GRAVITY AS A COUPLING OF TWO ELECTROMAGNETIC FIELDS

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Abstract: In this work we discuss the possibility to formulate gravity as a coupling of two electromagnetic fields of equal magnitude but opposite direction.

In this work we will show that a general theory of gravity can be formulated as a coupling of two electromagnetic fields of equal magnitude but opposite direction. In this case since electromagnetism is regarded as more fundamental than gravitation therefore we adopt the view that electromagnetism should be described in its own right as an independent affine structure of a spacetime manifold without reference to any other possible metric structures, such as the Riemannian metric structure used to describe the gravitational field [1]. In field theory this approach is equivalent to postulating independent physical fields on the same background Minkowski spacetime. The spacetime manifold can be endowed with a geometric structure from which a particular affine or metric structure can be chosen to describe a physical field. Unless it is equipped with a geometric structure, the background spacetime manifold cannot describe physical laws [2]. For example, Einstein general theory of relativity can be formulated in terms of a principal frame bundle associated with a tangent bundle over an *n*-dimensional manifold [3,4]. On the other hand, in order to describe electrodynamics geometrically, an affine connection is introduced into the differentiable spacetime manifold. The introduction of such a connection can be carried out by adopting a heuristic approach modelled on parallel transport of a vector field and its covariant derivative. We assume that the change δV^α in the components of a vector V^α under an infinitesimal parallel displacement is of the form [5]

$$\delta V^\mu = -\beta \Lambda_\nu V^\mu dx^\nu \tag{1}$$

in which case the covariant derivatives are defined as

$$\nabla_\nu V^\mu = \frac{\partial V^\mu}{\partial x^\nu} + \beta \Lambda_\nu V^\mu \tag{2}$$

$$\nabla_\nu V_\mu = \frac{\partial V_\mu}{\partial x^\nu} - \beta \Lambda_\nu V_\mu \tag{3}$$

so that they transform like a tensor under general coordinate transformations. Here the quantity Λ_ν is an affine connection of the spacetime manifold, which will be identified with the electromagnetic four-vector potential. The quantity β is an arbitrary dimensional constant. The transformation law for the affine connection Λ_ν can be deduced from the transformation properties of the covariant derivative given in Equations (2) and (3). Under a general coordinate transformation $x'^{\mu} = x'^{\mu}(x^{\nu})$, the connection Λ_{ν} transforms as

$$\Lambda'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \Lambda_{\nu} \tag{4}$$

As expected, the affine connection Λ_{μ} transforms like a covariant vector under a general coordinate transformation. The generalisation of Equations (2) and (3) can be obtained from the definition of the covariant derivative

$$\nabla_{\sigma} A^{\mu_1 \cdots \mu_m} = \frac{\partial A^{\mu_1 \cdots \mu_m}}{\partial x^{\sigma}} + m\beta \Lambda_{\sigma} A^{\mu_1 \cdots \mu_m} \tag{5}$$

$$\nabla_{\sigma} A_{\nu_1 \dots \nu_n} = \frac{\partial A_{\nu_1 \dots \nu_n}}{\partial x^{\sigma}} - n\beta \Lambda_{\sigma} A_{\nu_1 \dots \nu_n} \tag{6}$$

$$\nabla_{\sigma} A^{\mu_1 \cdots \mu_m}_{\nu_1 \dots \nu_n} = \frac{\partial A^{\mu_1 \cdots \mu_m}_{\nu_1 \dots \nu_n}}{\partial x^{\sigma}} + (m-n)\beta \Lambda_{\sigma} A^{\mu_1 \cdots \mu_m}_{\nu_1 \dots \nu_n} \tag{7}$$

It is noted that the covariant derivative of a mixed tensor having equal number of superscripts and subscripts is identical to its ordinary derivative. It should be reiterated that the electromagnetic structure is assumed to be an independent structure, which defines a curved spacetime. This must not be considered as an additional structure arising from the postulate of gauge invariance, as in Weyl's theory which assumes a change $\delta l = -l\phi_{\mu} dx^{\mu}$ of the length $l = g_{\mu\nu} \xi^{\mu} \xi^{\nu}$ of a vector ξ^{μ} under parallel transport. In the latter case $g_{\mu\nu}$ represents a Riemannian spacetime structure of gravitation and ϕ_{μ} is a four-vector function which is identified with the four-vector potential of an electromagnetic field [6]. The electromagnetic spacetime of our case is assumed to exist by itself, independent of any other spacetime structures, such as the gravitational field. The purpose of the introduction of the connection Λ_{μ} is to construct a non-Riemannian spacetime manifold which can be used to represent electromagnetism alone. In this way an appropriate topological structure of the manifold can be related to the quantum dynamics of a particle in spacetime. The formulation of a physical theory is normally required to be covariant only under some particular group of transformations, except for the general theory of relativity in which the formalism is based on the requirement of general covariance. However, in the present geometric formulation of electromagnetism, the geometrical object which plays the role of the Riemannian curvature tensor is covariant under general coordinate transformations. This curvature can be derived by considering the change $\Delta V_{\mu} = \oint \delta V_{\mu}$ of a vector V_{μ} parallel transported around an infinitesimal closed path. To first order, an infinitesimal closed path permits the components of the vector V_{μ} at points inside the path to be uniquely determined by their values on the path. By Stokes' theorem, it is found that

$$\Delta V_{\sigma} = \oint \Lambda_{\nu} V_{\sigma} dx^{\nu} = \frac{1}{2} \left(\frac{\partial \Lambda_{\mu}}{\partial x^{\nu}} - \frac{\partial \Lambda_{\nu}}{\partial x^{\mu}} \right) V_{\sigma} \Delta f^{\mu\nu} \tag{8}$$

where $\Delta f^{\mu\nu}$ is the area enclosed by a closed path [7]. Since V^{μ} is a vector and $\Delta f^{\mu\nu}$ is a tensor, and since ΔV_{σ} is also a vector, because it is the difference between the values of vectors at the same point after parallel displacement, the tensor character of the curvature, defined by the relation $F_{\mu\nu} = \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu}$ is determined from the quotient theorem in tensor calculus. The quantities $F_{\mu\nu}$ therefore form a tensor under general coordinate transformations. This result shows that if only the field strength of the electromagnetic field is considered significant, then the present geometric formulation of electromagnetism, like the general relativistic formulation of gravitation, is also covariant with respect to the general group of transformations. The curvature $F_{\mu\nu} = \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu}$ automatically satisfies the homogeneous equations of classical electrodynamics $\partial_{\alpha}F_{\mu\nu} + \partial_{\mu}F_{\nu\alpha} + \partial_{\nu}F_{\alpha\mu} = 0$. The result shows that the homogeneous equations of electrodynamics are geometrical rather than dynamical when the connection Λ_{μ} is considered as being a purely geometrical object. As usual, to determine the dynamics of the electromagnetic spacetime manifold, an action, that may or may not relate geometrical properties of the manifold to matter or charge, must be specified. If such an action is defined by the form $S = -\int\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \kappa\Lambda_{\mu}j^{\mu}\right)d^4x$, where κ is an arbitrary constant, then the variation of the action S with respect to the connection Λ_{μ} leads to the inhomogeneous equations of classical electrodynamics $\partial_{\mu}F^{\mu\nu} + \kappa j^{\nu} = 0$. The external current density j^{μ}, whose geometrical character is unknown, plays the role of the stress tensor in the field equations of general relativity.

In the above discussions, the spacetime structure of electromagnetism that has been described using the connection Λ_{μ} and the curvature $F_{\mu\nu}$ is entirely affine. However, an affine structure is not capable of providing a dynamical description of the motion of a particle in the field. This is exactly the case in classical electrodynamics where the Lorentz force must be added to the Maxwell field equations for a dynamical description of a charged particle. With a geometrical formulation of the physical field, the dynamics can be provided by introducing a metric tensor $g_{\mu\nu}$ onto the spacetime manifold through the defining relation $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$. When the spacetime manifold is endowed with a metric, a relationship between the metric and the connection can be obtained by demanding that the metric be covariantly constant, in the sense that the inner product of two vectors remains constant under parallel transport along a curve. This requirement leads to the condition

$$\nabla_{\sigma}g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} - 2\beta\Lambda_{\sigma}g_{\mu\nu} = 0 \tag{9}$$

Equation (9) can be rewritten in the form

$$\Lambda_{\mu} = \frac{g^{\lambda\sigma}}{2\beta}\frac{\partial g_{\lambda\sigma}}{\partial x^{\mu}} = \frac{1}{2\beta g}\frac{\partial g}{\partial x^{\mu}} = \frac{1}{2\beta}\frac{\partial \ln\sqrt{-g}}{\partial x^{\mu}} \tag{10}$$

where $g = \det\left(g_{\mu\nu}\right)$. The equation of motion of a charged particle in an electromagnetic spacetime manifold can be obtained from the requirement that the path of a particle is geodesic

$$\frac{d^2x^\mu}{ds^2}+\beta\Lambda_\nu\frac{dx^\nu}{ds}\frac{dx^\mu}{ds}=0 \tag{11}$$

where the parameter s is identified with the arc-length only when a metric exists. Since the affine connection Λ_μ is entirely geometrical, it is necessary to introduce some kind of relationship between the geometrical objects and the physical quantities in order to provide a possible dynamical description of the system consisting of particle and field. For example, if the following relationship is assumed

$$\beta\Lambda_\nu\frac{dx^\mu}{ds}=-\frac{q}{m}F_\nu^\mu \tag{12}$$

then the familiar form of Lorentz force law for the motion of a charged particle in an electromagnetic field is regained

$$\frac{d^2x^\mu}{ds^2}=\frac{q}{m}F_\nu^\mu\frac{dx^\nu}{ds} \tag{13}$$

However, since there is no physical basis for its introduction, the relationship given in Equation (12) should be considered as an intrinsic relationship between the geometrical field and the experimentally defined physical quantities that characterise the mass and the charge of a particle. We will discuss how to specify the mass and the charge of an elementary particle in terms of geometrical objects in further works. They may be viewed as manifestations of purely geometrical properties determined by the three-dimensional topological structure of the particle.

Having established electromagnetism as an affine structure on the spacetime manifold, we are now in the position to discuss how gravity can be formulated as a coupling of two electromagnetic fields of equal magnitude but opposite direction. Consider an asymmetric connection of the form $\Gamma_{\mu\nu}^\sigma=\Lambda_\mu^\sigma\Phi_\nu$. As shown below, the motivation for considering this form of asymmetric connection is that it allows a construction of an affine connection in terms of two electromagnetic fields. The quantity Φ_ν will then be identified with the four-vector potential of one electromagnetic field and the quantity Λ_μ^σ with the field strength of the second opposing field. When the affine connection is constructed from two electromagnetic fields, the resulting formulation is considered to give rise to genuine physical effects when the electromagnetic field is viewed as a physical field. However, due to the asymmetry of the connection, these effects cannot be identified with gravity, since the theory of general relativity requires a symmetric connection. To meet this requirement, it is necessary to reduce the physical Ricci tensor, which is formed by two electromagnetic fields, to a symmetric form. The affine connection of the particular form $\Gamma_{\mu\nu}^\sigma=\Lambda_\mu^\sigma\Phi_\nu$ reduces the Riemann curvature tensor $R_{\beta\mu\nu}^\alpha=\partial_\mu\Gamma_{\beta\nu}^\alpha-\partial_\nu\Gamma_{\beta\mu}^\alpha+\Gamma_{\lambda\mu}^\alpha\Gamma_{\beta\nu}^\lambda-\Gamma_{\lambda\nu}^\alpha\Gamma_{\beta\mu}^\lambda$ to the simpler form as

$$R_{\beta\mu\nu}^\alpha=\frac{\partial\left(\Lambda_\beta^\alpha\Phi_\nu\right)}{\partial x^\mu}-\frac{\partial\left(\Lambda_\beta^\alpha\Phi_\mu\right)}{\partial x^\nu} \tag{14}$$

The Ricci tensor defined by the relation $R_{\mu\nu}=R_{\mu\alpha\nu}^\alpha$ becomes

$$R_{\mu\nu} = \left(\frac{\partial\Phi_\nu}{\partial x^\sigma} - \frac{\partial\Phi_\sigma}{\partial x^\nu}\right)\Lambda_\mu^\sigma + \Phi_\nu \frac{\partial\Lambda_\mu^\sigma}{\partial x^\sigma} - \Phi_\sigma \frac{\partial\Lambda_\mu^\sigma}{\partial x^\nu} \tag{15}$$

The Ricci tensor in this form can be reduced to a symmetric form if the quantities Λ_μ^σ satisfy the relation $\partial\Lambda_\mu^\sigma/\partial x^\nu = \eta\Lambda_\mu^\sigma\Phi_\nu$, where η are arbitrary functions of the coordinate variables. The Ricci tensor then becomes

$$R_{\mu\nu} = \Lambda_\mu^\sigma F_{\sigma\nu} \tag{16}$$

where we have defined $F_{\mu\nu} = \partial_\mu\Phi_\nu - \partial_\nu\Phi_\mu$. It is seen that the reduced Ricci tensor given in Equation (16) is symmetric if the quantity Λ_μ^σ is the transpose of the quantity $F_{\mu\nu}$. The reduced Ricci tensor suggests that in order to incorporate it into electromagnetism, the quantity Φ_μ should be identified with the four-vector potential and the quantity $F_{\mu\nu}$ with the field strength of an electromagnetic field. In this case, if the reduced Ricci tensor is required to be symmetric, the quantity Λ_μ^ν is identified as the field opposite to the field $F_{\mu\nu}$. The reduced Ricci tensor therefore may be used as a counterpart of the energy-momentum tensor to form general relativistic field equations for gravitation because according to the Lorentz force law in classical electrodynamics two opposing electromagnetic fields of equal magnitude are considered to have no classical electromagnetic effects on a charged particle moving in the coupled fields. In terms of the field strengths Λ_μ^σ and $F_{\sigma\nu}$, the reduced Ricci tensor given in Equation (16) takes the explicit form as follows

$$\Lambda_\mu^\sigma = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{17}$$

$$F_{\sigma\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{18}$$

$$R_{\mu\nu} = \begin{pmatrix} E_1^2+E_2^2+E_3^2 & E_2B_3-E_3B_2 & E_3B_1-E_1B_3 & E_1B_2-E_2B_1 \\ E_2B_3-E_3B_2 & E_1^2+B_2^2+B_3^2 & E_1E_2-B_1B_2 & E_1E_3-B_1B_3 \\ E_3B_1-E_1B_3 & E_1E_2-B_1B_2 & B_1^2+E_2^2+B_3^2 & E_2E_3-B_2B_3 \\ E_1B_2-E_2B_1 & E_1E_3-B_1B_3 & E_2E_3-B_2B_3 & B_1^2+B_2^2+E_3^2 \end{pmatrix} \tag{19}$$

We have assumed that the dynamics of a charged particle in a region of spacetime, whose structure is determined by the affine connection $\Lambda_\mu^\sigma\Phi_\nu$ formed by two opposing electromagnetic fields, is governed by the reduced form of the Ricci tensor given in Equation (16). In order to determine the dynamical aspects of the particle in the spirit of general relativity, a new symmetrical metric tensor $g_{\mu\nu}$ is introduced according to the defining relation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. With the introduction of this symmetric metric tensor into the spacetime structure, it is now possible to construct field equations for the gravitational field in terms of differential geometry that reflects the structure of the physical quantity that determines the Ricci tensor as specified in Equation (19). It is possible to adopt Einstein field equations of general relativity as a postulating physical formulation to describe gravity, however, as shown in our other works on general relativity, in order to formulate gravity purely in terms of differential geometry, we can postulate field equations of general relativity by adopting Bianchi identities instead [8,9]. It is shown that the Ricci tensor satisfies the Bianchi identities

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} g^{\mu\nu} \nabla_\mu R \tag{20}$$

Even though Equation (20) is purely geometrical, it has a form of Maxwell field equations of the electromagnetic tensor, $\partial_\mu F^{\mu\nu} = \mu j^\nu$. If the quantity $\frac{1}{2} g^{\mu\nu} \nabla_\mu R$ can be perceived as a physical entity, such as a four-current of gravitational matter, then Equation (20) has the status of a dynamical law of a physical theory. With the assumption that the quantity $\frac{1}{2} g^{\mu\nu} \nabla_\mu R$ to be identified with a four-current of gravitational matter then a four-current $j^\nu = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as follows

$$j^\nu = \frac{1}{2} g^{\mu\nu} \nabla_\mu R \tag{21}$$

For a purely gravitational field, Equation (20) reduces to

$$\nabla_\mu R^{\mu\nu} = 0 \tag{22}$$

Using the identity $\nabla_\mu g^{\alpha\beta} \equiv 0$, Equation (22) implies

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{23}$$

where Λ is an undetermined constant. Using the identities $g_{\mu\nu} g^{\mu\nu} = 4$ and $g_{\mu\nu} R^{\mu\nu} = R$, we obtain $\Lambda = R/4$. With the new purely geometrical formulation of gravity, Einstein field equations given by the relation $R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$ can be interpreted as a definition of an energy-momentum tensor, as that of Maxwell theory of the electromagnetic field. From Equation (23), we obtain

$$T_{\mu\nu} = -\frac{\Lambda}{\kappa} g_{\mu\nu} \tag{24}$$

If we assume that the effect of the coupling of two opposing electromagnetic fields can be interpreted as a gravitational field, then the affine connection $\Gamma^\sigma_{\mu\nu} = \Lambda^\sigma_\mu \Phi_\nu$ gives rise to a geodesic equation of the form

$$\frac{d^2 x^\mu}{ds^2} + \Lambda^\mu_\nu \Phi_\sigma \frac{dx^\sigma}{ds} \frac{dx^\nu}{ds} = 0 \tag{25}$$

Equation (25) admits a linear first integral of the form [10]

$$\Phi_\sigma \frac{dx^\sigma}{ds} = -\frac{q}{m} \tag{26}$$

provided the quantities Φ_σ satisfy the condition $\nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu = 0$ and we have set the constant in the first integral equal to $-q/m$ for convenience. This condition identifies Φ_σ as a Killing vector field, which defines a direction of symmetry along which the motion leaves the spacetime geometry unchanged. The geodesic equation then has the form of the Lorentz force law

$$\frac{d^2x^\mu}{ds^2} = \frac{q}{m} \Lambda^\mu_\nu \frac{dx^\nu}{ds} \tag{27}$$

We may interpret this result as follows. When one of the electromagnetic fields drives the charged particle according to the laws of classical electrodynamics, the opposite field resists such motion of the particle and the resistance manifests itself as the mass of the particle via the linear first integral given in Equation (26).

References

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