A Note on Incompatibility of the Dirac-like Field Operator with the Majorana Anzatz

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Abstract
We investigate some subtle points of the Majorana(-like) theories.

1 Introduction.

Majorana deduced his theory of neutral particles, in fact, on the basis of the Dirac equation [1]. However, the quantum field theory has not yet been completed in 1937. The Dirac equation [2, 3, 4] is well known

$$[i\gamma^\mu \partial_\mu - m]\Psi(x) = 0 \tag{1}$$

to describe the charged particles of the spin 1/2. The γ^μ are the Clifford algebra matrices

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\,. \tag{2}$$

Usually, everybody uses the following definition of the field operator [5]:

$$\Psi(x) = \frac{1}{(2\pi)^3} \sum_h \int \frac{d^3\mathbf{p}}{2E_p} \sqrt{m} [u_h(\mathbf{p}) a_h(\mathbf{p}) e^{-ip\cdot x} + v_h(\mathbf{p}) b_h^\dagger(\mathbf{p})] e^{+ip\cdot x}]\,, \tag{3}$$

as given *ab initio*. After actions of the Dirac operator at $\exp(\mp i p_\mu x^\mu)$ the 4-spinors ($u-$ and $v-$) satisfy the momentum-space equations: $(\hat{p}-m)u_h(p) = 0$ and $(\hat{p}+m)v_h(p) = 0$, respectively; the h is the polarization index. It is easy to prove from the characteristic equations $Det(\hat{p} \mp m) = (p_0^2 - \mathbf{p}^2 - m^2)^2 = 0$ that the solutions should satisfy the energy-momentum relation $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$ with both signs of p_0.

2 The Construction of the Field Operators.

The general scheme of construction of the field operator has been presented in [6]. In the case of the $(1/2,0)\oplus(0,1/2)$ representation we have:

$$\Psi(x)=\frac{1}{(2\pi)^3}\int dp e^{-ip\cdot x}\tilde{\Psi}(p)\,. \tag{4}$$

From the Klein-Gordon equation we know:

$$(p^2-m^2)\tilde{\Psi}(p)=0\,. \tag{5}$$

Thus,

$$\tilde{\Psi}(p)=\delta(p^2-m^2)\Psi(p)\,. \tag{6}$$

Next,

$$\begin{aligned}\Psi(x)=\frac{1}{(2\pi)^3}\int dp\, e^{-ip\cdot x}\delta(p^2-m^2)(\theta(p_0)+\theta(-p_0))\Psi(p)=\\ =\frac{1}{(2\pi)^3}\int dp\left[e^{-ip\cdot x}\delta(p^2-m^2)\Psi^+(p)+e^{+ip\cdot x}\delta(p^2-m^2)\Psi^-(p)\right],\end{aligned} \tag{7}$$

where

$$\Psi^+(p)=\theta(p_0)\Psi(p)\,,\text{ and }\Psi^-(p)=\theta(p_0)\Psi(-p)\,, \tag{8}$$

$$\Psi^+(x) = \frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2E_p}e^{-ip\cdot x}\Psi^+(p)\,, \tag{9}$$

$$\Psi^-(x) = \frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2E_p}e^{+ip\cdot x}\Psi^-(p)\,. \tag{10}$$

We continue:

$$\begin{aligned}\Psi(x)=\frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)e^{-ip\cdot x}\Psi(p)=\\ =\frac{1}{(2\pi)^3}\sum_{h=\pm1/2}\int d^4p\,\delta(p_0^2-E_p^2)e^{-ip\cdot x}\sqrt{m}[u_h(p_0,\mathbf{p})a_h(p_0,\mathbf{p})]=\\ =\frac{\sqrt{m}}{(2\pi)^3}\int\frac{d^4p}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)][\theta(p_0)+\theta(-p_0)]e^{-ip\cdot x}\sum_{h=\pm1/2}u_h(p)a_h(p)\end{aligned} \tag{11}$$

$$
\begin{aligned}
&= \frac{\sqrt{m}}{(2\pi)^3}\sum_{h=\pm 1/2}\int \frac{d^4p}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)]\left[\theta(p_0)u_h(p)a_h(p)e^{-ip\cdot x}+\right.\\
&+ \left.\theta(p_0)u_h(-p)a_h(-p)e^{+ip\cdot x}\right] = \frac{\sqrt{m}}{(2\pi)^3}\sum_{h=\pm 1/2}\int \frac{d^3\mathbf{p}}{2E_p}\theta(p_0)\\
&\quad \left[u_h(p)a_h(p)|_{p_0=E_p}e^{-i(E_pt-\mathbf{p}\cdot\mathbf{x})}+u_h(-p)a_h(-p)|_{p_0=E_p}e^{+i(E_pt-\mathbf{p}\cdot\mathbf{x})}\right]
\end{aligned}
$$

During the calculations above we had to represent $1=\theta(p_0)+\theta(-p_0)$ in order to get positive- and negative-frequency parts. Moreover, during these calculations we did not yet assumed, which equation did this field operator (namely, the $u-$ spinor) satisfy, with negative- or positive- mass (except for the Klein-Gordon equation).

In general we should transform $u_h(-p)$ to the $v(p)$. The procedure is the following one [7, 8]. In the Dirac case we should assume the following relation in the field operator:

$$
\sum_h v_h(p)b_h^\dagger(p) = \sum_h u_h(-p)a_h(-p)\,. \tag{12}
$$

We know that [4][1]

$$
\begin{aligned}
\bar{u}_\mu(p)u_\lambda(p) &= +\delta_{\mu\lambda}\,, && (13)\\
\bar{u}_\mu(p)u_\lambda(-p) &= 0\,, && (14)\\
\bar{v}_\mu(p)v_\lambda(p) &= -\delta_{\mu\lambda}\,, && (15)\\
\bar{v}_\mu(p)u_\lambda(p) &= 0\,, && (16)
\end{aligned}
$$

but we need $\Lambda_{\mu\lambda}(p)=\bar{v}_\mu(p)u_\lambda(-p)$. By direct calculations, we find

$$
-b_\mu^\dagger(p) = \sum_\lambda \Lambda_{\mu\lambda}(p)a_\lambda(-p)\,. \tag{17}
$$

where $\Lambda_{\mu\lambda}=-i(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda}$, $\mathbf{n}=\mathbf{p}/|\mathbf{p}|$, and

$$
b_\mu^\dagger(p) = +i\sum_\lambda(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda}a_\lambda(-p)\,. \tag{18}
$$

Multiplying (12) by $\bar{u}_\mu(-p)$ we obtain

$$
a_\mu(-p) = -i\sum_\lambda(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda}b_\lambda^\dagger(p)\,. \tag{19}
$$

[1] μ and λ are the polarization indices. We use the notation of Ref. [4].

The equations are self-consistent.

Next, we can introduce the helicity operator, which commutes with the Dirac Hamiltonian, thus developing the theory in the helicity basis. The 2-eigenspinors of the helicity operator

$$\frac{1}{2}\boldsymbol{\sigma}\cdot\widehat{\mathbf{p}} = \frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{+i\phi} & -\cos\theta\end{pmatrix} \tag{20}$$

can be defined as follows [9, 10]:

$$\phi_{\frac{1}{2}\uparrow} = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{+i\phi/2}\end{pmatrix}, \quad \phi_{\frac{1}{2}\downarrow} = \begin{pmatrix}\sin\frac{\theta}{2}e^{-i\phi/2}\\ -\cos\frac{\theta}{2}e^{+i\phi/2}\end{pmatrix}, \tag{21}$$

for $\pm 1/2$ eigenvalues, respectively.

We can start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i. e., two degrees of freedom), Ref. [3]; $c = \hbar = 1$:

$$(E + \boldsymbol{\sigma}\cdot\mathbf{p})(E - \boldsymbol{\sigma}\cdot\mathbf{p})\phi = m^2\phi\,. \tag{22}$$

It can be re-written in the form of the system of two first-order equations for 2-spinors. Simultaneously, we observe that they may be chosen as eigenstates of the helicity operator which presents in (22). Namely,

$$\begin{aligned}
(E - (\boldsymbol{\sigma}\cdot\mathbf{p}))\phi_\uparrow &= (E-p)\phi_\uparrow = m\chi_\uparrow\,, && (23)\\
(E + (\boldsymbol{\sigma}\cdot\mathbf{p}))\chi_\uparrow &= (E+p)\chi_\uparrow = m\phi_\uparrow\,, && (24)\\
(E - (\boldsymbol{\sigma}\cdot\mathbf{p}))\phi_\downarrow &= (E+p)\phi_\downarrow = m\chi_\downarrow\,, && (25)\\
(E + (\boldsymbol{\sigma}\cdot\mathbf{p}))\chi_\downarrow &= (E-p)\chi_\downarrow = m\phi_\downarrow\,. && (26)
\end{aligned}$$

If the ϕ spinors are defined by the equation (21) then we can construct the corresponding $u-$ and $v-$ 4-spinors.

$$\begin{aligned}
u_\uparrow &= N_\uparrow^+\begin{pmatrix}\phi_\uparrow\\ \frac{E-p}{m}\phi_\uparrow\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{E+p}{m}}\phi_\uparrow\\ \sqrt{\frac{m}{E+p}}\phi_\uparrow\end{pmatrix}, && (27)\\
u_\downarrow &= N_\downarrow^+\begin{pmatrix}\phi_\downarrow\\ \frac{E+p}{m}\phi_\downarrow\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{m}{E+p}}\phi_\downarrow\\ \sqrt{\frac{E+p}{m}}\phi_\downarrow\end{pmatrix}, && (28)\\
v_\uparrow &= N_\uparrow^-\begin{pmatrix}\phi_\uparrow\\ -\frac{E-p}{m}\phi_\uparrow\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{E+p}{m}}\phi_\uparrow\\ -\sqrt{\frac{m}{E+p}}\phi_\uparrow\end{pmatrix}, && (29)\\
v_\downarrow &= N_\downarrow^-\begin{pmatrix}\phi_\downarrow\\ -\frac{E+p}{m}\phi_\downarrow\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{m}{E+p}}\phi_\downarrow\\ -\sqrt{\frac{E+p}{m}}\phi_\downarrow\end{pmatrix}, && (30)
\end{aligned}$$

where the normalization to the unit was again used.

We again define the field operator as in (3) except for the polarization index h, which answers now for the helicity. The commutation relations are assumed to be the standard ones [5, 6, 11, 12], except for adjusting the dimensional factor:

$$\left[a_\mu(\mathbf{p}), a_\lambda^\dagger(\mathbf{k})\right]_+ = 2E_p\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{\mu\lambda}\,, [a_\mu(\mathbf{p}), a_\lambda(\mathbf{k})]_+ = 0 = \left[a_\mu^\dagger(\mathbf{p}), a_\lambda^\dagger(\mathbf{k})\right]_+\,, \tag{31}$$

$$\left[a_\mu(\mathbf{p}), b_\lambda^\dagger(\mathbf{k})\right]_+ = 0 = \left[b_\mu(\mathbf{p}), a_\lambda^\dagger(\mathbf{k})\right]_+\,, \tag{32}$$

$$\left[b_\mu(\mathbf{p}), b_\lambda^\dagger(\mathbf{k})\right]_+ = 2E_p\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{\mu\lambda}\,, [b_\mu(\mathbf{p}), b_\lambda(\mathbf{k})]_+ = 0 = \left[b_\mu^\dagger(\mathbf{p}), b_\lambda^\dagger(\mathbf{k})\right]_+\,. \tag{33}$$

However, the attempt is now failed[2] to obtain the previous result (18) for $\Lambda_{\mu\lambda}(p)$. In this helicity case $\bar{v}_\mu(p)u_\lambda(-p) = i\sigma^y_{\mu\lambda}$. The content of this Section is taken from [13, 14, 15, 16, 17]. In the next Section we turn our attention to the neutral particle theory by E. Majorana.

3 Analysis of the Majorana Anzatz.

It is well known that "*particle=antiparticle*" in the Majorana theory. So, in the language of the quantum field theory we should have

$$b_\mu(E_p, \mathbf{p}) = e^{i\varphi} a_\mu(E_p, \mathbf{p})\,. \tag{34}$$

Usually, different authors use $\varphi = 0, \pm\pi/2$ depending on the metrics and on the forms of the 4-spinors and commutation relations.

So, on using (18) and the above-mentioned postulate we come to:

$$a_\mu^\dagger(p) = +ie^{i\varphi}(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} a_\lambda(-p)\,. \tag{35}$$

On the other hand, on using (19) we make the substitutions $E_p \to -E_p$, $\mathbf{p} \to -\mathbf{p}$ to obtain

$$a_\mu(p) = +i(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} b_\lambda^\dagger(-p)\,. \tag{36}$$

[2]Please do not be confused with signs during calculations. Remember, that $\sqrt{ab} \neq \sqrt{a}\sqrt{b}$ over the field of negative numbers [18].

The totally reflected (34) is $b_\mu(-E_p, -\mathbf{p}) = e^{i\varphi} a_\mu(-E_p, -\mathbf{p})$. Thus,

$$b_\mu^\dagger(-p) = e^{-i\varphi} a_\mu^\dagger(-p)\,. \qquad (37)$$

Combining with (36), we come to

$$a_\mu(p) = +ie^{-i\varphi}(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} a_\lambda^\dagger(-p)\,, \qquad (38)$$

and

$$a_\mu^\dagger(p) = -ie^{i\varphi}(\boldsymbol{\sigma}^*\cdot\mathbf{n})_{\mu\lambda} a_\lambda(-p)\,. \qquad (39)$$

This contradicts with the equation (35) unless we have the preferred axis in every inertial system.

Next, we can use another Majorana anzatz $\Psi = \pm e^{i\alpha}\Psi^c$ with usual definitions

$$C = \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix} \mathcal{K}\,, \quad \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \qquad (40)$$

Thus, on using $Cu_\uparrow^*(\mathbf{p}) = iv_\downarrow(\mathbf{p})$, $Cu_\downarrow^*(\mathbf{p}) = -iv_\uparrow(\mathbf{p})$ we come to other relations between creation/annihilation operators

$$a_\uparrow^\dagger(\mathbf{p}) = \mp ie^{-i\alpha} b_\downarrow^\dagger(\mathbf{p})\,, \qquad (41)$$

$$a_\downarrow^\dagger(\mathbf{p}) = \pm ie^{-i\alpha} b_\uparrow^\dagger(\mathbf{p})\,, \qquad (42)$$

which may be used instead of (34). Due to the possible signs $\pm$ the number of the corresponding states is the same as in the Dirac case that permits us to have the complete system of the Fock states over the $(1/2,0)\oplus(0,1/2)$ representation space in the mathematical sense.[3] However, in this case we deal with the self/anti-self charge conjugate quantum field operator instead of the self/anti-self charge conjugate quantum states. Please remember that it is the latter that answers for the neutral particles; the quantum field operator contains the information about more than one state, which may be either electrically neutral or charged.

[3]Please note that the phase factors may have physical significance in quantum field theories as opposed to the textbook nonrelativistic quantum mechanics, as was discussed recently by several authors.

4 Conclusions.

We conclude that something is missed in the foundations of both the original Majorana theory and its generalizations.

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References

[1] E. Majorana, Nuovo Cim. **14**, 171 (1937).

[2] P. A. M. Dirac, *Proc. Roy. Soc. Lond.* A**117**, 610 (1928).

[3] J. J. Sakurai, *Advanced Quantum Mechanics.* (Addison-Wesley, 1967).

[4] L. H. Ryder, *Quantum Field Theory.* (Cambridge University Press, Cambridge, 1985).

[5] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill Book Co., 1980).

[6] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields.* 2nd Edition. (Nauka, Moscow, 1973).

[7] V. V. Dvoeglazov, Hadronic J. Suppl. **18**, 239 (2003).

[8] Valeri V. Dvoeglazov, Int. J. Mod. Phys. B**20**, 1317 (2006).

[9] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskiĭ, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988), §6.2.5.

[10] V. V. Dvoeglazov, Fizika B**6**, 111 (1997).

[11] S. Weinberg, *The Quantum Theory of Fields. Vol. I. Foundations.* (Cambridge University Press, Cambridge, 1995).

[12] W. Greiner, *Field Quantization* (Springer, 1996), Chapter 10.

[13] Valeriy V. Dvoeglazov, in *Einstein and Others: Unification.* Ed. V. V. Dvoeglazov. (Nova Sci. Pubs., NY, USA, 2015), p. 211.

[14] Valeriy V. Dvoeglazov, Z. Naturforsch. A **71**, 345 (2016).

[15] V. V. Dvoeglazov, SFIN **22**-A1, 157 (2009); J. Phys. Conf. Ser. **284**, 012024 (2011); Bled Workshops **10**-2, 52 (2009); ibid. **11**-2, 9 (2010); in *Beyond the Standard Model. Proceedings of the Vigier Symposium.* (Noetic Press, Orinda, USA, 2005), p. 388.

[16] V. V. Dvoeglazov, Bled Workshops **14**-2, 199 (2013).

[17] V. V. Dvoeglazov, Int. J. Theor. Phys. **43**, 1287 (2004).

[18] M. Kline, *Mathematics: The Loss of Certainty.* (Oxford University Press, 1980).