Skyrme Model, Wess-Zumino Anomaly, Quark Model, and Consistent Symmetry Breaking

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Abstract

The original Skyrme lagrangian needs to be supplemented with a Wess-Zumino anomaly term to ensure proper quantzation. This is our Skyrme-Wess-Zumino model here. In this model, we show that the study of the electric charges is a very discriminating property. It provides powerful statements as to how the two flavour group SU(2) may be embedded in the three flavour group SU(3). The subsequent symmetry breaking is found to be very different from the one necessary in the SU(3) quark model. The Skyrme-Wess-Zumino model leads to a unique and unambiguos symmetry breaking process. It is known that all Irreducible Representations given by triangle diagrams for SU(3) are 3, 6, 10, 15, 21 etc. dimensional states. The triplet, being the lowest dimensional one, plays the most crucial and basic role here. This leads to composite Sakaton as emerging to become the proper Irreducible Representation of the flavour group SU(3) in the Skyrme-Wess-Zumino model.

Keywords: Topological Skyrme model, Wess-Zumino anomaly, QCD, SU(2) and SU(3) flavours, quark model, symmetry breaking, Sakaton

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The topological Skyrme model of the 1960's [1] has been focus of much activity in recent years [2-14]. Most of the work has been done for two- and three-flavours. The original Skyrme lagrangian needs to be supplemented with a Wess-Zumino anomaly term spplemented in the action, to ensure proper quantzation [2,3,4,13,14]. Hence our model in this paper shall be the original Skyrme lagrangian plus the Wess-Zumino anomaly term. This we call here as the Skyrme-Wess-Zumino model.

The connection between the Skyrme model and the well established quark models, is established through the studies of Quantum Chromodynamics (QCD) for arbitrary number of colours [15-19].

In our Skyrme-Wess-Zumino model we show that the stduy of the electric charges is a very discriminating property. It provides powerful statements as to how the two flavour group $SU(2)_F$, may be embedded in the three flavour group $SU(3)_F$. It leads to unique and unambiguos symmetry breaking process, as we discuss below. The quark model for arbtrary number of colours [15-19] provides consistency to the above statements regarding symmetry breaking.

The pure Skyrme Lagrangian is wrtten as [2-6],

$$L_S = \frac{{f_\pi}^2}{4} Tr(L_\mu L^\mu) + \frac{1}{32e^2} Tr[L_\mu, L_\nu]^2 \tag{1}$$

where $L_\mu = U^\dagger \partial_\mu U$. The topological current is,

$$W_\mu = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} Tr[L_\nu L_\alpha L_\beta] \tag{2}$$

This topological current is conserved, i.e. $\partial^\mu W_\mu = 0$ and giving a conserved topological charge $q = \int W_0 d^3x$.

Here $U(x)$ is an element of the group $SU(2)_F$,

$$U(x)^{SU(2)} = exp((i\tau^a \phi^a / f_\pi), \;\; (a = 1, 2, 3) \tag{3}$$

The solitonic structure present in the Lagrangian is obtained on making Skyrme ansatz as follows [2-6].

$$U_c(x)^{SU(2)} = exp((i/f_\pi \theta(r) \hat{r}^a \tau^a), \;\; (a = 1, 2, 3) \tag{4}$$

This $U_c(x)$ is called the Skyrmion. But on quantization, the two flavour model Skyrmion has a well known boson-fermion ambiguity [2-6]. This is rectified by going to three flavours. In that case we take,

$$U(x)^{SU(3)} = exp[\frac{i\lambda^a \phi^a(x)}{f_\pi}] \;\; (a = 1, 2..., 8) \tag{5}$$

with ϕ^a the pseudoscalar octet of π, K and η mesons. In our Skyrme-Wess-Zumino model this is supplemented with a Wess-Zumino (WZ) effective action [2-6]

$$\Gamma_{WZ} = \frac{-i}{240\pi^2} \int_\Sigma d^5x \epsilon^{\mu\nu\alpha\beta\gamma} Tr[L_\mu L_\nu L_\alpha L_\beta L_\gamma] \tag{6}$$

on surface Σ. Thus with this anomaly term, the effective action is.

$$S_{eff} = \frac{f_\pi^2}{4} \int d^4x \; Tr \; [L_\mu L^\mu] + n \; \Gamma_{WZ} \tag{7}$$

where the winding number n is an integer $n \in Z$, the homotopy group of mapping being $\Pi_5(SU(3)) = Z$.

Write effective action as,

$$S_{eff} = \frac{f_\pi^2}{4} \int d^4x \; Tr \; [\partial_\mu U \partial^\mu U^\dagger] + n \; \Gamma_{WZ} \tag{8}$$

Taking Q as charge operator, under a local electro-magnetic gauge transformation $h(x) = \exp(i\theta(x)Q)$ with small θ, one finds

$$\Gamma_{WZ} \to \Gamma_{WZ} - \int d^4x \partial_\mu x J^\mu(x) \tag{9}$$

whee J^μ is the Noether current arising from the WZ term. This coupling to the photon field is like,

$$J_\mu = \frac{1}{48\pi^2} \epsilon^{\mu\nu\alpha\beta} Tr[Q(L_\nu L_\alpha L_\beta - R_\nu R_\alpha R_\beta)] \tag{10}$$

where $L_\mu = U^\dagger \partial_\mu U$, $R_\mu = U \partial_\mu U^\dagger$. With the electromagnetic field A_μ present, the gauge invariant form of eqn. (8) is,

$$\hat{S_{eff}} = \frac{f_\pi^2}{4} \int d^4x \; Tr \; [L_\mu L^\mu] + n \; \hat{\Gamma_{WZ}} \tag{11}$$

This means that when replacing the LHS in eqn. (10) by $\hat{\Gamma_{WZ}}$, then the RHS has two new terms involving $F_{\mu\nu}F^{\mu\nu}$. This allows us to interpret J_μ with the current carried by quarks [2-6]. With the charge operator Q, J_μ is found to be isoscalar. To obtain the baryon current from eqn. (11), one replaces Q by $\frac{1}{N_c}$ (where N_c is the number of colours in $SU(N_c)$ - QCD for arbitrary number of colours), which is the baryon charge carried by each quark making up the baryon. For total antisymmetry, N_c number of quarks are needed to make up a baryon. Then $nJ_\mu \to J_\mu^B$ gives,

$$\begin{aligned} nJ_\mu^B(x) &= \frac{1}{48\pi^2} \left(\frac{n}{N_c}\right) \epsilon^{\mu\nu\alpha\beta} Tr[(L_\nu L_\alpha L_\beta - R_\nu R_\alpha R_\beta)] \\ &= \frac{1}{24\pi^2} \epsilon^{\mu\nu\alpha\beta} Tr[L_\nu L_\alpha L_\beta] \end{aligned} \tag{12}$$

This is the same as the topological current of Skyrme as given by eqn. (2). Thus the gauged WZ term gives rise to $J_\mu(x)$ which in turn gives the baryon charge. Thus though the WZ term Γ_{WZ} is zero for two-flavour case, but $J_\mu(x)$ still contibutes to the two-flavour case.

Next we embed the SU(2) Skyrme ansatz into $U(x)^{SU(3)}$ as follows for the SU(3) Skyrmion [13],

$$U_c(x)^{SU(2)} \ \rightarrow \ U_c(x)^{SU(3)} = \begin{pmatrix} U_c(x)^{SU(2)} & \\ & \\ & 1 \end{pmatrix} \tag{13}$$

Next we insert the identity,

$$U(\vec{r},t)^{SU(3)} = A(t) U(\vec{r})_c^{SU(3)} A^{-1}(t) \qquad A \in SU(3)_F \tag{14}$$

where A is the collective coordinate. Note that $U(\vec{r},t)$ is invariant under,

$$A \rightarrow A e^{iY\alpha(t)} \tag{15}$$

where

$$Y = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{16}$$

With this the quantum degrees of freedom manifest themselves in the WZ term (eqn. (7)) as,

$$L_{WZ} = -\frac{1}{2} N_c B(U_c) tr(Y A^{-1} A) \tag{17}$$

where $B(U_c)$ is the baryon number (winding number) of the classical configuration U_c. The gauge invariance leads to changing L_{WZ} to

$$L_{WZ} \rightarrow L_{WZ} + \frac{1}{3} N_c B(U_c) \dot{\alpha} \tag{18}$$

In the quantized theory A and Y are operators and from Noether's theorem one obtains (with Ψ as allowed quantum state)

$$\hat{Y}\Psi = \frac{1}{3} N_c B \Psi \tag{19}$$

This gives the right-hypercharge,

$$Y_R = \frac{1}{3} N_c B \tag{20}$$

where the baryon number B is necessarily an integer and colour N_c is an integer too [2,3,13]. With B = 1 and $N_c = 3$ one gets $Y_R = 1$. This identifies the nucleon hypercharge with the body-fixed hypercharge Y_R. This is the most significant and basic property of $SU(3)_F$ Skyrme-Wess-Zumino model.

The electric charge for $SU(3)_F$ in the Skyrme-Wess-Zumino model has been studied earlier[13,15]. In this paper let us concentrate on the structure of the electric charge in the $SU(2)_F$ model. As pointed out by Blachandran et. al. [13, p. 176] this has not been paid the attention it deserves. This because as we show below, it presents a serious challenge to the Skyrme lagrangian for two flavours. Following Balachandran et. al. [13], we define the electric charge operator in SU(2) as,

$$Q = \left(\begin{array}{cc} q_1 & 0 \\ 0 & q_2 \end{array} \right) \tag{21}$$

It induces the following transformation,

$$U(x) \rightarrow e^{i\epsilon_0 \Lambda Q} U(x) e^{-i\epsilon_0 \Lambda Q} = e^{\frac{i\epsilon_0 \Lambda \tau_3 (q_1 - q_2)}{2}} U(x) e^{\frac{-i\epsilon_0 \Lambda \tau_3 (q_1 - q_2)}{2}} \tag{22}$$

where ϵ_0 is the electromagnetic coupling constant. The Noether current associated with the above symmetry is,

$$\frac{J_\mu^{em}}{\epsilon_0} = \frac{iF_\pi^2}{8} \, Tr \, L_\mu (Q - U^\dagger Q U) - \frac{i}{8\epsilon_0^2} \, Tr \, [L_\nu, Q - U^\dagger Q U][L_\mu, L_\nu] \tag{23}$$

We obtain the gauge theory by replacing

$$\partial_\mu U \rightarrow D_\mu U = \partial_\mu U - i\epsilon_0 \Lambda_\mu [Q, U] \tag{24}$$

To obtain constraints on charges in eqn. (22), first expand on pion fields to obtain,

$$J_\mu^{em} = -i\epsilon_0 (q_1 - q_2)(\pi_- \partial_\mu \pi_+ - \pi_+ \partial_\mu \pi_-) + ... \tag{25}$$

From pion charges one gets

$$(q_1 - q_2) = 1 \tag{26}$$

Next the charges of baryons $N \; and \; \Delta$ with B=1 charge on using eqn. (15),

$$Q = \int d^4x \; J_0^{em}(\vec{x}, t) = \epsilon_0 L_\alpha \; Tr \tau_\alpha Q \tag{27}$$

From eqn. (22) we get,

$$Q = \epsilon_0 (q_1 - q_2) L_3 \tag{28}$$

On using eqn. (27),

$$Q = \epsilon_0 L_3 \tag{29}$$

As L_3 is the third component of the isospiun operator, we get (in units of ϵ_0),

$$Q(proton) = +\frac{1}{2} \;\; and \;\; Q(neutron) = -\frac{1}{2} \tag{30}$$

This is in complete disagreement with experiment. Thus the pure Skyrme Lagrangian eqn. (1) fails to provide correct electric charges to proton and neutron. As such this should be construed to mean that just the Skyrme lagrangian in itself, is not enough to give consistent description of the B=1 nucleon.

It needs another term to pull it out of this conundrum. And indeed we have the additional Wess-Zumino term to do the job. Again let the field U be transformed by an electric charge operator Q as, $U(x) \rightarrow e^{i\Lambda\epsilon_0 Q}U(x)e^{-i\Lambda\epsilon_0 Q}$,

Making $\Lambda = \Lambda(x)$ a local transformation the Noether current is [13]

$$J_\mu{}^{em}(x) = j_\mu{}^{em}(x) + j_\mu{}^{WZ}(x) \tag{31}$$

where the first one is the standard Skyrme term and the second is the Wess-Zumino term

$$j_\mu{}^{WZ}(x) = \frac{\epsilon_0 N_c}{48\pi^2}\epsilon_{\mu\nu\lambda\sigma}TrV^\nu V^\lambda V^\sigma(Q + U^\dagger QU) \tag{32}$$

Remember that even though the WZ term vanishes for two flavours, its resulting contribution to electric charge does not.

One finally obtains [13, p. 208],

$$j_\mu{}^{WZ}(x) = \frac{\epsilon_0}{2}(q_1 + q_2)N_c J_\mu(x) \tag{33}$$

The WZ term correction to the electric charge is therefore,

$$\frac{\epsilon_0}{2}(q_1 + q_2)N_c \int J_0(x)d^3x \tag{34}$$

Using eqn. (15) above,

$$\frac{\epsilon_0}{2}(q_1 + q_2)N_c B(U_c) \tag{35}$$

Remember the right hypercharge $Y_R = 1$ in eqn. (20) and subsequently B=1 for $N_c = 3$. We thus obtain the charges of N *and* Δ if we put,

$$q_1 + q_2 = \frac{1}{3} \tag{36}$$

Along with eqn. (26), we obtain the charges as,

$$q_1 = \frac{2}{3}\ , \quad q_2 = -\frac{1}{3} \tag{37}$$

These are the charges of u- and d- quarks, which make up, for example, the proton of three quarks with baryon number B=1. One also finds that for $SU(3)_F$, $Q_3 = Q_2 = -\frac{1}{3}$ [13,15].

Now note that in our Skyrme-Wess-Zumino model the starting point is the three flavour $SU(3)_F$ group. Hypercharge in eqn. (20) was its main result to provide proper irreducible representation of this group. However when we went down to the smaller two flavour group $SU(2)_F$, then too we ensured that eqn. (20) was still valid. We ultimately got the proper fractional charges of the u- and d- quarks of the isopin group. It is amazing that we are getting the fractional quark charges in the $SU(2)_F$ group itself.

This is opposite to what happens in the $SU(3)_F$ quark model. In the quark model, in the smaller SU(2)-isospin group one has integral charges for nucleon

N = $\binom{p}{n}$. One has to go to higher group $SU(3)_F$, to be able to get fractional charges for the quarks in the quark model.

This is a major difference between the quark model and the topological Skyrme-Wess-Zumino model. In quark model the three fractional charged quarks provide octet and decuptet states. In would be naive to expect that the above difference as to the fractional charges of quarks in the Skyrme-Wess-Zumino model, would do nothing more than just to reproduce the same octet and decuptet structures as in the quark model. In fact one may expect major differences between the two. Let us study as to what these may actually be.

In quark model there are three two-flavour groups; I-spin, U-spin and V-spin, which together constitute the bigger $SU(3)_F$ group [6]. One takes this $SU(3)_F$ symmetry as exact to start with and break it thereafter through a perturbation as,

$$H_{strong} = H_{SU(3)} + \epsilon H_{medium-strong} \tag{38}$$

The octet and decuptet masses are fitted with this expression.

In contrast, in our Skyrme-Wess-Zumino model, $SU(3)_F$ is a good symmetry to start with. Next as we showed above, one ensures that the bigger symmetry group $SU(3)_F$ is broken down to one good, two flavourr $SU(2)_F$ isospin group. This means that the other two flavour subgroups U-spin and V-spin are broken simultaneously. Hence we expect that this symmetry breaking should be through the pure $SU(3)_F$ diagonal generator. Thus

$$m(I,Y) = m_0 + aY \tag{39}$$

This may be studied group theoretically. In SU(3), assuming symmetry breaking of the spin 1/2 baryon octet as given by Okubo [20]

$$M_{(8)} = m_0 Tr(\bar{B}B) + \frac{1}{2}m_1(\{\bar{B},B\}\lambda_8) - \frac{1}{2}m_2 Tr([\bar{B},B]\lambda_8) \tag{40}$$

This gives the well known Gell-Mann-Okubo mass formula

$$m_8(I,Y) = m_0 + aY + b[I(I+1) - \frac{1}{4}Y^2] \tag{41}$$

which works very well for the spin 1/2 baryon octet in SU(3).

For the case of the 3/2 decuptet the above mass formula reduces to a much simpler and a successful form as,

$$m_{10}(I,Y) = m_0^{'} + a^{'}Y \tag{42}$$

The matching of eqns. (39) and (42) indicates that this should be what would hold good for our Skyrme-Wess-Zumino model here,

In fact Okubo has shown [20] that the above complex equation (eqns. (40) and (41)) reduces to the simpler form above (eqn. (39)) for all Irreducible Representations given as triangular diagrams like the decuptet. These are 3, 6, 10, 15, 21 etc, dimensional states. which (except for the first 3-dimensional one) are totally symmetric states of 2- 3-, 4-, 5- etc. quarks respectively. And thus this should be the key to our quest here. Remember that the octet is completely out from this. It works for decuptet though. But we discard it as being of higher dimension and of spin 3/2. The case of the first triangular diagram of dimension-3 and of spin 1/2, appears to be most promising, and thus we discuss it in detail below.

Firstly we take Sakata [21], who extended the group $SU(2)_I$ to $SU(2)_I \times U(1)_Y$, and took Λ as a representation of the U(1) group, and assuming that hadrons could be treated to be composites of these. Thus it was natural to take $\mathcal{S} = \begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$ as the fundamenatl representation of a larger $SU(3)_F$ group [22]. It is called Sakaton in anology with Nucleon of the isospin group. Note that the charges in Sakaton are all integral: 1,0,0 respectively. The Sakata Model predicted the mesons correctly as composites: $3 \times \bar{3} = 1 + 8$. However it failed to describe the baryons as $3 \times 3 \times \bar{3} = 3 + 3 + 6 + 15$. Also as both $p \ and \ \Lambda$ are neutral members of the fundamental triplet in Sakata model, they should have the same magnetic moment, $\mu_\Lambda = \mu_n$. This fails to match the expariment where, $\mu_\Lambda = -0.613 \ and \ \mu_n = -1.913$ in units of $\frac{e\hbar}{2m_p c}$, where mass is that of proton. Thus the fundamental triplet sakaton is rejected.

Now (p, n, Λ) can be composites which is symmetric under exchange in the SU(6) model. However, this belongs to an octet with other spin-1/2 baryons, arising in this SU(6) model. Hence it is not the triangle represenation, that we are seeking.

Hence bottom line is that in our Skyrme-Wess-Zumino model, the three quark masses should abide by the symmetry breaking eqns. (39) and (42). Thus $m_s = m_0 \ , \ \ and \ \ (m_u = m_d) = m_o + aY$. Taking 'a' as negative in magnitute ensures that, $m_s > (m_u = m_d)$. In quark model for arbitrary number of colours, for proton one takes [16,17,18,6] $N_c = 2k + 1$, and one has (k+1) number of u-quark with spin $\vec{S_u} = \frac{k+1}{2}$ and (k) number of d-quarks with spin $\vec{S_d} = \frac{k}{2}$ and $\vec{S_p} = \vec{S_u} + \vec{S_d} = \frac{1}{2}$. Similarly for neutron and Λ. Hence magnetic moments of our composite Sakaton, $\mathcal{S} = \begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$. as per the standard quark model calculations [16,17,18,6]

$$\begin{array}{ccc} Baryons & quark\ model & experiment \\ p & \frac{(4\mu_u - \mu_d)}{3} & 2.793 \\ n & \frac{(4\mu_d - \mu_u)}{3} & -1.913 \\ \Lambda & \mu_s & -0.614 \end{array} \tag{43}$$

Hence the success of the above quark model calculation proves that indeed it is composite sakaton as per the 3-dimensional triangular representation, which provides the proper Irreducible Representation of the $SU(3)_F$ in our Skyrme-Wess-Zumino model. It is not part of octet and decuptet as popularly believed so far [2-4, 13-14]. This is the same conclusion that we arrived recently through a different set of arguments [23] Thus in this paper, we have consolidated our recent result [23] that it is actually composite sakaton, which arises as the lowest excited state in the topological Skyrme-Wess-Zumino model.

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