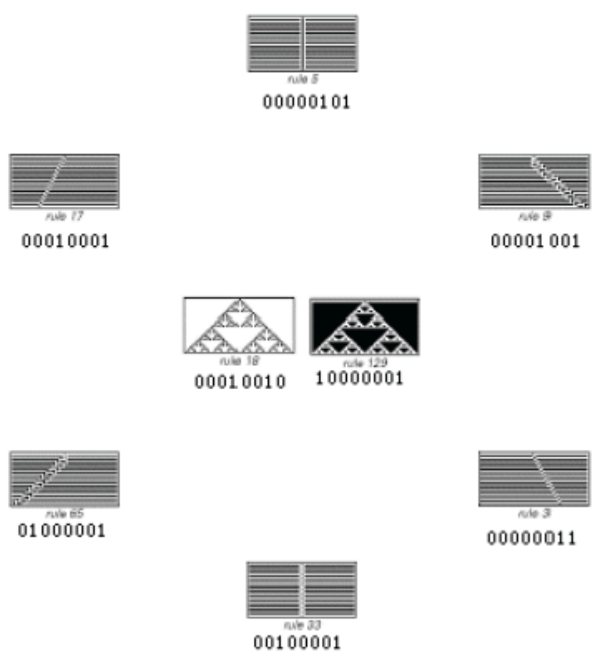
Rhombic Triacontahedra Jitterbug to Truncated Octahedra.
The Truncated Octahedra space-filling structure
is consistent with
256 Elementary Cellular Automata describing Physics based on 256-dim Cl(8)
and, by periodicity,
all tensor products of Cl(8) including Cl(8) x Cl(8) = Cl(16) containing E8

256 Elementary Cellular Automata

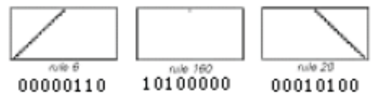


Some examples of physical interpretations of Elementary Cellular Automata are, from grade 2 representing bivector gauge bosons:

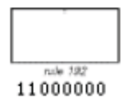
SU(3):



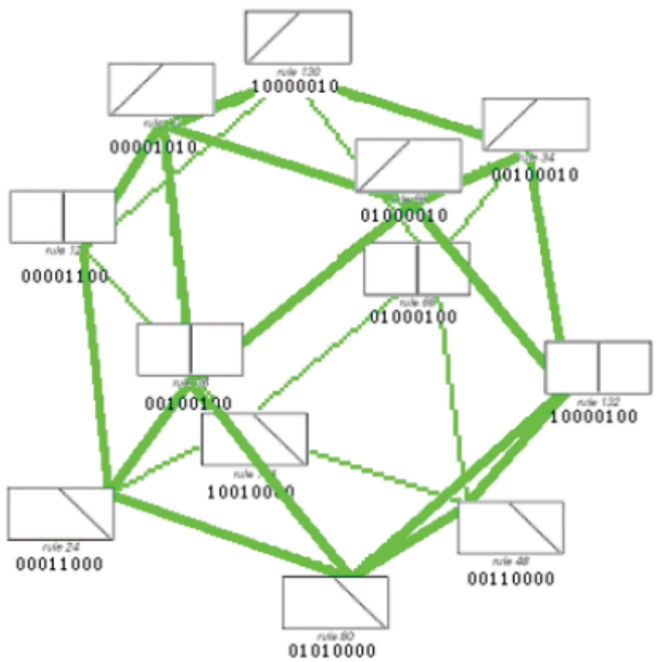
SU(2):

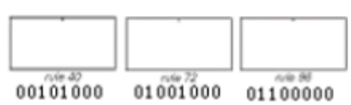


U(1):

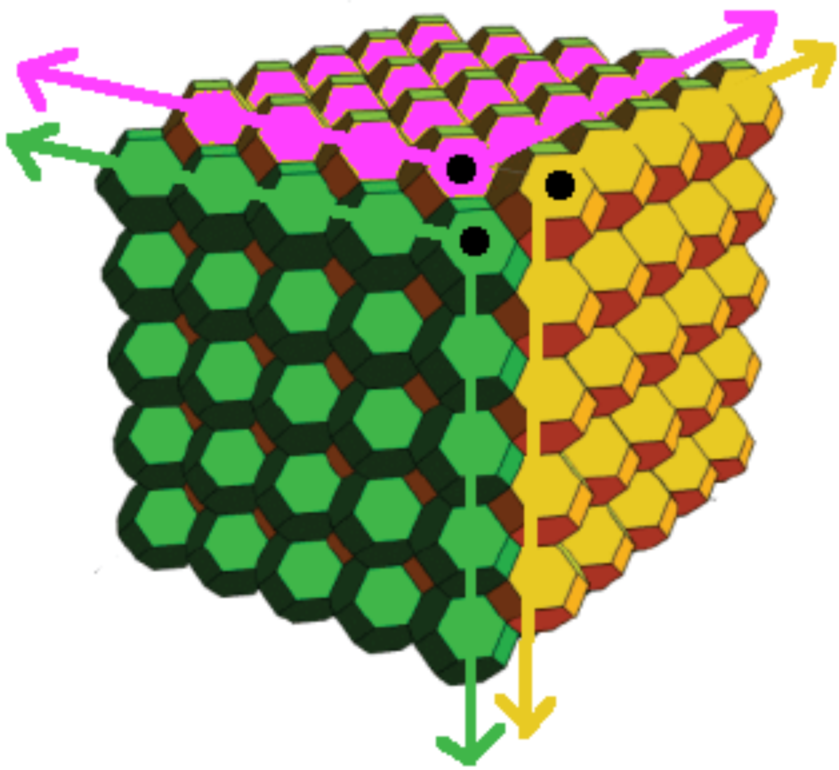


Conformal Gravity Spin(2,4) = SU(2,2):





Each of the 256 Cellular Automata can be represented by a triangular pyramid
and each of the 3 mutually perpendicular faces of the
3D Truncated Octahedra / Cuboctahedra structure can be seen as a triangular pyramid



Each of the 3 triangular pyramids (magenta, gold, green) can carry any
of the 256 Cellular Automata which correspond to the 256 elements of Cl(8).

The 8-dim Vectors of Cl(8) live in an 8-dim Integral Octonion Space E8 Lattice.
There are 7 Algebraically Independent E8 Integral Domain Lattices
corresponding to the 7 Imaginary Octonions i, j, k, e, ie, je, ke
of which 3 i, j, e are Algebraically independent.

Let the green triangular pyramid carry an E8i Lattice
and the gold triangular pyramid carry an E8j Lattice
and the magenta triangular pyramid carry an E8e lattice.

7 E8 lattice integral domains E8i E8j E8k E8e E8ie E8je E8ke
correspond to the 7 imaginary octonions i j k e ie je ke
Scale them so that their Inner Shells have Unit Radius because as Geoffrey Dixon said
"... the inner shell should ... consist of unit elements ...
[since]... O multiplication of ... unit elements is closed ...".

Consider the 240-vertex Unit Radius Inner Shells of E8 Lattice Integral Domains corresponding to the algebraic generators i j e of the imaginary octonions with coordinates of the form (E8i , E8j , E8e)

E8i itself has 240 vertices (x , 0 , 0)
E8j itself has 240 vertices (0 , x , 0)
E8e itself has 240 vertices (0 , 0 , x)

Then, consider the 240 + (240 + 16 x 240) = 4320 vertices
of Unit Radius Inner Shells of Λ16 Barnes-Wall Lattices
constructed from pairs of E8 Lattices using Dixon's XY-product
with X and Y in the pairs

E8i x E8j with 16x240 = 3840 new vertices (x , y , 0)
E8j x E8e with 16x240 = 3840 new vertices (x , 0 , y)
E8e x E8i with 16x240 = 3840 new vertices (0 , x , y)

Then, consider the 61,440 = 16x16x240 vertices of the second shell of Barnes-Wall Λ16 rescaled for Unit Radius constructed from triples of E8 Lattices using Dixon's XY-product with X and Y outside the E8i, E8j, E8e and their Λ16 Lattices

(E8i x E8j) x E8e with 16x16x240 = 61,440 vertices (x , y , z)
(E8j x E8e) x E8i with 16x16x240 = 61,440 vertices (x , y , z)
(E8e x E8i) x E8j with 16x16x240 = 61,440 vertices (x , y , z)

The total inner vertices = 3 x (240 + 3840 + 61,440) = 196,560
correspond to the **inner-shell vertices of the 24-dim Leech Lattice**

One Cell of E8 26-dimensional Bosonic String Theory with structure J(3,O)o
with Strings being physically interpreted as World-Lines
and massless spin-2 states are interpreted as carriers of Bohm Quantum Potential
can be described by taking the quotient of its
24-dimensional O+, O-, Ov subspace modulo the 24-dimensional Leech lattice.

Therefore:

3D Rhombic Triacontahedron -- Jitterbug --> 3D Truncated Octahedron
which fills 3D space with each node corresponding to
3 Elementary Sets of Cellular Automata (CA)
each of which corresponds to an E8 Lattice so that
the 3 Sets of CA represent a 24D Leech Lattice
underlying the structure of the 26D String Theory of E8 Physics AQFT
based on Strings as World-Lines and
massless spin-2 states as carriers of Bohm Quantum Potential

From 600-cell to Superposition of 8 E8 Lattices

Start with the 4D H4 QC QuasiLattice whose origin-neighbor vertices form a 600-cell

Radius	Interior	Sphere
1	0	120 

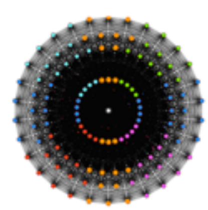
with unit Radius. It has 120 vertices whose physical interpretations are

32 blue = 4D M4 Minkowski part of 8D M4xCP2 Kaluza-Klein Spacetime
32 green and cyan = 4 Minkowski components of 8 Fermion Particles
32 red and magenta = 4 Minkowski components of 8 Fermion AntiParticles
24 yellow = D4g Root Vectors = 12 Root Vectors of SU(2,2) Conformal Gravity
+ 12 Ghosts of Standard Model SU(3)xSU(2)xU(1)

arranged, with respect to a circle-sphere projection to 2D,
in 4 circles of 30 vertices each.

Boyle and Steinhardt in arXiv 1608.08220 , arXiv 1608.08215 ,arXiv 1604.06426 say:
"... there is a unique reflection QL … quasilattice … Λ … in 4D …
Every vector … in Λ can … be written as an integer combination of the 120 H4 roots
… Λ must contain all the golden integers times each H4 root,
and all integer linear combinations of such vectors …
it is unique … H4 root QL … correspond[ing] to the … icosians...
the scaling group of the QL must be a subgroup of the scaling group of its 1D sublattice
... H4 contain[s] Z(sqrt(5)) ... scaling factor is the "golden ratio" ... (1 + sqrt(5)) / 2 ...".

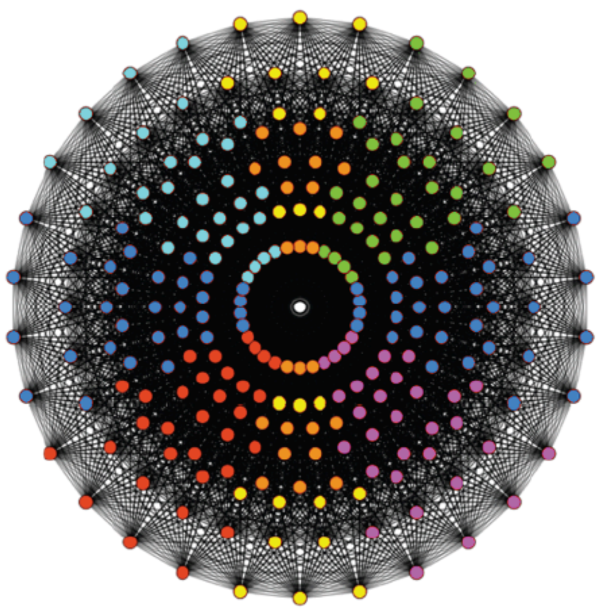
Therefore, the second shell of the 4D H4 QC QuasiLattice is also a 600-cell
whose expanded Radius is the "golden ratio" ... (1 + sqrt(5)) / 2 = 1.61
arranged in 4 circles of 30 vertices each with physical interpretations

1.61	120 	120 

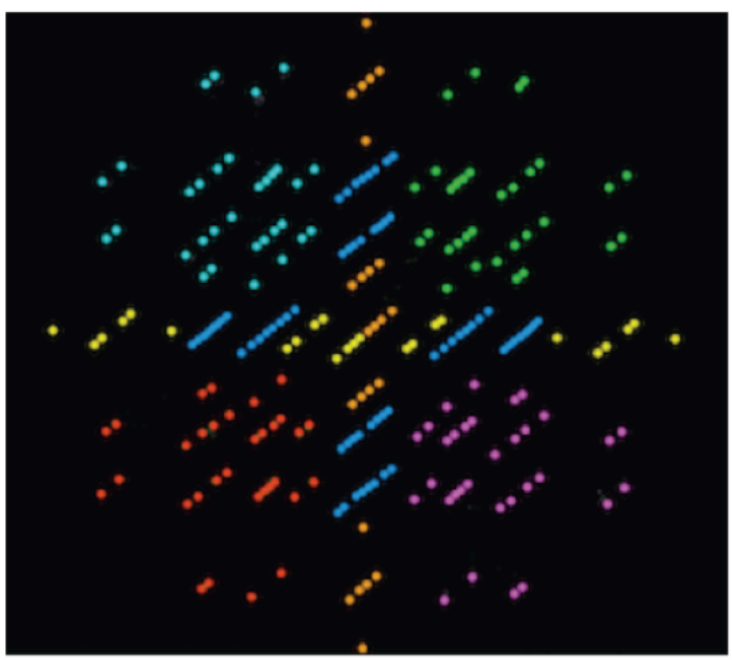
32 blue = 4D CP2 Internal Symmetry part of 8D M4xCP2 Kaluza-Klein Spacetime
32 green and cyan = 4 CP2 components of 8 Fermion Particles
32 red and magenta = 4 CP2 components of 8 Fermion AntiParticles
24 orange = D4sm Root Vectors = 8 Root Vectors of Standard Model SU(3)xSU(2)xU(1)
+ 16 Ghosts of U(2,2) of Conformal Gravity

Boyle and Steinhardt in arXiv 1608.08220 , arXiv 1608.08215 ,arXiv 1604.06426 say:
"... H4 root QL … corresponds to the … icosians,
which may be obtained by orthogonally projecting the E8 root lattice
on a maximally symmetric 4D subspace …
the 240 E8 roots project onto the parallel space to yield two copies of the 120 H4 roots
(an inner copy and an outer copy that is longer by T [Golden Ratio]) ...".

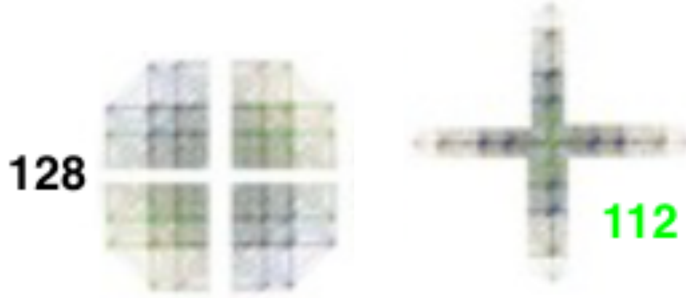
In other words, the first two shells of the 4D H4 QC QuasiLattice
form the 240 Root Vector first shell of an 8D E8 Latice with 8D norm 2x1 = 2
shown here in the circle-ball 2-dim projection with 8 circles of 30 vertices each



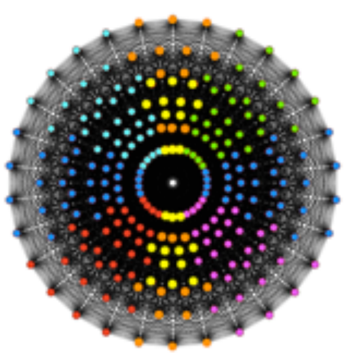
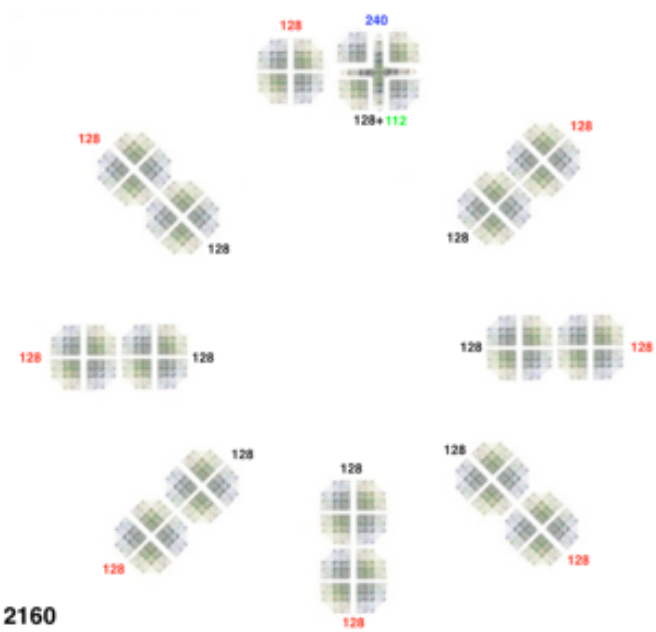
and in a square-cube 2-dim projection with the same physical interpretation color-coding



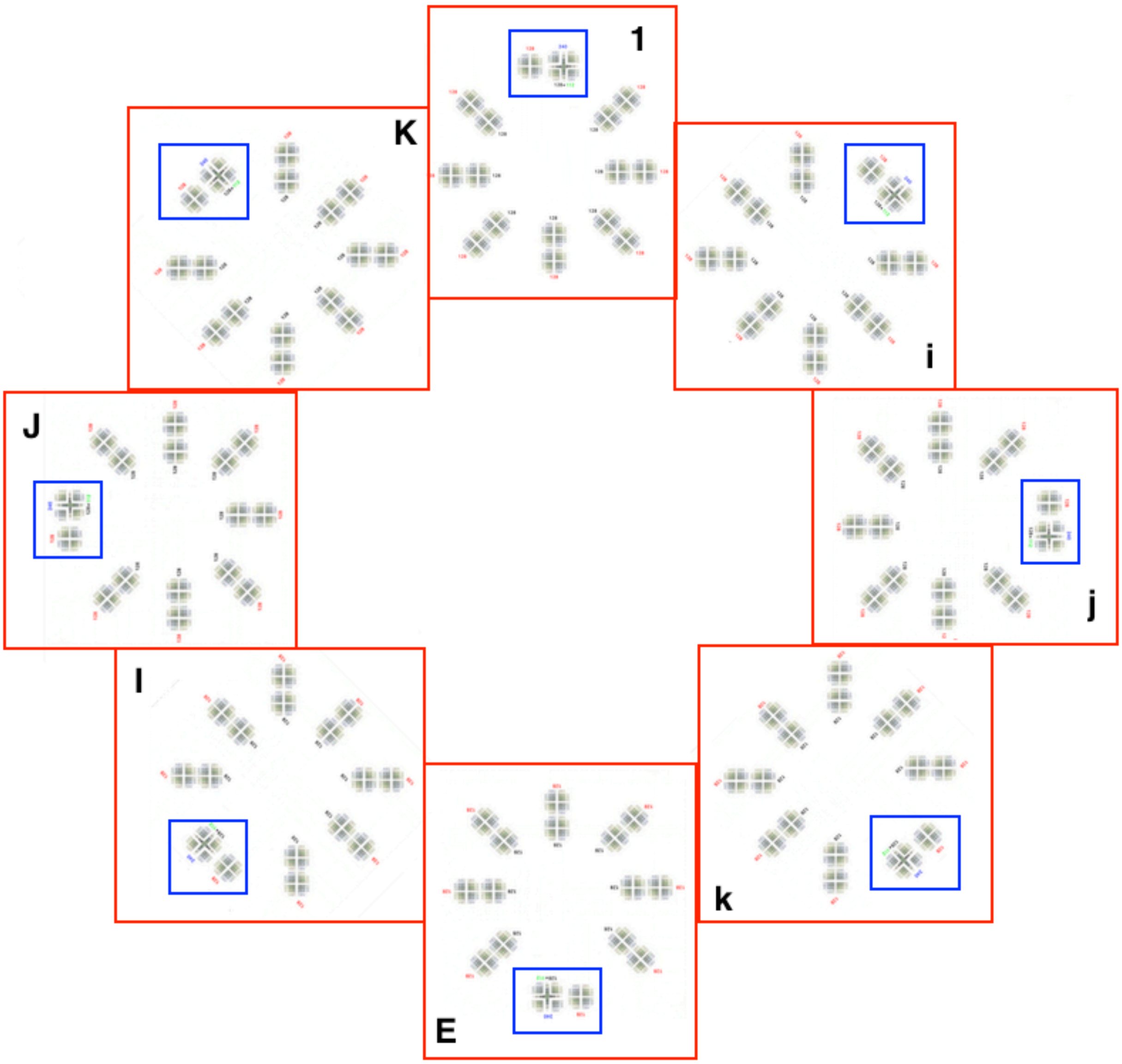
The second shell of the 8D E8 Latice, with 8D norm 2x2 = 4, has 2160 vertices:
8 pairs of 128-vertex Cl(16) half-spinors for 2048 vertices
and 112 vertices corresponding to Root Vectors of the D8 subalgebra of E8



The 112-vertex D8 combines with a left-handed 128-vertex Cl(16) half-spinor,
representing E8 Physics Fermion Particles, which are left-handed,
to form a 240-vertex configuration like the E8 Root Vector Gosset Polytope

8D E8 Shells 1 and 2	 240	 2160

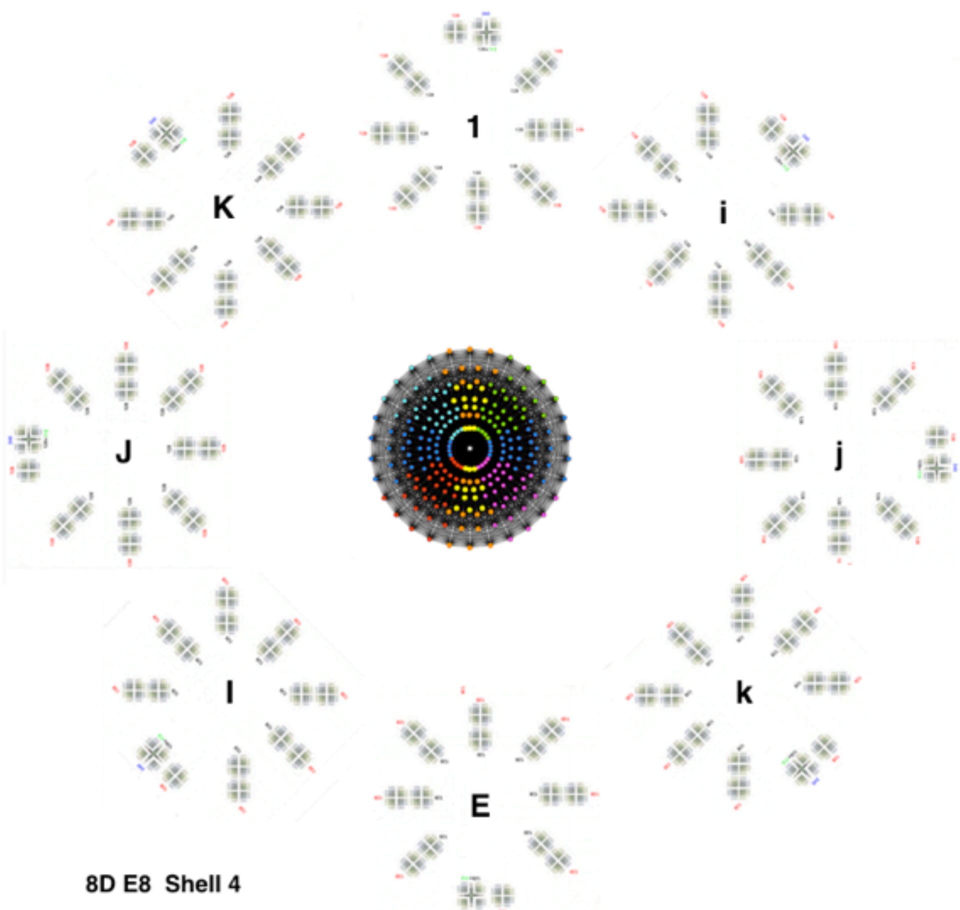
Since there are 8 pairs (left-handed and right-handed) of 128-vertex Cl(16) half-spinors in the second E8 shell, if you require the 112-vertex D8 to combine with the left-handed half-spinor to form a 240-vertex E8 Root Vector Polytope for an E8 Physics model with realistic left-handed half-spinors representing Fermion Particles,
then
there are 8 ways you can produce an E8 Lattice for E8 Physics.
7 of the 8 ways produce algebraically distinct independent Integral Domains, corresponding to the 7 imaginary Octonions i, j, k, E, I, J, K
The 8th way is not an algebraically independent Integral Domain and it corresponds to the real Octonion 1.



In E8 Physics, 8-dim Spacetime (described by the D8 / D4xD4 part of E8) is a Superposition of Spacetimes of each of those 8 E8 sets of 240 Root Vectors.

If you consider each of those 8 E8 sets of 240 Root Vectors
as a first shell of a second-order E8 Lattice
each having a second shell of 2160 Vertices
then
you get 8 x 2160 = 17280 Vertices
and
if you add 240 Vertices of a first-shell set of E8 Root Vectors
then
you get the 17280 + 240 = 17520 Vertices
of the 4th shell of an E8 Lattice

as described by the E8 theta series
Wikipedia: "... the number of E_8 lattice vectors of norm $2n$ is 240 times the sum of the cubes of the divisors of n. The first few terms of this [theta] series are given by (sequence A004009 in ... OEIS) ... 240 ... 2160 ... 6720 ... 17520 ... 30240 ... 60480 ...".



**8D E8 Shell 4 shows explicitly not only the 240 Root Vectors of E8 Physics
but also
the 8-fold Octonionic Superposition Structure of E8 Physics.**