# A boundary value problem of the partial differential-integral

#### equations and their applications

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### **Abstract**

We study the boundary value problem of a partial differential-integral equations that have many applications in finance and insurance. We will solve a boundary value problem of the partial differential-integral equations by using the solution of conjugate equation and reflection method and apply it to determine the probability of company bankruptcy in insurance mathematics.

 $Keyword:$  partial differential - integral equation, boundary value problem

### **Introduction**

The partial differential - integral equations is now more in the application of financial and insurance. In many papers have considered the boundary value problem of the differential equations,however,a few papers have considered the boundary value problem of the partial differential-integral equations. If  $s$  integral equations is now more in the application of financial and<br>pers have considered the boundary value problem of the differential<br>few papers have considered the boundary value problem of the<br>prail equations.<br>m

In [1~5], authors considered the unique existence of the partial differential-integral equations by using viscosity analysis and in [5], author considered the maximum value problem of the partial differential-integral equations.

Firstly, we study the boundary value problem of a partial differential-integral equations and secondly, apply its result in insurance mathematics.

# 1. The boundary value problem of a partial differential-integral equations

Let consider the following boundary value problem of a partial differential-integral equations:

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the partial differential-integral equations.  
study the boundary value problem of a partial differential-integral equations  
by, apply its result in insurance mathematics.  
**uundary value problem of a partial differential-integral equations**  
ler the following boundary value problem of a partial differential-integral  

$$
\left(\frac{\partial}{\partial s} + L\right)u(s, x) = 0, \quad (s, x) \in [0, T] \times R_+^2
$$

$$
(1)
$$

$$
u(T, x) = 1, \quad x \in R_+^2, \quad u\left(s, (x_1, 0)^t\right) = 0, \quad s \in [0, T], \quad x_1 \in R_+^1
$$

$$
u\left(s, (0, x_2)^{'}\right) = 0, \quad s \in [0, T], \quad x_1 \in R^1_+,
$$
  
\n
$$
x) \in C^2\left(R^2_+\right) \text{ operator } L \text{ define as following:}
$$
\n
$$
u(x_1, x_2, x_3) = 0, \quad u(x_1, x_2, x_3) = 0, \quad u(x_2, x_3) = 0, \quad u(x_3, x_2) = 0, \quad u(x_4, x_5) = 0.
$$

where for  $f(x) \in C^2(R_+^2)$  operator  $L$  define as following:

$$
u\left(s,\left(0,x_{2}\right)^{'}\right)=0, \quad s \in\left[0,T\right], \quad x_{1} \in R_{+}^{1},
$$
\n(2)  
\nwhere for  $f(x) \in C^{2}\left(R_{+}^{2}\right)$  operator *L* define as following:  
\n
$$
Lf\left(x\right)=\sum_{i=1}^{2} a_{i} \cdot \frac{\partial f}{\partial x_{i}}+\frac{1}{\tau}\sum_{i=1}^{2} \sum_{j=1}^{2} a_{i,j} \frac{\partial^{2} f\left(x\right)}{\partial x_{i} \partial x_{j}}+a_{0} f\left(x\right)
$$
\n
$$
+\sum_{k=1}^{d} \int_{R^{2}}\left[f\left(x+c_{k}\left(z\right)\right)-f\left(x\right)-\sum_{i=1}^{2} c_{ik}\left(z\right) \frac{\partial f}{\partial x_{i}} I_{\left\{z\right\}}\left(z\right)\right] v_{k}\left(dz\right).
$$
\nAssumption:  
\n(H1)  $a_{i}, i=0,1,2$  are constant and  $c_{ik}(z), i=1,2, k=\overline{1,d}$  are satisfied

# Assumption:

(H1) 
$$
a_i
$$
,  $i = 0,1,2$  are constant and  $c_{ik}(z)$ ,  $i = 1,2$ ,  $k = \overline{1,d}$  are satisfied  
\n
$$
\int_{R^2} |c_{ik}(z)|v_k(dz) < \infty
$$
\n(H2)  $\forall \theta \in R^2 (\theta \neq 0)$ ,  $\exists \lambda > 0$ ;  $\sum_{i=1}^2 \sum_{j=1}^2 a_{ij} \theta_i \theta_j \ge \lambda |\theta|^2$   
\n(H2)  $v(\cdot)$  is the finite measure on  $B^2$ ,  $v(\{0\}) = 0$  and can express  
\n $\lambda_k(dz) = \lambda_k p_k(z)dz$ ,  $z \in R^2$  (3)  
\n, where  $P_k(z)$  is the hyper-density function on  $R^2$ . Namely,  
\n $p_k(z) \ge 0$ ,  $\int_{R^2} p_k(z)dz = 1$ .  
\nThen, the operator  $L$  can rewrite as following:  
\n $(\overline{L}f)(x) = \sum_{i=1}^2 \widetilde{a}_i \frac{\partial f(x)}{\partial x} + \frac{1}{\tau} \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \widetilde{a}_0 f(x) + \int_{R} f(z) \cdot \sum_{k=1}^d \lambda_k \widetilde{p}_k(z - x)dz$  (4)

(H2)  $|V(\cdot)|$  is the finit measure on  $|B^2|$ ,  $|V(\{0\})=0$  and can express

$$
\lambda_k(dz) = \lambda_k p_k(z)dz, \quad z \in R^2 \tag{3}
$$

where  $P_k(z)$  is the hyper-density function on  $R^2$ . Namely,

$$
p_{k}(z) \geq 0, \qquad \int\limits_{R^2} p_{k}(z)dz = 1.
$$

Then, the operator  $L$  can rewrite as following:

$$
\left(\overline{L}f\right)(x) = \sum_{i=1}^{2} \widetilde{a}_{i} \frac{\partial f(x)}{\partial x} + \frac{1}{\tau} \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \frac{\partial^{2} f(x)}{\partial x_{i} \partial x_{j}} + \widetilde{a}_{0} f(x) + \int_{R} f(z) \cdot \sum_{k=1}^{d} \lambda_{k} \widetilde{p}_{k} (z - x) dz
$$
\n(4)

$$
\tilde{a}_{i} = a_{i} - \sum_{k=1}^{d} \lambda_{k} \int_{R} c_{ik} (z) p_{k} (z) dz
$$
\n
$$
\tilde{a}_{0} = a_{0} - \sum_{k=1}^{d} \lambda_{k}
$$
\n
$$
\tilde{p}_{k} (z - x) = p_{k} (c_{k}^{-1} (z - x)) \frac{d}{dz} c_{k}^{-1} (z - x)
$$
\n(5)\n\npption (H2), we can rewrite the matrix  $\begin{pmatrix} a_{ij} \end{pmatrix}$  into the diagonal-matrix  $\sigma = (\sigma_{i}^{2})$ 

From assumption (H2), we can rewrite the matrix ij <sup>a</sup> into the diagonal-matrix <sup>2</sup> <sup>i</sup> by using appropriate variable transformation.

Comfortably, will express  $\widetilde{a}_i$  into  $a_i$ .

Let's consider following two operators;

$$
A_s = \frac{\partial}{\partial t} + \sum a_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^2 \sigma_i^2 \frac{\partial^2}{\partial x_i^2} + a_0
$$
 (6)

$$
A_s^* = -\frac{\partial}{\partial t} - \sum a_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^2 \sigma_i^2 \frac{\partial^2}{\partial x_i^2} + a_0
$$
 (7)

**Theorem 1**: For  $\tau \le t \le T$ ,  $x, \xi \in R_+^2 = [0, \infty] \times [0, \infty]$ , the solution of the partial differential equation

$$
A_s^* k(t, x; \tau, \xi) = 0
$$
\n
$$
k(t, (0, x_2)) = k(t, (x_1, 0)) = 0
$$
\n
$$
k(\tau, x; \tau, \xi) = \delta(\xi - x)
$$
\n(9)

is as following. Namely,

$$
\partial t \longrightarrow \partial x_i \quad 2 \xrightarrow{i=1} \partial x_i^2 \longrightarrow
$$
\n(6)  
\n
$$
A_s^* = -\frac{\partial}{\partial t} - \sum a_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^2 \sigma_i^2 \frac{\partial^2}{\partial x_i^2} + a_0
$$
\n(7)  
\n**Theorem 1**: For  $\tau \le t \le T$ ,  $x, \xi \in R_+^2 = [0, \infty] \times [0, \infty]$ , the solution of the partial differential  
\nquation  
\n
$$
A_s^* k(t, x; \tau, \xi) = 0
$$
\n(8)  
\n
$$
k(t, (0, x_2)) = k(t, (x_1, 0)) = 0
$$
\n
$$
k(\tau, x; \tau, \xi) = \delta(\xi - x)
$$
\n(9)  
\nis as following. Namely,  
\n
$$
k(t, x; \tau, \xi) = \exp{\{\beta(t - \tau) + (\alpha, x - \xi)\}}
$$
\n
$$
\times \prod_{i=1}^2 \frac{1}{\sqrt{2\pi(t - \tau)}\sigma_i} \left\{ e^{-\frac{(x_i - \xi_i)^2}{2(t - \tau)\sigma_i^2}} - e^{-\frac{(x_i + \xi_i)^2}{2(t - \tau)\sigma_i^2}} \right\}
$$
\n(10)

Proof: Let's assume that the solution of the equation (8) on boundary condition (9) exist,and apply following transformation:

$$
s(t, x; \tau, \xi) = e^{-(\alpha, x)} k(t, x; \tau, \xi)
$$

Then for  $\alpha = (\alpha_1, \alpha_2), a = (a_1, a_2)$  the equation (8) is following:

Proof: Let's assume that the solution of the equation (8) on boundary condition (9)  
\nxist,and apply following transformation:  
\n
$$
s(t, x; \tau, \xi) = e^{-(\alpha, x)} k(t, x; \tau, \xi)
$$
\nThen for  $\alpha = (\alpha_1, \alpha_2), \alpha = (\alpha_1, \alpha_2)$  the equation (8) is following:  
\n
$$
\frac{\partial s}{\partial t} - \left[ (\alpha, \alpha) + \frac{1}{2} \alpha^T \sigma^2 \alpha + \alpha_0 \right] s + (-\alpha_1 + \sigma_1^2 \alpha_1) \frac{\partial s}{\partial x_1} +
$$
\n
$$
+ (-\alpha_2 + \sigma_2^2 \alpha_2) \frac{\partial s}{\partial x_2} + \left[ \sigma_1^2 \frac{\partial^2 s}{\partial x_2^2} + \sigma_2^2 \frac{\partial^2 s}{\partial x_2^2} \right] = 0
$$
\nNow, for  $\tau \le t \le T$ ,  $x, \xi \in R_+^2$  instituting  $\alpha_i$  and  $\beta$  into (11), (9), we obtain the  
\nallowing equation:  
\n
$$
\left[ -\frac{\partial}{\partial t} + \frac{1}{\tau} \left[ \sigma_1^2 \frac{\partial^2}{\partial x_1^2} + \sigma_2^2 \frac{\partial^2}{\partial x_2^2} \right] + \beta \right] s(t, x; \tau, \xi) = 0
$$
\n
$$
s(t, (0, x_2), r, \xi) = s(t, (x_1, 0), r, \xi) = 0
$$
\n
$$
s(t, x; \tau, \xi) = e^{-(\alpha, x)} \delta(x - \xi)
$$
\n(13)  
\n
$$
\alpha_i = \frac{\alpha_i}{\sigma_i^2}, \quad i = 1, 2, \quad \beta = a_0 + (\alpha, \alpha) + \frac{1}{2} \alpha^T \sigma^2 \alpha
$$
\nLet's obtain the solution of above equation by expanding the domain of equation (12) that satisfies the boundary condition (13) the  $\frac{R_i}{\sigma}$  onto  $\frac{R}{\sigma}$ 

Now, for  $\tau \leq t \leq T$ ,  $x, \xi \in R^2$  instituting  $\alpha_i$  and  $\beta$  into (11), (9), we obtain the

following equation:

$$
\left[ -\frac{\partial}{\partial t} + \frac{1}{\tau} \left[ \sigma_1^2 \frac{\partial^2}{\partial x_1^2} + \sigma_2^2 \frac{\partial^2}{\partial x_2^2} \right] + \beta \right] s(t, x; \tau, \xi) = 0
$$
\n(12)

$$
s(t, (0, x_2); \tau, \xi) = s(t, (x_1, 0); \tau, \xi) = 0
$$
  

$$
s(\tau, x; \tau, \xi) = e^{-(\alpha, x)} \delta(x - \xi)
$$
 (13)

**Where**  $=\frac{a_i}{\sigma^2}$ ,  $i=1, 2$ , i  $i = \frac{a_i}{\sigma_i^2}$ 2  $0^{(u,u)}$  2  $=a_0+(a,\alpha)+\frac{1}{2}\alpha^T$ 

Let's obtain the solution of above equation by expanding the domain of equation (12) that

.

is satisfied the boundary condition (13) the  $\overline{R}_+$  onto  $\overline{R}_+$ .

Namely, when  $S$  is the solution of equation (12), we define the expanded solution  $\Gamma$  as following:

$$
\left[-\frac{\partial}{\partial t} + \frac{1}{\tau}\left[\sigma_1^2 \frac{\partial^2}{\partial x_1^2} + \sigma_2^2 \frac{\partial^2}{\partial x_2^2}\right] + \beta\right] s(t, x; \tau, \xi) = 0
$$
\n(12)  
\n
$$
s(t, (0, x_2) ; \tau, \xi) = s(t, (x_1, 0) ; \tau, \xi) = 0
$$
\n
$$
s(\tau, x; \tau, \xi) = e^{-(\alpha, x)} \delta(x - \xi)
$$
\n(13)  
\n
$$
\alpha_i = \frac{a_i}{\sigma_i^2}, \quad i = 1, 2, \quad \beta = a_0 + (a, \alpha) + \frac{1}{2} \alpha^T \sigma^2 \alpha
$$
\nLet's obtain the solution of above equation by expanding the domain of equation (12) that  
\ns satisfied the boundary condition (13) the  $R_+$  onto  $R$ .  
\nNamely, when  $S$  is the solution of equation (12), we define the expanded solution  $\Gamma$  as  
\nallowing:  
\n
$$
s(t, (x_1, x_2) ; \tau, \xi) = \begin{cases}\ns(t, (x_1, x_2) ; \tau, \xi) & x_1 \ge 0, x_1 \ge 0 \\
-s(t, (-x_1, x_2) ; \tau, \xi) & x_1 < 0, x_1 \ge 0 \\
-s(t, (-x_1, x_2) ; \tau, \xi) & x_1 < 0, x_1 < 0 \\
s(t, (-x_1, -x_2) ; \tau, \xi) & x_1 < 0, x_1 < 0 \\
s(t, (-x_1, -x_2) ; \tau, \xi) & x_1 < 0, x_1 < 0\n\end{cases}
$$
\n(14)  
\nAlso, the condition (13) become as following initial condition:

Also, the condition (13) become as following initial condition:

$$
\Gamma(\tau, x; \tau, \xi) = \varphi(x, \xi) = \begin{cases} e^{-(\alpha, x)} \delta(x - \xi) & , & x_1 \ge 0, \quad x_2 \ge 0 \\ -e^{-\alpha_1 x_1 + \alpha_2 x_2} \delta(x_1 - \xi_1, x_2 + \xi_2), & x_1 \ge 0, \quad x_2 < 0 \\ -e^{\alpha_1 x_1 - \alpha_2 x_2} \delta(x_1 + \xi_1, x_2 - \xi_2), & x_1 < 0, \quad x_2 \ge 0 \\ e^{\alpha_1 x_1 + \alpha_2 x_2} \delta(x_1 + \xi_1, x_2 + \xi_2), & x_1 < 0, \quad x_2 < 0 \\ \end{cases} \tag{15}
$$
\nTherefore, we can consider the boundary value problem (12),(13) to following initial value  
\n
$$
\Gamma(\tau, x; \tau, \xi) = \varphi(x, \xi) = \begin{cases} e^{-(\alpha, x)} \delta(x - \xi) & , & x_1 \ge 0, \quad x_2 \ge 0 \\ e^{\alpha_1 x_1 + \alpha_2 x_2} \delta(x_1 + \xi_1, x_2 + \xi_2), & x_1 < 0, \quad x_2 < 0 \\ \end{cases}
$$

Therefore, we can consider the boundary value problem (12),(13) to following initial value problem:

$$
\Gamma(\tau, x; \tau, \xi) = \varphi(x, \xi) = \begin{cases} e^{-(\alpha, x)} \delta(x - \xi) & x_1 \ge 0, \quad x_2 \ge 0 \\ -e^{-(\alpha, x_1 + \alpha_2 x_2)} \delta(x_1 - \xi_1, x_2 + \xi_2), & x_1 \ge 0, \quad x_2 < 0 \\ -e^{(\alpha, x_1 - \alpha_2 x_2)} \delta(x_1 + \xi_1, x_2 - \xi_2), & x_1 < 0, \quad x_2 \ge 0 \\ e^{(\alpha, x_1 + \alpha_2 x_2)} \delta(x_1 + \xi_1, x_2 + \xi_2), & x_1 < 0, \quad x_2 < 0 \end{cases} \tag{15}
$$
\nTherefore, we can consider the boundary value problem (12),(13) to following initial value problem:

\n
$$
\left[ -\frac{\partial}{\partial t} + \frac{1}{\tau} \left( \sigma_1^2 \frac{\partial^2}{\partial x_1^2} + \sigma_2^2 \frac{\partial^2}{\partial x_2^2} \right) + \beta \right] \Gamma(t, x; \tau, \xi) = 0 \tag{16}
$$
\n
$$
\Gamma(\tau, x; \tau, \xi) = \varphi(x, \xi)
$$
\n(17)

The solution of this initial value problem is well-known. Namely, the solution is obtained as following:

$$
\Gamma(r, x; r, \xi) = \varphi(x, \xi) = \begin{cases} e^{-(\alpha, x)} \delta(x - \xi) , & x_1 \ge 0, & x_2 \ge 0 \\ -e^{(\alpha, x_1 + \alpha_2, x_2)} \delta(x_1 + \xi_1, x_2 + \xi_2) , & x_1 \ge 0, & x_2 < 0 \\ -e^{(\alpha, x_1 + \alpha_2, x_2)} \delta(x_1 + \xi_1, x_2 - \xi_2) , & x_1 < 0, & x_2 \ge 0 \\ e^{(\alpha, x_1 + \alpha_2, x_2)} \delta(x_1 + \xi_1, x_2 + \xi_2) , & x_1 < 0, & x_2 < 0 \\ e^{(\alpha, x_1 + \alpha_2, x_2)} \delta(x_1 + \xi_1, x_2 + \xi_2) , & x_1 < 0, & x_2 < 0 \end{cases}
$$
\n(15)  
\n**Therefore, we can consider the boundary value problem (12),(13) to following initial value problem:**\n
$$
\left[ -\frac{\partial}{\partial t} + \frac{1}{\tau} \left( \sigma_1^2 \frac{\partial^2}{\partial x_1^2} + \sigma_2^2 \frac{\partial^2}{\partial x_2^2} \right) + \beta \right] \Gamma(t, x; r, \xi) = 0
$$
\n(16)  
\n
$$
\Gamma(r, x; r, \xi) = \varphi(x, \xi)
$$
\n
$$
\Gamma(r, x; r, \xi) = \int_{R^2} \varphi(x - y, \xi) \prod_{i=1}^2 \frac{1}{\sqrt{2\pi\sigma_i^2(t - r)}} e^{-\frac{y_i^2}{2\sigma_i^2(t - s)}} e^{\rho(t - r)} dy_1 dy_2 =
$$
\n
$$
= \exp\{- (\alpha, \xi)\} \frac{1}{2\pi(t - \tau) \sigma_1 \sigma_2} \left[ \exp\left\{-\frac{\frac{(x_1 - \xi_1)^2}{\sigma_1^2} + \frac{(x_2 - \xi_2)^2}{\sigma_2^2}}{2(t - \tau)}\right\} - \exp\left\{-\frac{\frac{(x_1 - \xi_1)^2}{\sigma_1^2} + \frac{(x_2 + \xi_2)^2}{\sigma_2^2}}{2(t - \tau)}\right\} - \exp\left\{-\frac{\frac{(x_1 + \xi_1)^2}{\sigma_1^2}
$$

When reduce to  $\overline{R}_{+}^{2}$  the domain in equation (18), we obtain that

$$
k(t, x; \tau, \xi) = e^{(\alpha, x)}s(t, x; \tau, \xi)
$$
  
\n
$$
= \exp\{( \alpha, x - \xi) + \beta(t - \tau) \} \prod_{i=1}^{2} \frac{1}{\sqrt{2\pi(t - \tau)}\sigma_{i}} \left[ e^{-\frac{(x_{i} - \xi_{i})^{2}}{2(t - \tau)\sigma_{i}^{2}} - e^{-\frac{(x_{i} + \xi_{i})^{2}}{2(t - \tau)\sigma_{i}^{2}}}} \right]
$$
  
\n**Theorem 2.** The differential-integral equation (1) equal to following integral equation:  
\n
$$
u(\tau, \xi) = \int_{\tau}^{\tau} \int_{R_{\tau}^{2}} u(t, x)G(t, x; \tau, \xi)dxdt + F(\tau, \xi)
$$
\n(19)  
\nWhere the notation  $F(\tau, \xi)$ ,  $G(t, x, \tau, \xi)dx$   
\n
$$
G(t, x; \tau, \xi) = \int_{\tau=1}^{d} k(T, x; \tau, \xi)dx
$$
  
\n
$$
G(t, x; \tau, \xi) = \sum_{i=1}^{d} \lambda_{i} \int_{R_{\tau}^{2}} k(t, z; \tau, \xi)p_{i}(z - x)dz
$$
  
\nwhere the notation  $k$  is denoted as the equation (10) of theorem 1.

Theorem 2. The differential-integral equation (1) equal to following integral equation:

.

$$
u(\tau,\xi) = \int_{\tau}^{T} \int_{R_{+}^{2}} u(t,x)G(t,x;\tau,\xi)dxdt + F(\tau,\xi)
$$
\n(19)

Where the notation  $F(\tau \xi)$ ,  $G(t \times \tau \xi)$ are denoted as following:

(19)  
\nWhere the notation 
$$
F(\tau \xi)
$$
,  $G(t \times \tau \xi)$  are denoted as following:  
\n
$$
F(\tau, \xi) = \int_{R_1^2} k(T, x; \tau, \xi) dx
$$
\n
$$
G(t, x; \tau, \xi) = \sum_{l=1}^d \lambda_l \int_{R_1^2} k(t, z; \tau, \xi) p_l(z - x) dz
$$
\n(20)  
\nwhere the notation  $k$  is denoted as the equation (10) of theorem 1.  
\nProof: Let the solution  $u$  of equation (1) with respect to operator (4) show equation (19).  
\n
$$
K(\lambda)A(u(\cdot)) = k(\lambda)A(u(\cdot)) - uA^*k(\cdot)) =
$$
\n
$$
\frac{\partial}{\partial t}(u(\cdot)k(\cdot)) + \sum_{i=1}^2 a_i \frac{\partial u}{\partial x_i} + \frac{1}{\tau} \sum_{i=1}^2 \sigma_i^2 \frac{\partial}{\partial x_i} \left(k(\cdot) \frac{\partial u(\cdot)}{\partial x_i} - u(\cdot) \frac{\partial k(\cdot)}{\partial x_i}\right)
$$
\n(21)  
\nFrom the boundary and initial condition of  $u$ ,  $k$ , each term of equation (21) is expressed  
\n
$$
S = \int_{0}^{R_1} f(u(\cdot)k(\cdot)) + \int_{0}^{R_2} f(u(\cdot)) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} + \int_{0}^{R_3} f(u(\cdot)) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} + \int_{0}^{R_3} f(u(\cdot)) \frac{\partial u}{\partial x_i} \frac{\partial u}{
$$

where the notation  $k$  is denoted as the equation (10) of theorem 1.

**Proof:** Let the solution  $u$  of equation (1) with respect to operator (4) show equation (19). With respect to operator  $A, A^*$ , we obtain following equation:

$$
= \frac{\partial}{\partial t} \big( u(\cdot) k(\cdot) \big) + \sum_{i=1}^{2} a_i \frac{\partial u}{\partial x_i} + \frac{1}{\tau} \sum_{i=1}^{2} \sigma_i^2 \frac{\partial}{\partial x_i} \bigg( k(\cdot) \frac{\partial u(\cdot)}{\partial x_i} - u(\cdot) \frac{\partial k(\cdot)}{\partial x_i} \bigg) \tag{21}
$$

From the boundary and initial condition of  $u$ ,  $k$ , each term of equation (21) is expressed as following:

$$
G(t, x; \tau, \xi) = \sum_{l=1}^{d} \lambda_l \int_{R} k(t, z; \tau, \xi) p_l(z - x) dz
$$
\n(20)  
\nwhere the notation  $k$  is denoted as the equation (10) of theorem 1.  
\n**Proof**: Let the solution  $u$  of equation (1) with respect to operator (4) show equation (19).  
\nWith respect to operator  $A$ ,  $A^*$ , we obtain following equation:  
\n
$$
k(\cdot)A(u(\cdot)) = k(\cdot)A(u(\cdot)) - uA^*(k(\cdot)) =
$$
\n
$$
= \frac{\partial}{\partial t}(u(\cdot)k(\cdot)) + \sum_{i=1}^{2} a_i \frac{\partial u}{\partial x_i} + \frac{1}{\tau} \sum_{i=1}^{2} \sigma_i^2 \frac{\partial}{\partial x_i} \left(k(\cdot) \frac{\partial u(\cdot)}{\partial x_i} - u(\cdot) \frac{\partial k(\cdot)}{\partial x_i}\right)
$$
\n(21)  
\nFrom the boundary and initial condition of  $u$ ,  $k$ , each term of equation (21) is expressed  
\nis following:  
\n
$$
\int_{R_{+}^2}^T \int_{\tau}^{\tau} \frac{\partial}{\partial t}(u(\cdot)k(\cdot))dtdx = \int_{R_{+}^2} u(\cdot)k(\cdot) \Big|_{\tau}^T dx =
$$
\n
$$
= \int_{R_{+}^2} k(T, x; \tau, \xi) dx - \int_{R_{+}^2} u(\tau, x) \delta(x - \xi) dx =
$$

$$
= \int_{R_+^2} k(T, x; \tau, \xi) dx - u(\tau, \xi)
$$
\n<sup>(22)</sup>

$$
= \int_{R_{+}^{2}} k(T, x; \tau, \xi) dx - u(\tau, \xi)
$$
\n
$$
\int_{R_{+}^{2}}^{T} \int_{R_{+}^{2}} \frac{\partial (k(\cdot)u(\cdot))}{\partial x_{1}} dx dt = \int_{\tau}^{T+\infty} k(\cdot)u(\cdot) \Big|_{0}^{+\infty} dtdx_{2} = 0
$$
\n
$$
\int_{\tau}^{T} \int_{R_{+}^{2}} \frac{\partial}{\partial x_{i}} \Bigg[ k(\cdot) \frac{\partial u(\cdot)}{\partial x_{i}} - u(\cdot) \frac{\partial k(\cdot)}{\partial x_{i}} \Bigg] dx dt = 0
$$
\n(24)\n
$$
-(24) \text{ and } (1), \text{ we obtain}
$$
\n
$$
\int_{R_{+}^{2}} k(\cdot) A(u(\cdot)) dx dt = \int_{R_{+}^{2}} k(T, x; \tau, \xi) dx - u(\tau, \xi)
$$
\n
$$
\int_{R_{+}^{2}} k(T, x; \tau, \xi) A(u(\cdot)) dx dt =
$$
\n(25)

$$
\int_{\tau}^{T} \int_{R_{i}^{2}} \frac{\partial}{\partial x_{i}} \left[ k(\cdot) \frac{\partial u(\cdot)}{\partial x_{i}} - u(\cdot) \frac{\partial k(\cdot)}{\partial x_{i}} \right] dx dt = 0
$$
\n(24)

From  $(21)$  - $(24)$  and  $(1)$ , we obtain

$$
\int_{\tau}^{T} \int_{R_{+}^{2}} k(\cdot) A(u(\cdot)) dx dt = \int_{R_{+}^{2}} k(T, x; \tau, \xi) dx - u(\tau, \xi)
$$
\n(25)

$$
= \int_{R_{+}^{2}} k(T, x; \tau, \xi) dx - u(\tau, \xi)
$$
\n(22)  
\n
$$
\int_{R_{+}^{2}} \int_{\frac{\partial X}{\partial x_{i}}} \frac{\partial(k(\cdot)u(\cdot))}{\partial x_{i}} dxdt = \int_{\tau}^{T+\infty} b(\cdot)u(\cdot)^{-1} \frac{\partial}{\partial t} dx = 0
$$
\n(23)  
\n
$$
\int_{\tau}^{T} \int_{R_{+}^{2}} \frac{\partial}{\partial x_{i}} \left[ k(\cdot) \frac{\partial u(\cdot)}{\partial x_{i}} - u(\cdot) \frac{\partial k(\cdot)}{\partial x_{i}} \right] dxdt = 0
$$
\n(24)  
\nFrom (21)-(24) and (1), we obtain  
\n
$$
\int_{\tau}^{T} k(\cdot)A(u(\cdot))dxdt = \int_{R_{+}^{2}} k(T, x; \tau, \xi)dx - u(\tau, \xi)
$$
\n(25)  
\n
$$
\int_{\tau}^{T} k(T, x; \tau, \xi)A(u(\cdot))dxdt =
$$
\n
$$
= \int_{\tau}^{T} \int_{R_{+}^{2}} k(T, x; \tau, \xi) \left( -\sum_{k} \lambda_{k} \int_{R_{+}^{2}} u(t, z)p_{1}(z - x)dz \right) dxdt =
$$
\n
$$
= -\int_{\tau}^{T} \int_{R_{+}^{2}} k(T, x; \tau, \xi) dxdt
$$
\n(26)  
\nTherefore, by substituting equation (26) to (27), we obtain equation (19).  
\n
$$
X = \int_{\tau}^{T} \int_{R_{+}^{2}} u(t, x)G(t, x; \tau, \xi) dxdt
$$
\n(28)  
\nConversely, the solution of equation (19) show one of (1).  
\n
$$
X = \int_{\tau}^{T} G(t, x; \tau, \xi) dxdt \leq |\lambda| \int_{\tau}^{T} k(t, x; \tau, \xi) dxdt < \infty
$$
\n
$$
\int_{\tau}^{T} G(t, x; \tau, \xi) dxdt \leq |\lambda| \int_{\tau}^{T} k(t, x; \tau, \xi) dxdt < \infty
$$
\n
$$
B = \int_{\tau}^{T} \int_{R_{+}^{2}}^{R} k(t, x; \tau
$$

Therefore, by substituting equation (26) to (27), we obtain equation (19). Namely, the solution of equation (1) is solution of (19). Conversely, let the solution of (19) show one of (1).

Assume that  $u$  is solution of (19). From equation (10), integral kernel  $G$  of (19) is as following:

$$
\int_{\tau}^{T} \int_{R_{+}^{2}} G(t, x; \tau, \xi) dx dt \leq |\lambda| \int_{\tau}^{T} \int_{R_{+}^{2}} k(t, x; \tau, \xi) dx dt
$$

G is integrable in domain  $\Sigma = \{ (t \xi) \in [\tau T] \times R_+ \}$ .

,

Therefore, for arbitrary  $\xi \in R_{+,O} \leq \tau \leq T_{,if} u$  is the solution of (19), it must satisfy following equation: for arbitrary  $\xi \in R_+$ ,  $o \le \tau \le T$ , if  $u$  is the solution of (19), it must satisfy<br>ation:<br> $(t, x; \tau, \xi) \left[ A(u(t, x)) + \sum_{\substack{\lambda_k \ k_i^2}} u(t, z) p_k(z - x) dz \right] dx dt = 0$ <br>that the braket part of integrade is zero, namely, equation (1) is sati

$$
\int_{\tau}^{T} \int_{R_{+}^{2}} k(t, x; \tau, \xi) \left[ A(u(t, x)) + \sum_{k} \lambda_{k} \int_{R_{+}^{2}} u(t, z) p_{k}(z - x) dz \right] dx dt = 0
$$
\n(27)

This shows that the braket part of integrade is zero, namely, equation (1) is satisfied. The proof is completed.

**Lemma 1:** If  $\mathit{P}(x)$  is density function of regular distribution and  $\mathit{P}_2(x)$  is arbitrary hyper-density function,  $p_1 * p_2(x)$  is infinite time differential in usual meaning. Where  $*$ is synthesis multiple symbol. book that the braket part of integrade is zero, namely, equation (1) is satisfied.<br>
1: If  $P(x)$  is density function of regular distribution and  $P_2(x)$  is arbitrary<br>
Instity function,  $P_1 * P_2(x)$  is infinite time differen

**Lemma 2:**  $F$ ,  $G$  of equation (20) in theorem 2 are infinite time differential on domain  $\Sigma = \{(t, \xi) \in [\tau, T] \times R_+^2\}$ sper-density function,  $P_1 * P_2(x)$  is infinite time differential in usual meaning. Where \*<br>synthesis multiple symbol.<br>Lemma 2:  $F_1 G$  of equation (20) in theorem 2 are infinite time differential on domain<br> $\mathcal{L} = \{(t, \xi) \$ 

We omit proof here.

**Theorem 3:** On domain  $\Sigma$  the unique solution of integral equation (19) is as following:

$$
u(\tau,\xi) = F(\tau,\xi) + \sum_{k=1}^{\infty} \int_{\tau}^{T} \int_{R_{+}^{2}} G^{*k}(t,x;\tau,\xi) F(t,x) dx dt
$$
\n(28)

where

$$
G^{*1}(t, x; \tau, \xi) = G(t, x; \tau, \xi)
$$
  

$$
G^{*k}(t, x; \tau, \xi) = \int_{\tau}^{T} G(s, z; \tau, \xi) G^{*(k-1)}(t, x; s, z) dz ds
$$

 $G$  equal to equation (20) of theorem 2.

Proof: By means of lemma 2 and convergence property condition of progressive formula it is sufficient to show that inequality (29) hold:

,

$$
\max_{\{\tau,\xi\}\in\Sigma} \int_{\tau}^{T} \int_{R_{+}^{2}} |G(t, x; \tau, \xi)| dt dx \le 1
$$
\nequation (20) of theorem 2,

\n
$$
\int_{\Gamma} \int_{\Gamma} |G(t, x; \tau)| dt + \int_{\Gamma} \int_{\Gamma} |G(t, x; \tau)| dt
$$

From the equation (20) of theorem 2,

$$
\max_{\tau,\xi} \int_{\tau}^{T} \int_{\tau} |G(t, x; \tau, \xi)| dt dx \le 1
$$
\n
$$
\text{equation (20) of theorem 2,}
$$
\n
$$
\int_{\tau}^{T} \int_{R_{+}^{2}} |G(t, x; \tau, \xi)| dt dx \le |a_{0}| \int_{\tau}^{T} \int_{R_{+}^{2}} k(t, x; \tau, \xi) dx dt
$$
\n
$$
\text{for hand, the equation (10) can rewrite as following:}
$$
\n
$$
= \frac{e^{-a_{0}(t-s)}}{2\pi\sigma_{1}\sigma_{2}(t-\tau)} \left( \exp\left\{-\frac{(y_{1} - \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} - \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-s)} \right\} \right)
$$
\n
$$
\frac{(y_{1} - \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} + \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)} \left\{ e^{-\frac{2a_{1}\xi_{1}}{\sigma_{2}^{2}}} - \frac{2\sigma_{1}^{2}(t-\tau)}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} - \xi_{2} - a_{1}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)} \right\} e^{-\frac{2a_{1}\xi_{1}}{\sigma_{1}^{2}}} + \frac{(y_{1} + \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} + \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)} e^{-\frac{2a_{1}\xi_{1}}{\sigma_{1}^{2}} - \frac{2a_{2}\xi_{2}}{\sigma_{1}^{2}}}
$$
\n
$$
\frac{(y_{1} + \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} + \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)} e^{-\frac{2a_{1}\xi_{1}}{\sigma_{1}^{2}} - \frac{2a_{2}\xi_{2
$$

 $\sim$ 

On the other hand, the equation (10) can rewrite as following:

$$
\max_{(\tau,\xi)\in\Sigma} \int_{\tau}^{T} \int_{R_{+}^{2}} |G(t, x; \tau, \xi)| dt dx \le 1
$$
\nFrom the equation (20) of theorem 2,

\n
$$
\int_{\tau}^{T} \int_{R_{+}^{2}} G(t, x; \tau, \xi) dt dx \le |a_{0}| \int_{\tau}^{T} \int_{R_{+}^{2}} k(t, x; \tau, \xi) dx dt
$$
\nOn the other hand, the equation (10) can rewrite as following:

\n
$$
k(t, y; \tau, \xi) = \frac{e^{-a_{0}(t-s)}}{2\pi\sigma_{1}\sigma_{2}(t-\tau)} \left( \exp\left\{-\frac{(y_{1} - \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} - \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)}\right\}
$$
\n
$$
-\exp\left\{-\frac{(y_{1} - \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} + \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)}\right\} e^{-\frac{2a_{2}\xi_{2}}{\sigma_{2}^{2}} - \frac{2a_{1}\xi_{2}}{\sigma_{1}^{2}(t-\tau)}} \approx \frac{2a_{1}\xi_{2}}{\sigma_{1}^{2}(t-\tau)} \left( \frac{2a_{1}\xi_{2}}{\sigma_{2}^{2}(t-\tau)} - \frac{2a_{2}\xi_{2}}{\sigma_{2}^{2}(t-\tau)} \right)
$$
\n(29)

$$
-\exp\left\{-\frac{(y_1-\xi_1-a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)}-\frac{(y_2+\xi_2-a_2(t-\tau))^2}{2\sigma_2^2(t-\tau)}\right\}e^{-\frac{2a_2\xi_2}{\sigma_2^2}}-
$$

$$
-\exp\left\{-\frac{(y_1+\xi_1-a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)}-\frac{(y_2-\xi_2-a_1(t-\tau))^2}{2\sigma_2^2(t-\tau)}\right\}e^{-\frac{2a_1\xi_1}{\sigma_1^2}}+
$$

From the equation (20) of theorem 2,  
\n
$$
\int_{\tau}^{T} \int_{R_{+}^{2}} G(t, x; \tau, \xi) dtdx \leq |a_{0}| \int_{\tau}^{T} \int_{R_{+}^{2}} k(t, x; \tau, \xi) dxdt
$$
\nOn the other hand, the equation (10) can rewrite as following:  
\n
$$
k(t, y; \tau, \xi) = \frac{e^{-a_{0}(t-s)}}{2\pi\sigma_{1}\sigma_{2}(t-\tau)} \left( \exp\left\{-\frac{(y_{1} - \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} - \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)}\right\} \right)
$$
\n
$$
-\exp\left\{-\frac{(y_{1} - \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} + \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)}\right\} e^{-\frac{2a_{3}\xi_{2}}{\sigma_{2}^{2}} - \frac{1}{2\sigma_{1}^{2}(t-\tau)} - \exp\left\{-\frac{(y_{1} + \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} - \xi_{2} - a_{1}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)}\right\} e^{-\frac{2a_{1}\xi_{1}}{\sigma_{1}^{2}} + \frac{1}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{1} + \xi_{1} - a_{1}(t-\tau))^{2}}{2\sigma_{1}^{2}(t-\tau)} - \frac{(y_{2} + \xi_{2} - a_{2}(t-\tau))^{2}}{2\sigma_{2}^{2}(t-\tau)}\right\} e^{-\frac{2a_{1}\xi_{1}}{\sigma_{1}^{2}} - \frac{2a_{2}\xi_{2}}{\sigma_{2}^{2}} - \frac{1}{2\sigma_{2}^{2}}.
$$
\n(31)  
\nNow,  $N(d)$ ,  $d_{j}(t \tau \xi_{j})$ ,  $d_{j}(t \tau \xi_{j})$  are denoted as following:

Now,  $N(d)$   $_{,}d_{,j}(t\;\tau\;\xi_{j})$   $_{,}d_{,j}(t\;\tau\;\xi_{j})$  are denoted as following:

$$
k(t, y; \tau, \xi) = \frac{e^{-a_0(t-s)}}{2\pi\sigma_1\sigma_2(t-\tau)} \Biggl\{ \exp\Biggl\{ -\frac{(y_1 - \xi_1 - a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)} - \frac{(y_2 - \xi_2 - a_2(t-\tau))^2}{2\sigma_2^2(t-s)} \Biggr\}
$$
  
\n
$$
- \exp\Biggl\{ -\frac{(y_1 - \xi_1 - a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)} - \frac{(y_2 + \xi_2 - a_2(t-\tau))^2}{2\sigma_2^2(t-\tau)} \Biggr\} e^{-\frac{2a_2\xi_2}{\sigma_2^2}}
$$
  
\n
$$
- \exp\Biggl\{ -\frac{(y_1 + \xi_1 - a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)} - \frac{(y_2 - \xi_2 - a_1(t-\tau))^2}{2\sigma_2^2(t-\tau)} \Biggr\} e^{-\frac{2a_1\xi_1}{\sigma_1^2}} +
$$
  
\n
$$
+ \exp\Biggl\{ -\frac{(y_1 + \xi_1 - a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)} - \frac{(y_2 + \xi_2 - a_2(t-\tau))^2}{2\sigma_2^2(t-\tau)} \Biggr\} e^{-\frac{2a_1\xi_1 - 2a_2\xi_2}{\sigma_1^3}}
$$
  
\nNow,  $N(d), d, (t, \tau, \xi_1)$  are denoted as following:  
\n
$$
N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{\tau}} dx
$$
  
\n
$$
d_{1, j}(t; \tau, \xi_j) = -\frac{(\xi_j + \alpha_j\sigma_j^2(t-\tau))}{\sigma_j\sqrt{t-\tau}}
$$
  
\n
$$
d_{2, j}(t; \tau, \xi_j) = \frac{-(\xi_j - \alpha_j\sigma_j^2(t-\tau))}{\sigma_j\sqrt{t-\tau}}
$$
  
\n $j = 1, 2$   
\nBy substituting equation (31) into (30), we have

By substituting equation (31) into (30), we have

$$
\begin{vmatrix}\n\int_{\mathbb{R}} \int_{\mathbb{R}^2} k(t, x; \tau, \xi) dx dt \\
\int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{-a_0(t-z)} \prod_{j=1}^2 \left[ 1 - N(d_j) - e^{-2\alpha_j \xi_j} (1 - N(d_2)) \right] dt\n\end{vmatrix}
$$
\n(33)  
\nWe define  $\mathcal{S}(\tau, \xi_j)$  so that  $\mathcal{S}(\tau, \xi_j) = \left| - N(d_{1j}) - e^{-2\alpha_j \xi_j} (1 - N(d_{2j})) \right|$ .  
\nThen, function  $\mathcal{S}(\tau, \xi_j)$  monotonously increase with respect to  $\xi_j$ ,  $\mathcal{S}(\tau, 0) = 0$  and  $\mathcal{S}(\tau, +\infty) = 1$ .  
\nTherefore, from equation (33), we obtain that  
\n
$$
\left| \int_{\tau}^{\tau} \int_{\mathbb{R}^2_+} k(t, x; \tau, \xi) dx dt \right| \le \left| \int_{\tau}^{\tau} e^{-a_0(t-\tau)} dt \right| = \frac{1}{a_0} (1 - e^{-a_0(t-\tau)}) \le \frac{1}{a_0}
$$
\n(34)  
\nFrom equation (30), (34), equation (29) is satisfied. Therefore we can apply progressive  
\normula of second kinds of integral equations.

Then, function  $\,{{\mathscr E}}(\tau\,\,\xi_j)\,$  monotonously increase with respect to  $\, \breve{\xi}_j\,$  ,  $\,g(\tau,0)\!=\!0\,$   $\,$  and and  $g(\tau, +\infty) = 1$ . define  $g(r \xi_j)$  so that  $g(r, \xi_j) = |-N(d_{1j}) - e^{-2\alpha_j \xi_j} (1 - N(d_{2j}))|$ .<br>
In function  $g(r \xi_j)$  monotonously increase with respect to  $\xi_j$ ,  $g(r, 0) = 0$  and<br>  $+\infty$ ) = 1<br>
refore, from equation (33), we obtain that<br>  $\left|\int_{r}^{r} \int_{R_r}$ 

Therefore, from equation (33), we obtain that

$$
\left| \int_{\tau}^{T} \int_{R_{+}^{2}} k(t, x; \tau, \xi) dx dt \right| \leq \left| \int_{\tau}^{T} e^{-a_0(t-\tau)} dt \right| = \frac{1}{a_0} \left( 1 - e^{-a_0(t-\tau)} \right) \leq \frac{1}{a_0}
$$
\n(34)

From equation (30), (34), equation (29) is satisfied. Therefore we can apply progressive formula of second kinds of integral equations.

Let the operator  $\Phi: C((0,T]\times R_+^2) \to C\big((\tau,T]\times R_+^2\big)$  be

$$
\Phi(\varphi) = \lambda \int_{\tau}^{T} \int_{R_{+}^{2}} G(t, x; \tau, \xi) \varphi(t, x) dx dt + F(\tau, \xi)
$$

Then, equation (19) equal to operation equation  $u = \Phi u$ .

Now, let 
$$
u^0(\tau, \xi) = F(\tau, \xi)
$$
 (35)

 $\mathcal{L}^{\text{max}}$ 

be zero-order approximation. By applying

$$
u^{(k+1)} = \Phi(u^{(k)})
$$
\n(36)

we obtain the equation (28).

On the other hand, because the operator  $\Phi$  is reduced operator, it has the uniquefixed point and therefore, approximation formula (36) converges to the unique solution.

Their applications in insurance mathematics

The two-dimensional risk model we consider in this paper can be formally stated as

$$
\begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t + \begin{pmatrix} \sigma_1 w_r(t) \\ \sigma_2 w_r(t) \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^{M_1(t)} z_{1k} \\ \sum_{k=1}^{M_2(t)} z_{2k} \end{pmatrix}
$$
\nWhere  $M_1(t)$ ,  $M_2(t)$  are the number of claims between time 0 and t which follows a

Where  $M_1(t)$ ,  $M_2(t)$  are the number of claims between time 0 and t, which follows a

Poisson process with parameter λ.  $\{z_{ik}\}$  are claim size random variables as in the univariate risk model. For simplicity, we assume that  $\{X1k, k= 1,2,...\}$  and  $\{X2k, k= 1,2,...\}$ are independent, and furthermore, both of them are also independent of M1(t), M2(t).

 $u_i$ the initial surplus of eath insurance company,  $\overline{c_i}$  the rate at which the premiums are received. dependent, and furthermore, both of them are also independent of M1(t), M2(t).<br>
initial surplus of eath insurance company,  $c_i$  the rate at which the premiums are<br>
d.<br>
is the surplus of  $i$ -th insurance company at time t v the initial surplus of eath insurance company,  $c_i$  the rate at which the pre<br>
sceived.<br>  $R_i(t)$  is the surplus of  $i$  -th insurance company at time  $t \ge 0$ .<br>
in this paper, we consider the following type of ruin:<br>  $T = \inf$ 

 $R_i(t)$  is the surplus of  $i$  -th insurance company at time t  $\geq 0$ .

In this paper, we consider the following type of ruin:

$$
T = \inf \{ t \mid \min \{ R_1(t), R_2(t) \} < 0 \}.
$$

With the time of ruin defined, the corresponding probability of ruin is denoted by

$$
\Phi(u_1, u_2) = P\{T < \infty \ | (R_1(0), \ R_2(0)) = (u_1, u_2) \}.
$$

Set  $R(t) = (R_1(t), R_2(t))^T$ . We can rewrite the equation (37) as following:

$$
R_i(t)
$$
 is the surplus of *i* -th insurance company at time t ≥ 0.  
\nIn this paper, we consider the following type of ruin:  
\n
$$
T = \inf \{t | \min \{R_1(t), R_2(t)\} < 0\}
$$
\nWith the time of ruin defined, the corresponding probability of ruin is denoted by  
\n
$$
\Phi(u_1, u_2) = P\{T < \infty | (R_1(0), R_2(0)) = (u_1, u_2) \}
$$
\nSet  $R(t) = (R_1(t), R_2(t))^T$ . We can rewrite the equation (37) as following:  
\n
$$
\begin{cases}\n dR(t) = c \cdot dt + \sigma \cdot dw(t) + \int \alpha(z)(\mu - \nu)(dt, dz) + \int \alpha(z)\mu(dt, dz) \\
 |z| < 1\n\end{cases}
$$
\n
$$
R(0) = u
$$
\nWhere  
\n
$$
c = (c_1, c_2)^T \quad \sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \alpha(z) = \begin{pmatrix} \alpha_{11}(z) & \alpha_{12}(z) \\ \alpha_{21}(z) & \alpha_{22}(z) \end{pmatrix} \quad u = (u_1, u_2)^T
$$

**Where** 

$$
R_i(t)
$$
 is the surplus of *i*-th insurance company at time t ≥ 0.  
\nIn this paper, we consider the following type of ruin:  
\n
$$
T = \inf \{t \mid \min \{R_1(t), R_2(t)\} < 0\}
$$
\nWith the time of ruin defined, the corresponding probability of ruin is denoted by  
\n
$$
\Phi(u_1, u_2) = P\{T < \infty \mid (R_1(0), R_2(0)) = (u_1, u_2)\}
$$
\nSet  $R(t) = (R_1(t), R_2(t))^T$ . We can rewrite the equation (37) as following:  
\n
$$
\begin{cases}\ndR(t) = c \cdot dt + \sigma \cdot dw(t) + \int \alpha(z)(\mu - v)(dt, dz) + \int \alpha(z)\mu(dt, dz) \\
R(0) = u\n\end{cases}
$$
\nWhere  
\n
$$
c = (c_1, c_2)^T \quad \sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \alpha(z) = \begin{pmatrix} \alpha_{11}(z) & \alpha_{12}(z) \\ \alpha_{21}(z) & \alpha_{22}(z) \end{pmatrix} \quad u = (u_1, u_2)^T \\
11\n\end{cases}
$$

.

 $W(t) = (W_1(t), W_2(t))^T$ are independent brownian motion,

$$
\nu(t) = (w_1(t), w_2(t))^T
$$
 are independent brownian motion,  

$$
\mu(dt, dz) = (\mu_1(dt, dz), \cdots, \mu_d(dt, dz))^T
$$
 is Poisson probability measure that  $\mu_i$  are

independent and have adjustment  $\ E\mu_i(dt,dz)\!=\!\nu_i(dz)dt$  .

$$
w(t) = (w_1(t), w_2(t))'
$$
 are independent brownian motion,  
\n
$$
\mu(dt, dz) = (\mu_1(dt, dz), \dots, \mu_d(dt, dz))'
$$
 is Poisson probability measure that  $\mu_i$  are  
\nndependent and have adjustment  $E\mu_i(dt, dz) = v_i(dz)dt$ .  
\nThen,  $R(t)$  is Levy process. On  $0 < s$  and  $x \in R^2_+$ , let ruin time  $T_{s,x}$  define  
\n $T_{s,x} = \inf\{t \ge s, R_1(t) < 0 \vee R_2(t) < 0 | R(s) = x\}$   
\nand let ruin probability of  $T$  time before define  $\Psi(s, x; T) = P\{T_{s,x} \le T\}$ ,  
\nsurvival probability is expressed as  $\Phi(s, x; T) = 1 - \Psi(s, x; T)$ .  
\n $\Phi$  satisfy the following partial differential-integral equation:  
\n $\frac{\partial}{\partial s}\Phi(s, x) + L\Phi(s, x) = 0$   
\nBy means of definition of  $\Phi$ , it satisfy  
\n $\Phi(s, (x_1, 0)) = 0, \Phi(s, (0, x_2)) = 0, \Phi(T, (x_1, x_2)) = 1$ 

and let ruin probabiliy of  $\left\{ T\right.$  time before define  $\left\{ \Psi (s, x; T)\!=\! P\!\left\{ T_{s, x}\leq T\right\} \right\} _{s}$ 

survival probability is expressed as  $\left(\Phi(s, x; T) \right) = 1 - \Psi(s, x; T)$ 

 $\Phi$  satisfy the following partial differential-integral equation:

$$
\frac{\partial}{\partial s}\Phi(s,x) + L\Phi(s,x) = 0
$$

By means of definition of  $\Phi$ , it satisfy

$$
\Phi(s,(x_1,0))=0
$$
,  $\Phi(s,(0,x_2))=0$ ,  $\Phi(T,(x_1,x_2))=1$ ,

where the operator  $L$  is

Then, 
$$
R(t)
$$
 is Levy process. On  $0 < s$  and  $x \in R_{+}$ , let ruin time  $T_{s,x}$  define  
\n
$$
T_{s,x} = \inf\{t \ge s, R_1(t) < 0 \vee R_2(t) < 0 | R(s) = x\}
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\n
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\frac{\partial}{\partial s}\Phi(s, x) + L\Phi(s, x) = 0
$$
\nBy means of definition of  $\Phi$ , it satisfy  
\n
$$
\Phi(s, (x_1, 0)) = 0, \Phi(s, (0, x_2)) = 0, \Phi(T, (x_1, x_2)) = 1
$$
\nwhere the operator  $L$  is  
\n
$$
L\Phi(s, x) = \sum c_i \frac{\partial \Phi}{\partial x_i} + \frac{1}{\tau} \sum \sigma_i^2 \frac{\partial^2 \Phi}{\partial x_i^2} +
$$

Therefore, the survival probability  $\Phi$  until time  $T$  can be obtained by means of equation (28) as following:

$$
\Phi(\tau,\xi)=F(\tau,\xi)+\sum_{k=1}^{\infty}\int\limits_{\tau}\int\limits_{R_{+}^{2}}G^{(k)}(t,x;\tau,\xi)F(t,x)dxdt
$$

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