On Schrödinger equations equivalent to constant coefficient equations

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This paper shows that the solution of some classes of Schrödinger equations may be performed in terms of the solution of equations of constant coefficients. In this context, it has been possible to generate new exactly solvable potentials and to show that the Schrödinger equation for some well known potentials may also be solved in terms of elementary functions.

Keywords: Schrödinger equation, constant mass, nonlocal transformation.

The analytical integration of Schrödinger equations leads in general to express bound and scattering state solutions in terms of special functions of mathematical physics so that these solutions may be sometimes very complicated mathematical formulas. Even the discrete bound state solutions of the prototype of dynamical systems, that is, of the linear harmonic oscillator, must be explicited in terms of special functions called Hermite polynomials. To compute analytically these solutions, many mathematical methods for solving eigenvalue problems are used in the literature. In this way several methods like contact transformation, point transformation and nonlocal transformation, which allow one to map the initial Schrödinger equation into an equation with well known solution, are widely used to solve the Schrödinger equation with constant mass as well as with position-dependent mass. However, there appears reasonable to ask whether these methods may be used to map the Schrödinger equation into the free particle equation or in general into an equation with constant coefficients. Such a problem is very interesting since it may lead to compute the general solution to the Schrödinger equation in terms of elementary functions with well-known analytical properties. The underlined problem has been examined effectively by some authors. Thus the problem of finding Schrödinger equations with time-dependent potentials which can be mapped into the free particle equation has been explored under nonlocal transformations [1]. Under point transformation Boyer [2] has been able to show that the Schrödinger equation with time-independent potential may be mapped into the free particle equation if the potential is a quadratic polynomial. Recently it has been shown that the Schrödinger equation with position-dependent mass can be mapped by variable transformation into equations of constant coefficients to compute bound and scattering state solutions in terms of elementary functions [3]. In this perspective the problem of finding Schrödinger equation with diverse types of time-independent potentials which can be mapped into equations of constant coefficients, or precisely into the free particle equation, is an interesting question for the mathematical physics since it may also lead to detect new exactly solvable potentials. To be specific, in this work, the question to be answered is to ask whether the Schrödinger equation may be mapped into constant coefficient equations in general, and in particular into the free particle equation with time-independent potential which is not a quadratic polynomial. The present work assumes such a prediction. To demonstrate, the required nonlocal transformation is clearly formulated and applied to map the free particle equation and the constant coefficient equation into general classes of Schrödinger equations (section 2) so that examples of potentials in addition to the quadratic polynomial potential may be highlighted (section 3). Finally these results are discussed (section 4) and a conclusion for the work is carried out.

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1 Classes of Schrödinger equations

This part is devoted to solve in a straightforward fashion the mathematical problem of interest by application of nonlocal transformation to the constant coefficient equation and to the free particle equation. In this way the appropriate classes of Schrödinger equations are carried out so that involved time-independent potentials may be generated.

1.1 Mathematical problem

Let

$$y''(\tau) + cy(\tau) = 0 \tag{1.1}$$

be the constant coefficient equation and the free particle equation when the constant $c = 0$, where prime means a differentiation with respect to the argument.

For $c = 0$, the solution of (1.1) may take the form

$$y(\tau) = A\tau + B \tag{1.2}$$

where A and B are arbitrary constants. If $c > 0$, then the general solution to (1.1) reads

$$y(\tau) = K_1 sin(\sqrt{c}\tau) + K_2 cos(\sqrt{c}\tau) \tag{1.3}$$

where K_1 and K_2 are arbitrary parameters.

For $c < 0$, the general solution to (1.1) may be written

$$y(\tau) = K_3 sinh(\tau\sqrt{-c}) + K_4 cosh(\tau\sqrt{-c}) \tag{1.4}$$

where K_3 and K_4 are arbitrary parameters. Therefore the following problem may be stated: Find the new classes of Schrödinger equations whose the general solution may be expressed in terms of (1.2), (1.3) or (1.4) with time-independent potential.

To achieve this goal, it is needed first to define appropriately the nonlocal transformation to be considered and secondly to show that its application to (1.1) may lead to a Schrödinger differential equation of the form [4]

$$u''(x) + [E - V(x)]u(x) = 0 \tag{1.5}$$

where $u(x)$ is the wave function, E is the spectral parameter and $V(x)$ denotes the time-independent potential.

1.2 Nonlocal transformation of the constant coefficient equation

The nonlocal transformation of the constant coefficient equation is necessary to establish the class of Schrödinger equations which admit general solutions in terms of (1.3) or (1.4). Thus define the change of variables

$$y(\tau) = u(x)e^{l\varphi(x)}, \quad d\tau = e^{\gamma\varphi(x)}dx \tag{1.6}$$

From (1.6), it follows

$$u(x) = y(\tau)e^{-l\varphi(x)} \tag{1.7}$$

where l and γ are arbitrary parameters and $\varphi(x)$ an arbitrary function. Then consider the following theorem.

Theorem 1. *Let $\gamma = 2l$. Then by application of nonlocal transformation (1.6), equation (1.1) is reducible to*

$$u''(x) + \left[l\varphi''(x) - l^2\varphi'(x)^2 + ce^{4l\varphi(x)}\right]u(x) = 0 \tag{1.8}$$

Proof. Using the nonlocal transformation (1.6), one may compute

$$y'(\tau) = [u'(x) + lu(x)\varphi'(x)]\, e^{(l-\gamma)\varphi(x)}$$

from which it follows after a few mathematical manipulations

$$y''(\tau) = \left[u''(x) + (2l-\gamma)\varphi'(x)u'(x) + \left[l\varphi''(x) + l(l-\gamma)\varphi'(x)^2\right]u(x)\right]e^{(l-2\gamma)\varphi(x)} \tag{1.9}$$

The substitution of (1.9) into (1.1) knowing (1.6) yields the differential equation

$$u''(x) + (2l-\gamma)\varphi'(x)u'(x) + \left[l\varphi''(x) + l(l-\gamma)\varphi'(x)^2 + ce^{2\gamma\varphi(x)}\right]u(x) = 0 \tag{1.10}$$

Putting $\gamma = 2l$, one may arrive at the differential equation (1.8).

The equation (1.8) may take the form of Schrödinger equation (1.5) once the function $\varphi(x)$ is convenently choosen, so that (1.8) defines a new class of Schrödinger equations which may be mapped into the constant coefficient equation (1.1) where $c \neq 0$. Therefore the equation (1.8) may take the form of (1.5) under the conditions that

i)

$$\varphi(x) = q_1x + q_2x^2 + \int f(x)dx \tag{1.11}$$

where $f(x)$ is an arbitrary function of x, and q_1 and q_2 are arbitrary parameters so that q_1 and q_2 do not simultaneously vanish. In this situation (1.8) becomes

$$\begin{aligned} & u''(x) + [2lq_2 - l^2q_1^2 - 4l^2q_2x^2 - 4l^2q_2x(q_1+f(x)) \\ - \; & l^2f(x)(2q_1+f(x)) + lf'(x) \\ + \; & ce^{4l(q_1x+q_2x^2+\int f(x)dx)}]u(x) = 0 \end{aligned} \tag{1.12}$$

where

$$\begin{cases} E = 2lq_2 - l^2q_1^2, \\ V(x) = 4l^2q_2x^2 + 4l^2q_2x(q_1+f(x)) + l^2f(x)(2q_1+f(x)) - \\ lf'(x) - ce^{4l(q_1x+q_2x^2+\int f(x)dx)} \end{cases} \tag{1.13}$$

ii)

$$\varphi(x) = ln(q+f(x)) \tag{1.14}$$

where $q \neq 0$.

Thus (1.8) may be expressed as

$$u''(x) + \left[\frac{lf''(x)}{q+f(x)} - \frac{l(1+l)f'(x)^2}{(q+f(x))^2} + c[q+f(x)]^{4l}\right]u(x) = 0 \tag{1.15}$$

where

$$E = cq^{4l}, \quad V(x) = -\frac{lf''(x)}{q+f(x)} + \frac{l(1+l)f'(x)^2}{(q+f(x))^2} - c[q+f(x)]^{4l} + cq^{4l} \tag{1.16}$$

The differential equation (1.8) is very interesting from the physical point of view since the functional choice $\varphi(x) = \frac{1}{4}ax$, where a is a control parameter, $q_1 = \frac{a}{4}$, $q_2 = 0$ and $f(x) = 0$, leads to the important Schrödinger equation with the purely exponential potential [4]

$$u''(x) + \left[-\frac{l^2a^2}{16} + ce^{alx}\right]u(x) = 0 \tag{1.17}$$

where $E = -\frac{l^2a^2}{16}$ and $V(x) = -ce^{alx}$. However the detailed study of this potential will be carried out in a subsequent work. Now, if $\varphi(x) = \frac{b}{4}x^2$, where b is a control parameter, then (1.8) reduces to

$$u''(x) + \left[\frac{lb}{2} - \frac{l^2b^2}{4}x^2 + ce^{blx^2}\right]u(x) = 0 \tag{1.18}$$

where $E = \frac{lb}{2}$, and $V(x) = \frac{l^2b^2}{4}x^2 - ce^{blx^2}$, when (1.18) is compared with (1.5).

The general solution to (1.18) may be expressed, knowing (1.3) for $c > 0$, as

$$u(x) = \left[K_1 sin\left(\sqrt{c}\int e^{\frac{lb}{2}x^2}dx\right) + K_2 cos\left(\sqrt{c}\int e^{\frac{lb}{2}x^2}dx\right)\right]e^{-\frac{lb}{4}x^2} \tag{1.19}$$

For $c < 0$, the general solution to (1.18), knowing (1.4), may take the expression

$$u(x) = \left[K_3 sinh\left(\sqrt{-c}\int e^{\frac{lb}{2}x^2}dx\right) + K_4 cosh\left(\sqrt{-c}\int e^{\frac{lb}{2}x^2}dx\right)\right]e^{-\frac{lb}{4}x^2} \tag{1.20}$$

If $\varphi(x) = \frac{1}{4}ln(ax+b)$, and $l = 2$, then the corresponding Schrödinger equation (1.8) takes the expression

$$u''(x) + \left[-\frac{3a^2}{4(ax+b)^2} + c(ax+b)^2\right]u(x) = 0 \tag{1.21}$$

The exact general solution may be written, for $c > 0$, as

$$u(x) = \left[K_1 sin\left(\sqrt{c}(\frac{1}{2}ax^2+bx)\right) + K_2 cos\left(\sqrt{c}(\frac{1}{2}ax^2+bx)\right)\right]\frac{1}{\sqrt{ax+b}} \tag{1.22}$$

where $E = b^2c$.

1.3 Nonlocal transformation of the free particle equation

Now, consider the following theorem, as a consequence of the theorem 1.

Theorem 2. *Let* $c = 0$. *Then (1.8) becomes*

$$u''(x) + \left[l\varphi''(x) - l^2\varphi'(x)^2\right]u(x) = 0 \tag{1.23}$$

Proof. It is easy to see that the theorem 2 is a special case ($c = 0$) of theorem 1. Equation (1.23) is the nonlocal transformation of (1.1), that is of the free particle equation. This equation may take the form of (1.5) under the condition that

$$\varphi'(x) = q + f(x) \tag{1.24}$$

where q is an arbitrary constant and $f(x)$ is an arbitrary function of x. From (1.24) one may obtain the condition on $\varphi(x)$, that is

$$\varphi(x) = qx + \int f(x)dx \tag{1.25}$$

In this context, (1.23) becomes

$$u''(x) + \left[-l^2q^2 + lf'(x) - l^2f(x)^2 - 2l^2qf(x)\right]u(x) = 0 \tag{1.26}$$

The comparison of (1.26) with (1.5) allows one to write

$$E = -l^2q^2 \tag{1.27}$$

and

$$V(x) = -lf'(x) + l^2f(x)^2 + 2l^2qf(x) \tag{1.28}$$

If $f(x) = \beta x$, where β is an arbitrary constant, the spectral parameter is defined as $E = -l^2q^2 + l\beta$, and the potential $V(x)$ must read $V(x) = l^2\beta^2x^2 + 2l^2q\beta x$, which is a quadratic polynomial, as highlighted by Boyer [2]. If the potential is defined as $V(x) = l^2\beta^2x^2 + 2l^2q\beta x - l\beta$, then E must be $E = -l^2q^2$. The time-independent potential (1.28) defines the desired new class of Schrödinger equations (1.26) which may be mapped into the free particle equation (1.1) where $c = 0$. In this situation the general solution to the Schrödinger equation (1.26) may take, after (1.7), the form

$$u(x) = \left[A\int e^{2l\varphi(x)}dx + B\right]e^{-l\varphi(x)} \tag{1.29}$$

where $\varphi(x)$ is given by (1.25).

Consider now some illustrative potentials.

2 Examples

Example 1

Let $\varphi(x) = qx + ln(ax^{\alpha})$, where α is an arbitrary parameter. Then, the Schrödinger equation (1.26) reduces to

$$u''(x) + \left[-l^2q^2 - \frac{l\alpha(l\alpha+1)}{x^2} - \frac{2l^2\alpha q}{x}\right]u(x) = 0 \tag{2.1}$$

where $E = -l^2q^2$, and the potential takes the form

$$V(x) = \frac{l\alpha(l\alpha+1)}{x^2} + \frac{2l^2\alpha q}{x} \tag{2.2}$$

Such a potential (2.2) is a special case of the singular Coulomb potential [5] which arises in Kepler problem and has been used by Kratzer in molecular physics [6]. For $l\alpha = -1$, $V(x) = -\frac{2lq}{x}$ becomes the condition under which the solution of the Schrödinger equation for the Coulomb potential may be expressed in terms of solution of the free particle equation.

The general solution to (2.1) may be written in the form

$$u(x) = a^{-l}\left[a^{2l}A\int x^{2l\alpha}e^{2lqx}dx + B\right]x^{-l\alpha}e^{-lqx} \tag{2.3}$$

Example 2

Consider now $\varphi(x) = qx + \frac{a}{b}e^{bx}$. Then the corresponding Schrödinger equation (1.26) becomes

$$u''(x) + \left[-l^2q^2 - l^2a^2e^{2bx} + la(b-2lq)e^{bx}\right]u(x) = 0 \tag{2.4}$$

Therefore the spectral parameter $E = -l^2q^2$, and

$$V(x) = l^2a^2e^{2bx} - la(b-2lq)e^{bx} \tag{2.5}$$

The potential (2.5) is a special case of the generalized Morse potential [5].

If $q = b$, then (2.5) reduces to

$$V(x) = l^2a^2e^{2bx} - lab(1-2l)e^{bx} \tag{2.6}$$

From (2.6) one may recover for $b = 1$, the Morse potential [7]

$$V(x) = l^2a^2e^{2x} - la(1-2l)e^{x} \tag{2.7}$$

In other words, the Schrödinger equation with the generalized Morse potential can be mapped into the free particle equation under the form (2.5). Now one may write the general solution to (2.4) as

$$u(x) = \left[A\int e^{2l(qx+\frac{a}{b}e^{bx})}dx + B\right]e^{-l(qx+\frac{a}{b}e^{bx})} \tag{2.8}$$

3 Discussion

The importance of mappings for solving differential equations has been widely underlined in the literature. Several transformations like contact transformation, point transformation and nonlocal transformation have been found to be a powerfull mathematical tool for solving exactly in closed form solutions linear as well as nonlinear differential equations. As such, these transformation methods have been intensively used to investigate the Schrödinger equation with a great variety of potentials. However the results are often to map such an equation into the hypergeometric type equation leading to a general solution in terms of special functions. In this way there appears appropriate to ask whether the Schrödinger equation can be mapped into the constant coefficient equation which can lead to express the bound and scattering state solutions in terms of elementary functions. It is easy to see that a few works are performed in this regard, due to the difficulty to find the convenient transformation of variables. Previous works have shown that the Schrödinger equation can be mapped into the free particle equation if and only if the time-independent potential is a quadratic polynomial [1, 2]. The present work has been able to extend this result using a nonlocal transformation. Thus it offers the possibility to detect new exactly solvable potentials but also well-known potentials for which the Schrödinger equation can be mapped into the free particle equation. In this perspective the Schrödinger equation with the Morse potential and with the singular Coulomb potential for example, have been mapped to the free particle equation. Due to the proposed nonlocal transformation, it has also been possible to show existence of a new class of Schrödinger equations which can be mapped into the constant coefficient equation of the second order leading to express the eigenstate solutions in terms of elementary solutions. On the basis of these findings this work may be concluded.

Conclusion

Some works in the literature have been devoted to investigate Schrödinger equations which may be mapped into differential equations of constant coefficients, more precisely into the free particle equation, which may allow solutions in terms of elementary functions. This work has been designed to enlarge this class of Schrödinger equations using a nonlocal transformation of the constant coefficient equation. In so doing a new class of Schrödinger equations which can be exactly and explicitly solved in terms of trigonometric functions but also in terms of hyperbolic sine function has been highlighted. In this regard a new class of Schrödinger equations which may be mapped into the free particle equation has been also established. As a major finding it has been noted that some Schrödinger equations with well known potentials can be mapped into constant coefficient equations.

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