The Volkov solution of the Dirac equation with the Higgs field

Miroslav Pardy
Department of Physical Electronics
Kotlářská 2, 611 37 Brno, Czech Republic
Masaryk University
e-mail:pamir@physics.muni.cz

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Abstract

We determine the power radiation formula of the electron moving in the plane wave Higgs potential from the Volkov solution of the Dirac equation. The Higgs potential is here the vector extension of the scalar Higgs potential. The Higgs bosons mass is involved in the power radiation formula. The article represents the unification of the particle and the laser physics.

1 Introduction

The Higgs mechanism is a process that gives mass to the massless elementary particles. According to this theory, particles get mass by interacting with the Higgs field that permeates all space.

The authors of the Higgs mechanism is also Anderson, Brout, Englert, Guralnik, Hagen, Higgs and Kibble. In 2013 Peter Higgs and Francois Englert obtained the Nobel Prize in Physics for the theoretical discovery of a mechanism that enables the understanding of the origin of the mass of subatomic particles. The ATLAS and CMS experiments at the CERN Large Hadron Collider confirmed the Higgs mechanism.

The nonrelativistic mechanism was proposed in 1962 by Anderson. A similar but distinct effect had previously been studied by Stueckelberg.

The Higgs mechanism occurs whenever a charged field has a vacuum expectation value. In the nonrelativistic context, this is the Landau model of a superconductor. In the relativistic model, the condensate is a scalar field, and it is relativistically invariant.

2 The Higgs particle

The Higgs mechanism is the mechanism of the spontaneous broken symmetry and it can be demonstrated by the Lagrangian for the scalar field φ (Kane, 1987):

$$\mathcal{L} = T - V = \frac{1}{2}\partial_\nu\varphi\partial^\nu\varphi - \left(\frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4\right)..., \tag{1}$$

where μ, λ are some constants.

The potential energy has its minimum for $\mu^2 < 0$ at points

$$\varphi_{min} = \pm\sqrt{-\frac{\mu^2}{\lambda}} \equiv \pm v. \tag{2}$$

We can write around the minimal point that

$$\varphi(x) = v + \eta(x). \tag{3}$$

Then after insertion of eq. (3) into the original Lagrangian (1) we get approximately

$$\mathcal{L} \approx \frac{1}{2}\partial_\nu\eta\partial^\nu\eta - \left(\lambda v^2\eta^2 + \lambda v\eta^3 + \frac{1}{4}\lambda\eta^4\right) + const \quad . \tag{4}$$

It is elementary to see that the mass term in the last Lagrangian is the mass term of the field η and it is:

$$m_\eta^2 = 2\lambda v^2 = -2\mu^2 > 0. \tag{5}$$

So, mass of a particle was hidden in the Higgs potential.

3 Volkov solution of the Dirac equation

Volkov solution of the Dirac equation with the plane wave was presented in 1935 (Volkov, 1935). To be pedagogically clear, we follow the method of derivation and metric convention of well known textbook (Berestetskii et al., 1989).

The Dirac equation with the plane wave potential is as follows:

$$(\gamma(p - eA) - m)\psi = 0. \tag{6}$$

where

$$A^\mu = A^\mu(\varphi); \quad \varphi = kx = \omega t - \mathbf{k}.\mathbf{x}. \tag{7}$$

We suppose that the four-potential satisfies the Lorentz gauge condition

$$\partial_\mu A^\mu = k_\mu \left(A^\mu\right)' = \left(k_\mu A^\mu\right)' = 0, \tag{8}$$

where the prime denotes derivative with regard to φ. From the last equation follows

$$kA = const = 0, \tag{9}$$

because we can put the constant to zero. The tensor of electromagnetic field is

$$F_{\mu\nu} = k_\mu A'_\nu - k_\nu A'_\mu. \tag{10}$$

Instead of the linear Dirac equation (6), we consider the quadratic equation, which we get by multiplication of the linear equation by operator $(\gamma(p - eA) + m)$ (Berestetzkii et al., 1989). We get:

$$\left[(p - eA)^2 - m^2 - \frac{i}{2} eF_{\mu\nu}\sigma^{\mu\nu}\right]\psi = 0. \tag{11}$$

Using $\partial_\mu(A^\mu\psi) = A^\mu\partial_\mu\psi$, which follows from eq. (8), and $\partial_\mu\partial^\mu = \partial^2 = -p^2$, with $p_\mu = i(\partial/\partial x^\mu) = i\partial_\mu$, we get the quadratic Dirac equation for the four potential of the plane wave:

$$[-\partial^2 - 2ie(A\partial) + e^2A^2 - m^2 - ie(\gamma k)(\gamma A')]\psi = 0. \tag{12}$$

We are looking for the solution of the last equation in the form:

$$\psi = e^{-ipx}F(\varphi). \tag{13}$$

After insertion of eq. (13) into eq. (12), we get with $(k^2 = 0)$

$$\partial^\mu F = k^\mu F', \quad \partial_\mu\partial^\mu F = k^2 F'' = 0, \tag{14}$$

the following equation for $F(\varphi)$

$$2i(kp)F' + [-2e(pA) + e^2A^2 - ie(\gamma k)(\gamma A')]F = 0. \quad (15)$$

The integral of the last equation is of the form (Berestetzkii et al., 1989):

$$F = \exp\left\{-i\int_0^{kx}\left[\frac{e(pA)}{(kp)} - \frac{e^2}{2(kp)}A^2\right]d\varphi + \frac{e(\gamma k)(\gamma A)}{2(kp)}\right\}\frac{u}{\sqrt{2p_0}}, \quad (16)$$

where $u/\sqrt{2p_0}$ is the arbitrary constant bispinor.

Al powers of $(\gamma k)(\gamma A)$ above the first are equal to zero, since

$$(\gamma k)(\gamma A)(\gamma k)(\gamma A) =$$

$$-(\gamma k)(\gamma k)(\gamma A)(\gamma A) + 2(kA)(\gamma k)(\gamma A) = -k^2A^2 = 0, \quad (17)$$

where we have used eq. (4) and relation $k^2 = 0$. Then we can write:

$$\exp\left\{e\frac{(\gamma k)(\gamma A)}{2(kp)}\right\} = 1 + \frac{e(\gamma k)(\gamma A)}{2(kp)}. \quad (18)$$

So, the solution is of the form:

$$\psi_p = R\frac{u}{\sqrt{2p_0}}e^{iS} = \left[1 + \frac{e}{2kp}(\gamma k)(\gamma A)\right]\frac{u}{\sqrt{2p_0}}e^{iS}, \quad (19)$$

where u is an electron bispinor of the corresponding Dirac equation

$$(\gamma p - m)u = 0 \quad (20)$$

and we shall take it to be normalized by condition $\bar{u}u = 2m$. The mathematical object S is the classical Hamilton-Jacobi function, which was determined in the form:

$$S = -px - \int_0^{kx}\frac{e}{(kp)}\left[(pA) - \frac{e}{2}A^2\right]d\varphi. \quad (21)$$

The current density is

$$j^\mu = \bar{\psi}_p\gamma^\mu\psi_p, \quad (22)$$

where $\bar{\Psi}$ is defined as the transposition of (19), or,

$$\bar{\psi}_p = \frac{\bar{u}}{\sqrt{2p_0}}\left[1 + \frac{e}{2kp}(\gamma A)(\gamma k)\right]e^{-iS}. \quad (23)$$

After insertion of Ψ_p and $\bar{\Psi}_p$ into the current density, we have:

$$j^\mu = \frac{1}{p_0}\left\{p^\mu - eA^\mu + k^\mu\left(\frac{e(pA)}{(kp)} - \frac{e^2A^2}{2(kp)}\right)\right\}. \tag{24}$$

4 The Volkov solution of the Dirac equation with the vector Higgs potential

Let the model for the Higgs mechanism be described by the Dirac equation with the vector field

$$A_\mu = \alpha_\mu\varphi^2 + \beta_\mu\varphi^4, \tag{25}$$

which means that the mass of the Higgs particle is hidden in the four-potential (4). At this moment we get the generalized Higgs mechanism with the four Higgs particle forming the four-vector Higgs boson. The equation for the mass of the Higgs boson can be obtained using eq. (3).

and let us find the Volkov solution of the Dirac equation with such vector potential. The corresponding tensor of electromagnetic field follows by the elementary operations:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2(kx)\left(\alpha_\nu k_\mu - \alpha_\mu k_\nu\right) + 4(kx)^3\left(\beta_\nu k_\mu - \beta_\mu k_\nu\right), \tag{26}$$

which means that electron moves in the non-constant electromagnetic field with the non-constant components $\mathbf{E}$ and $\mathbf{H}$.

So, we get with eq. (21) and (23):

$$S = -px - \int_0^{kx}\frac{e}{(kp)}\left[(pA) - \frac{e}{2}(A)^2\right]d\varphi =$$

$$-px - \int_0^{kx}\frac{e}{(kp)}\left[(p(\alpha_\mu\varphi^2 + \beta_\mu\varphi^4)) - \frac{e}{2}(\alpha_\mu\varphi^2 + \beta_\mu\varphi^4)^2\right]d\varphi =$$

$$-px - \int_0^{kx}\frac{e}{(kp)}\left[(p\alpha\varphi^2 + p\beta\varphi^4) - \frac{e}{2}(\alpha^2\varphi^4 + 2\alpha\beta\varphi^6 + \beta^2\varphi^8)\right]d\varphi \tag{27}$$

After elementary integration we get for S:

$$S = -px - \frac{e}{(kp)}\left[\left(\frac{p\alpha}{3}\varphi^3 + \frac{p\beta}{5}\varphi^5\right) - \frac{e}{2}\left(\frac{1}{5}\alpha^2\varphi^5 + \frac{2}{7}\alpha\beta\varphi^7 + \frac{1}{9}\beta^2\varphi^9\right)\right], \tag{28}$$

where $\varphi = kx = k_\mu x^\mu = \omega t - \mathbf{k}.\mathbf{x}$.

At this moment the mass of the Higgs boson is hidden in action (28) as can be easily see.

5 Photon emission

Quantum field theory works only with the matrix elements and power spectrum must be determined from the correct definition of the matrix element in case that electron is moving in potential (25).

It is possible to show in the quantum field theory, that the corresponding S-matrix element which describes transition from the state ψ_p to $\psi_{p'}$ with simultaneous emission of photon with polarization e' and four-momentum $k'^{\mu} = (k'_0, \mathbf{k}') = (\omega', \mathbf{k}')$ is given by the following expression (Berestetzkii et al., 1989), with $k' \to -k', S \to -S$, to be in accord with the Ritus article (Ritus, 1979):

$$M = e\int d^4x \bar{\psi}_{p'}(\gamma e'^*)\psi_p \frac{e^{-ik'x}}{\sqrt{2\omega'}}, \tag{29}$$

where $\psi_p, \bar{\psi}_p$ are given by the relations (19) and (23).

So, we see that the matrix element (29) cannot be expressed in the simple form in order to get the spectral distribution of photons generated by electron moving in the Higgs potential. The analogical problem was solved by Khalilov et al. (Khalilov et al., 1995). While the matrix element (29) has the classical limit in case of the classical potentials (Landau et al., 1988), the situation with the Higgs potential is not the classical one.

On the other hand, we can get the global information of photons (and Higgs boson) using the three invariants from which the formula for the total radiation is composed. Such invariant are as follows (Berestetzkii et al., 1989):

$$\chi^2 = -\frac{e^2}{m^6}\left(F_{\mu\nu}p^{\nu}\right)^2; \quad f = \frac{e^2}{m^4}\left(F_{\mu\nu}\right)^2; \quad g = \frac{e^2}{m^4}\varepsilon_{\lambda\mu\nu\varrho}F^{\lambda\mu}F^{\nu\varrho}. \tag{30}$$

It may be easy to show that (Berestetzkii et al., 1989):

$$\chi^2 = \frac{e^2}{m^6}\left\{(\mathbf{p}\times\mathbf{H} + p_0\mathbf{E})^2 - (\mathbf{p}\cdot\mathbf{E})^2\right\} \tag{31}$$

The expansion of the total energy photon formula $I = I(\chi, f, g)$ has the Taylor expansion for $\chi \ll 1$:

$$I = I(\chi, f, g) = I(0) + \left(\frac{\partial I}{\partial\chi}\right)_{\chi=0}\chi + \left(\frac{\partial^2 I}{\partial\chi^2}\right)_{\chi=0}\chi^2 + ... \tag{32}$$

Let us calculate the explicite form of χ and χ^2 for the Higgs potential (25).

$$\chi^2 = -\frac{e^2}{m^6}\left(2(kx)\left(\alpha_\nu k_\mu - \alpha_\mu k_\nu\right) + 4(kx)^3\left(\beta_\nu k_\mu - \beta_\mu k_\nu\right)p^\nu\right)^2 \tag{33}$$

$$f = \frac{e^2}{m^4}\left(2(kx)\left(\alpha_\nu k_\mu - \alpha_\mu k_\nu\right) + 4(kx)^3\left(\beta_\nu k_\mu - \beta_\mu k_\nu\right)\right)^2 \tag{34}$$

$$g = \frac{e^2}{m^4}\varepsilon_{\lambda\mu\nu\varrho}\left[2(kx)\left(\alpha^\nu k^\mu - \alpha^\mu k^\nu\right) + 4(kx)^3\left(\beta^\nu k^\mu - \beta^\mu k_\nu\right)\right] \times$$

$$\left[2(kx)\left(\alpha^\nu k^\mu - \alpha^\mu k^\nu\right) + 4(kx)^3\left(\beta^\nu k^\mu - \beta^\mu k^\nu\right)\right] \tag{35}$$

Now let us determine parameters in the four-Higgs potential in such a way that :

$$I(0) = 0; \quad \left(\frac{\partial I}{\partial \chi}\right)_{\chi=0} = 0; \quad \left(\frac{\partial^3 I}{\partial \chi^3}\right)_{\chi=0} = 0. \tag{36}$$

Then, we have approximately:

$$I(\chi, f, g) \approx \alpha\chi^2 + \beta\chi^4, \tag{37}$$

which formally corresponds to the original Higgs-boson potential with the corresponding mass generation. However, at this moment the identity is only formal and we cannot say that the Higgs boson mass is hidden in eq. (37).

6 Discussion

Let us as remark in conclusion that the Higgs boson masswas created by the spontaneous broken of symmetry (leading to the Lagrange function (4)). In case of the Volkov solution of the Dirac equation with the Higgs potential $A_\mu = \alpha_\mu\varphi^2 + \beta_\mu\varphi^4$ we can also consider the broken symmetry. So, the existence of the broken symmetry is involved in the Higgs potential and it means that the Higgs mass is involved in the final spectral formula. The present elementary theory can be generalized to the situation with massive photons (Pardy, 2004).

It is not excluded that the article will play the positive role in the unification of particle physics with laser physics of CERN and ELI.

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