On the distributional expansions of powered extremes from Maxwell distribution*

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Abstract. In this paper, asymptotic expansions of the distributions and densities of powered extremes for Maxwell samples are considered. The results show that the convergence speeds of normalized partial maxima relies on the powered index. Additionally, compared with previous result, the convergence rate of the distribution of powered extreme from Maxwell samples is faster than that of its extreme. Finally, numerical analysis is conducted to illustrate our findings.

Keywords. Asymptotic expansion; density; Maxwell distribution; powered extreme.

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1 Introduction

In extreme value theory, researchers recently focus on investigating the quality of convergence of normalized $\max\{X_k, 1 \leq k \leq n\} := M_n$ of a sample. For the convergence rate of normalized M_n, general cases were discussed by Smith [1], Leadbetter et al. [2], Galambos [3] and de Haan and Resnick [4], and specific cases were considered by Hall [5, 6], Nair [7], Liao and Peng [8], Lin et al. [9, 10], Du and Chen [11, 12], and Huang et al. [13]. Hall [6] derived the asymptotics of distribution of normalized $|M_n|^t$, the powered extremes for given power index $t > 0$. Zhou and Ling [14] improved Hall' results and proved that the convergence speed of distributions and densities of extremes depends on the power index. Nair [7] established the asymptotic expansions of normalized maximum from normal samples. Liao et al. [15] and Jia et al. [16] generalized Nair's work to skew-normal distribution and general error distribution, respectively.

Since the Maxwell distribution was proposed by James Clerk Maxwell [17], a variety of applications of it in physics (in particular in statistical mechanics) have been found; see Shim and Gatignol [18], Tomer and Panwar [19] and Shim [20] and some statisticians and reliability engineers have investigated the statistical properties of it as well, see [13, 21–27].

The aim of this paper is to investigate the distributional tail representation of $|X|^t$ with X following Maxwell distribution and the limiting distribution of normalized $|M_n|^t$, and obtain asymptotic expansions of distribution and density of powered maximum from Maxwell distribution.

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Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with marginal cumulative distribution function (cdf) F obeying the Maxwell distribution (abbreviated as $F \sim MD$), and as before let $M_n = \max\{X_i, 1 \leq i \leq n\}$ denote the partial maximum of $\{X_n, n \geq 1\}$. The probability density function (pdf) of the MD is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{x^2}{\sigma^3} \exp\left(-\frac{x^2}{2\sigma^2}\right), \; x > 0, \tag{1.1}$$

where $\sigma > 0$ is the scale parameter. Figure 1 presents the graph of pdf of Maxwell distribution. It shows that with the scale parameter increasing, the tail of pdf of MD becomes much heavier.

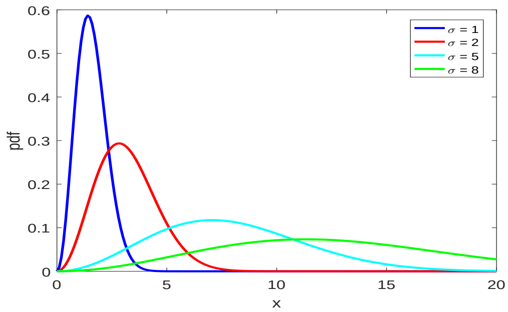


Figure 1: Probability density function of Maxwell distribution

Liu and Liu [21] showed that $F \in D(\Lambda)$, i.e., the max-domain of attraction of Gumbel extreme value distribution and the normalizing constants a_n and b_n can be given by

$$a_n = \sigma^2 b_n^{-1} \tag{1.2}$$

and

$$\sqrt{\frac{\pi}{2}} \frac{\sigma}{b_n} \exp\left(\frac{b_n^2}{2\sigma^2}\right) = n \tag{1.3}$$

such that

$$\lim_{n\to\infty} \mathbb{P}(M_n \leq a_n x + b_n) = \Lambda(x) = \exp\{-\exp(-x)\}. \tag{1.4}$$

The paper is constructed as follows. Section 2 presents auxiliary lemmas with proofs. The main results are given in Section 3. Numerical studies presented in Section 4 compare the precision of the true values with its approximations. Section 5 provides the proofs of main results.

2 Auxiliary results

To prove the main results, the following auxiliary lemmas are needed.

Lemma 2.1. *Let $F(x)$ and $f(x)$ respectively represent the cdf and the pdf of MD with $\sigma > 0$, respectively. For large x, we have*

$$1 - F(x) = \sigma^2 x^{-1} f(x) \left[1 + \sigma^2 x^{-2} - \sigma^4 x^{-4} + 3\sigma^6 x^{-6} + O(x^{-8})\right]. \tag{2.1}$$

The proof of Lemma 2.1 is derived by integration by parts.

The following lemma gives the distributional tail representation of X^t with $X \sim MD$.

Lemma 2.2. *Suppose that $0 < t \neq 2$. Let $F_t(x)$ denote the cdf of X^t with $X \sim MD$. Then for large x, we get*

$$1 - F_t(x) = C_t(x) \exp\left\{ -\int_1^x \frac{g_t(u)}{\tilde{f}_t(u)} du \right\}, \tag{2.2}$$

where

$$C_t(x) \to \frac{2}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2\sigma^2}\right) \text{ as } x \to \infty,$$
$$g_t(x) = 1 - \sigma^2 x^{-2/t} \to 1 \text{ as } x \to \infty,$$

and

$$\tilde{f}_t(x) = \sigma^2 t x^{1-\frac{2}{t}} \text{ with } \tilde{f}_t'(x) \to 0 \text{ as } x \to \infty. \tag{2.3}$$

Proof. Combining with (2.1), we get

$$\begin{aligned} 1 - F_t(x) &= 2\frac{\sigma^2 f(x^{\frac{1}{t}})}{x^{\frac{1}{t}}} \left[1 + \sigma^2 x^{-\frac{2}{t}} - \sigma^4 x^{-\frac{4}{t}} + 3\sigma^6 x^{-\frac{6}{t}} + O(x^{-\frac{8}{t}})\right] \qquad (2.4) \\ &= \frac{2}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^{\frac{2}{t}}}{2\sigma^2} + \frac{1}{t}\log x\right) \left[1 + \sigma^2 x^{-\frac{2}{t}} - \sigma^4 x^{-\frac{4}{t}} + 3\sigma^6 x^{-\frac{6}{t}} + O(x^{-\frac{8}{t}})\right] \\ &= C_t(x) \exp\left(-\int_1^x \frac{g_t(u)}{\tilde{f}_t(u)} du\right) \left[1 + \sigma^2 x^{-\frac{2}{t}} - \sigma^4 x^{-\frac{4}{t}} + 3\sigma^6 x^{-\frac{6}{t}} + O(x^{-\frac{8}{t}})\right] \end{aligned}$$

with $\tilde{f}_t(x) = \sigma^2 t x^{1-\frac{2}{t}}$, $g_t(x) = 1 - \sigma^2 x^{-2/t}$ and $C_t(x) \to \frac{2}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2\sigma^2}\right)$ as $x \to \infty$.

□

Applying the result of Lemma 2.2 and Corollary 1.7 [28], the following result holds.

Proposition 2.1. *Under the conditions of Lemma 2.2, we have $F_t(x) \in D(\Lambda)$, where $D(\Lambda)$ is the domain of $\Lambda(x) = \exp\{-\exp(-x)\}$.*

Then, our aim is to select the suitable normalizing constants which ensure that the distribution of maximum tends to its extreme value limit. A combination of (1.3) and (2.4), we obtain that $d_n = b_n^t$. It follows from (2.3) that

$$c_n = \tilde{f}_t(d_n) = \sigma^2 t b_n^{t(1-\frac{2}{t})} = \sigma^2 t b_n^{t-2}. \tag{2.5}$$

The following work is to find the special normalizing constants c_n and d_n for the case of powered index $t = 2$. Similarly, it is necessary to establish the distributional tail representation of X^2 with $X \sim MD$.

Lemma 2.3. *Assume that $t = 2$. Let $F_2(x)$ stand for the cdf of X^2 with $X \sim MD$. Then for large x, we get*

$$1 - F_2(x) = C_2(x) \exp\left\{ -\int_1^x \frac{g_2(u)}{\tilde{f}_2(u)} du \right\}, \tag{2.6}$$

where

$$C_2(x) \to \frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2\sigma^2}\right) \ \ as\ x \to \infty,$$

$$g_2(x) = 1 + \frac{\sigma^4}{x^2} \to 1 \ \ as\ x \to \infty,$$

and

$$\tilde{f}_2(x) = 2\sigma^2\left(1 + \frac{\sigma^2}{x}\right) \ \ with\ \tilde{f}_2'(x) \to 0 \ \ as\ x \to \infty. \tag{2.7}$$

Proof. Similar to the case of $t \neq 2$, we get

$$\begin{aligned}
1 - F_2(x) &= 2\frac{\sigma^2 f(x^{\frac{1}{2}})}{x^{\frac{1}{2}}}\left[1 + \sigma^2 x^{-1} - \sigma^4 x^{-2} + 3\sigma^6 x^{-3} + O(x^{-4})\right] \\
&= 2\frac{\sigma^2 f(x^{\frac{1}{2}})}{x^{\frac{1}{2}}}(1 + \sigma^2 x^{-1})\left[1 - \sigma^4 x^{-2}(1 + \sigma^2 x^{-1})^{-1} + 3\sigma^6 x^{-3}(1 + \sigma^2 x^{-1})^{-1} + O(x^{-4})\right] \qquad (2.8) \\
&= \frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\exp\left[-\frac{x}{2\sigma^2} + \frac{1}{2}\log x + \log\left(1 + \frac{\sigma^2}{x}\right)\right]\left[1 - \sigma^4 x^{-2} + 4\sigma^6 x^{-3} + O(x^{-4})\right] \\
&= \frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2\sigma^2}\right)\exp\left(-\int_1^x \frac{g_2(u)}{\tilde{f}_2(u)}du\right)\left[1 - \sigma^4 x^{-2} + 4\sigma^6 x^{-3} + O(x^{-4})\right]
\end{aligned}$$

with $g_2(x) = 1 + \sigma^4 t^{-2}$ and $\tilde{f}_2(x) = 2\sigma^2(1 + \sigma^2 t^{-1})$, where the third equality follows from the fact that $(1+x)^a = 1 + ax + (a(a-1)/2)x^2 + O(x^3)$ for all $a \in \mathbb{R}$, as $x \to 0$.

□

Similar to the case of $t \neq 2$, we have the following result:

Proposition 2.2. *Under the assumptions of Lemma* 2.3*, we get* $F_2(x) \in D(\Lambda)$*, where* $D(\Lambda)$ *is the domain of* $\Lambda(x) = \exp\{-\exp(-x)\}$.

Now we discuss how to find the constants c_n, d_n. Analogous to the case of $t \neq 2$, we may make choice of $d_n = b_n^2$ and $c_n = \tilde{f}_2(d_n) = 2\sigma^2(1 + \sigma^2 b_n^{-2})$. Inspired by c_n, now change

$$\begin{aligned}
\bar{d}_n &= b_n^2 + 2\sigma^4 b_n^{-2}, \\
\bar{c}_n &= \tilde{f}_2(\bar{d}_n) \\
&= 2\sigma^2[1 + \sigma^2 b_n^{-2} - 2\sigma^6 b_n^{-6} + O(b_n^{-10})] \\
&\sim 2\sigma^2(1 + \sigma^2 b_n^{-2}).
\end{aligned} \tag{2.9}$$

Let

$$T_n(x,t) = F^{n-1}((c_n x + d_n)^{1/t}) - (1 - F((c_n x + d_n)^{1/t}))^{n-1}.$$

The following lemmas present the expansions of the two terms of densities of $(|M_n|^t - d_n)/c_n$.

Lemma 2.4. *For normalizing constants* c_n *and* d_n *determined by (2.5) and* $0 < t \neq 2$*, we have*

$$T_n(x,t) = \Lambda(x)\left\{1 - A_1(t,x)e^{-x}b_n^{-2} + \left(\frac{1}{2}A_1^2(t,x)e^{-x} - A_2(t,x)\right)e^{-x}b_n^{-4} + O(b_n^{-6})\right\} \tag{2.10}$$

as $n \to \infty$, where

$$A_1(t,x) = \sigma^2 \left(1 + x + \frac{(t-2)x^2}{2}\right) \tag{2.11}$$

and

$$A_2(t,x) = \sigma^4 \left(\frac{(t-2)^2 x^4}{8} + \frac{1}{6}(t-2)(5-2t)x^3 - \frac{x^2}{2} - x - 1\right). \tag{2.12}$$

Proof. Let $\delta_n(x,t) = (c_n x + d_n)^{1/t}$. One can easily see that $c_n x + d_n > 0$ for large n and fixed $x \in \mathbb{R}$. By (1.3), for large n, we have $b_n^2 \sim 2\sigma^2 \log n$. Then, by (2.5), we have

$$\delta_n^a(x,t) = b_n^a \left[1 + \frac{a\sigma^2 x}{b_n^2} + \frac{a(a-t)\sigma^4 x^2}{2b_n^4} + \frac{a(a-t)(a-2t)\sigma^6 x^3}{6b_n^6} + O(b_n^{-8})\right], \tag{2.13}$$

where it follows from the fact that

$$(1+x)^a = 1 + ax + (a(a-1)/2)x^2 + (a(a-1)(a-2)/6)x^3 + O(x^4),$$

for $a \in \mathbb{R}$, as $x \to 0$. Then, we get

$$\begin{aligned}
\frac{\sigma^2 f(\delta_n(x,t))}{\delta_n(x,t)} &\overset{\text{(a)}}{=} \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} b_n \left[1 + \frac{\sigma^2 x}{b_n^2} + \frac{(1-t)\sigma^4 x^2}{2b_n^4} + \frac{(1-t)(1-2t)\sigma^6 x^3}{6b_n^6} + O(b_n^{-8})\right] \\
&\quad \times \exp\left\{-\frac{b_n^2}{2\sigma^2}\left[1 + \frac{2\sigma^2 x}{b_n^2} + \frac{(2-t)\sigma^4 x^2}{b_n^4} + \frac{(2-t)(2-2t)\sigma^6 x^3}{3b_n^6} + O(b_n^{-8})\right]\right\} \\
&\overset{\text{(b)}}{=} \frac{\sigma^2 f(b_n)}{b_n} e^{-x} \left[1 + \frac{\sigma^2 x}{b_n^2} + \frac{(1-t)\sigma^4 x^2}{2b_n^4} + O(b_n^{-6})\right] \\
&\quad \times \left[1 - \frac{(2-t)\sigma^2 x^2}{2b_n^2} - \frac{(2-t)(1-t)\sigma^4 x^3}{3b_n^4} + \frac{(2-t)^2\sigma^4 x^4}{8b_n^4} + O(b_n^{-6})\right] \\
&\overset{\text{(c)}}{=} n^{-1} e^{-x} \left\{1 + \frac{\sigma^2 x}{b_n^2}\left(1 + \frac{1}{2}(t-2)x\right)\right. \\
&\quad \left. + \frac{\sigma^4 x^2}{b_n^4}\left[\frac{1}{8}(t-2)^2 x^2 + \frac{1}{6}(t-2)(5-2t)x + \frac{1-t}{2}\right] + O(b_n^{-6})\right\}
\end{aligned} \tag{2.14}$$

where (a) follows from (2.13) with $a = 1$ and 2, (b) is from the fact that $e^x = 1 + x + x^2/2 + O(x^3)$, as $x \to 0$ and (c) is due to (1.3). Furthermore, we get

$$\begin{aligned}
&1 + \sigma^2 \delta_n^{-2}(x,t) - \sigma^4 \delta_n^{-4}(x,t) + O(\delta_n^{-6}(x,t)) \\
&\overset{\text{(a)}}{=} 1 + \frac{\sigma^2}{b_n^2}\left[1 - \frac{2\sigma^2 x}{b_n^2} + O(b_n^{-4})\right] - \frac{\sigma^4}{b_n^4}\left[1 + O(b_n^{-2})\right] + O(b_n^{-6}) \\
&= 1 + \frac{\sigma^2}{b_n^2} - \frac{\sigma^4}{b_n^4}(1+2x) + O(b_n^{-6}),
\end{aligned} \tag{2.15}$$

where (a) is from (2.13) with $a = -2$ and -4. By Lemma 2.1, we get

$$\begin{aligned}
1 - F(\delta_n(x,t)) &= \frac{\sigma^2 f(\delta_n(x,t))}{\delta_n(x,t)} \left[1 + \sigma^2 \delta_n^{-2}(x,t) - \sigma^4 \delta_n^{-2}(x,t) + O(\delta_n^{-6}(x,t))\right] \\
&\overset{\text{(a)}}{=} n^{-1} e^{-x} \left\{1 + \frac{\sigma^2}{b_n^2}\left[1 + x + \frac{1}{2}(t-2)x^2\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma^4}{b_n^4}\left[\frac{1}{8}(t-2)^2x^4+\frac{1}{6}(t-2)(5-2t)x^3-\frac{x^2}{2}-x-1\right]+O(b_n^{-6})\Bigg\} \\
&=: n^{-1}e^{-x}\left[1+A_1(t,x)b_n^{-2}+A_2(t,x)b_n^{-4}+O(b_n^{-6})\right], \qquad (2.16)
\end{aligned}
$$

where (a) is due to (2.14) and (2.15). Accordingly,

$$
\begin{aligned}
F^{n-1}(\delta_n(x,t)) &= \exp\{(n-1)\log[1-(1-F(\delta_n(x,t)))]\} \\
&\overset{\text{(a)}}{=} \Lambda(x)\exp\left[-A_1(t,x)e^{-x}b_n^{-2}-A_2(t,x)e^{-x}b_n^{-4}+O(b_n^{-6})\right] \\
&\overset{\text{(b)}}{=} \Lambda(x)\left\{1-A_1(t,x)e^{-x}b_n^{-2}+\left(\frac{1}{2}A_1^2(t,x)e^{-x}-A_2(t,x)\right)e^{-x}b_n^{-4}+O(b_n^{-6})\right\}, \qquad (2.17)
\end{aligned}
$$

and

$$
(1-F(\delta_n(x,t)))^{n-1}=\left\{\frac{e^{-x}}{n}\left[1+O(b_n^{-2})\right]\right\}^{n-1}=o(b_n^{-\eta}),\ \eta\ge 6, \qquad (2.18)
$$

where (a) is from the fact that $\log(1-x)=-x+O(x^2)$, as $x\to 0$, and (b) follows from that Taylor's expansion of e^x. The desired result follows by (2.17) and (2.18).

□

Lemma 2.5. *For the normalizing constants c_n and d_n determined by (2.5) and $0<t\neq 2$, we have*

$$
\begin{aligned}
n\frac{d}{dx}F((c_nx+d_n)^{1/t}) &= e^{-x}\Bigg\{1+\frac{\sigma^2x}{b_n^2}[3-t-(2-t)x] \\
&+\frac{\sigma^4x^2}{b_n^4}\left[\frac{1}{2}(3-t)(3-2t)+(t-2)\left(\frac{11}{6}-\frac{5}{6}t\right)x+\frac{1}{8}(t-2)^2x^2\right]+O(b_n^{-6})\Bigg\}, \qquad (2.19)
\end{aligned}
$$

as $n\to\infty$.

Proof. It is not hard to check that

$$
n\frac{d}{dx}F((c_nx+d_n)^{1/t})=\frac{1}{t}nc_n(c_nx+d_n)^{1/t-1}f((c_nx+d_n)^{1/t}).
$$

Therefore, we get

$$
\begin{aligned}
n\frac{d}{dx}F((c_nx+d_n)^{1/t}) &\overset{\text{(a)}}{=} \frac{n}{\sigma}b_n\sqrt{\frac{2}{\pi}}\left[1+\frac{(3-t)\sigma^2x}{b_n^2}+\frac{(3-t)(3-2t)\sigma^4x^2}{2b_n^4}+O(b_n^{-6})\right] \\
&\times\exp\left\{-\frac{b_n^2}{2\sigma^2}\left[1+\frac{2\sigma^2x}{b_n^2}+\frac{(2-t)\sigma^4x^2}{b_n^4}+\frac{(2-t)(2-2t)\sigma^6x^3}{3b_n^6}+O(b_n^{-8})\right]\right\} \\
&\overset{\text{(b)}}{=} nf(b_n)\frac{\sigma^2}{b_n}e^{-x}\left[1+\frac{(3-t)\sigma^2x}{b_n^2}+\frac{(3-t)(3-2t)\sigma^4x^2}{2b_n^4}+O(b_n^{-6})\right] \\
&\times\left[1-\frac{(2-t)\sigma^2x^2}{2b_n^2}-\frac{(2-t)(1-t)\sigma^4x^3}{3b_n^4}+\frac{(2-t)^2\sigma^4x^4}{8b_n^4}+O(b_n^{-6})\right] \\
&\overset{\text{(c)}}{=} e^{-x}\Bigg\{1+\frac{\sigma^2x}{b_n^2}[3-t-(2-t)x]
\end{aligned}
$$

$$+\frac{\sigma^4x^2}{b_n^4}\left[\frac{1}{2}(3-t)(3-2t)+(t-2)\left(\frac{11}{6}-\frac{5}{6}t\right)x+\frac{1}{8}(t-2)^2x^2\right]+O(b_n^{-6})\Big\},$$

where (a) follows from (2.13) with $a=3-t$, (2.5) and (2.14) for the expansion of $f(\delta_n(x,t))$ with $\delta_n(x,t)=(c_nx+d_n)^{1/t}$, (b) is from the fact that $e^x=1+x+x^2/2+O(x^3)$, as $x\to0$ and (c) is due to (1.3). The proof is complete.

□

Lemma 2.6. *For the normalizing constants c_n and d_n determined by (2.9) and $t=2$, we have*

$$T_n(x,t)=\Lambda(x)\left[1-B_1(t,x)e^{-x}b_n^{-4}-B_2(t,x)e^{-x}b_n^{-6}+O(b_n^{-8})\right], \tag{2.20}$$

as $n\to\infty$, where

$$B_1(t,x)=-\sigma^4\left(x^2+x+\frac{1}{2}\right) \tag{2.21}$$

and

$$B_2(t,x)=\sigma^6\left(\frac{4}{3}x^3+2x^2-2x+\frac{7}{3}\right). \tag{2.22}$$

Proof. The proof of the case of $t=2$ is similar to the case of $0<t\neq2$. Note that $c_n=2\sigma^2(1+\sigma^2b_n^{-2})$, $d_n=b_n^2+2\sigma^4b_n^{-2}$ for $t=2$. So, we get

$$\delta_n(x,2)=(c_nx+d_n)^{1/2}=b_n[1+2\sigma^2b_n^{-2}x+2\sigma^4(x+1)b_n^{-4}]^{1/2}=:\beta_n.$$

Then, we have

$$\beta_n^a=b_n^a\left[1+\frac{a\sigma^2x}{b_n^2}+\frac{a\sigma^4}{b_n^4}\left(1+x-\frac{2-a}{2}x^2\right)-\frac{a(2-a)\sigma^6x}{b_n^6}\left(1+x-\frac{4-a}{6}x^2\right)+O(b_n^{-8})\right]. \tag{2.23}$$

Further, we get

$$\begin{aligned}\frac{\sigma^2f(\beta_n)}{\beta_n}&\overset{(a)}{=}\sqrt{\frac{2}{\pi}\frac{b_n^2}{\sigma^3}}\exp\left(-\frac{b_n^2}{2\sigma^2}\right)\frac{\sigma^2}{b_n}e^{-x}\\&\quad\times\left[1+\frac{\sigma^2x}{b_n^2}+\frac{\sigma^4}{b_n^4}\left(1+x-\frac{1}{2}x^2\right)-\frac{\sigma^6x}{b_n^6}\left(1+x-\frac{1}{2}x^2\right)+O(b_n^{-8})\right]\\&\quad\times\left[1-\frac{\sigma^2(1+x)}{b_n^2}+\frac{\sigma^4(1+x)^2}{2b_n^4}-\frac{\sigma^6(1+x)^3}{6b_n^6}+O(b_n^{-8})\right]\\&\overset{(b)}{=}n^{-1}e^{-x}\left[1-\frac{\sigma^2}{b_n^2}-\frac{\sigma^4}{b_n^4}\left(x^2-x-\frac{3}{2}\right)+\frac{\sigma^6}{b_n^6}\left(\frac{4x^3}{3}-x^2-3x-\frac{7}{6}\right)+O(b_n^{-8})\right],\end{aligned} \tag{2.24}$$

where (a) is from (2.23) with $a=1$ and 2 and $e^x=1+x+x^2/2+O(x^3)$, as $x\to0$, and (b) is due to (1.3). Besides, applying (2.23) with $a=-2$, -4 and -6, we get

$$\begin{aligned}&1+\sigma^2\beta_n^{-2}-\sigma^4\beta_n^{-4}+3\sigma^6\beta_n^{-6}+O(\beta_n^{-8})\\=&1+\sigma^2b_n^{-2}\left[1-\frac{2\sigma^2x}{b_n^2}-\frac{2\sigma^4}{b_n^4}\left(1+x-2x^2\right)+O(b_n^{-6})\right]\end{aligned}$$

$$
\begin{aligned}
&-\sigma^4 b_n^{-4}\left[1-\frac{4\sigma^2 x}{b_n^2}+O(b_n^{-4})\right]+3\sigma^6 b_n^{-6}(1+O(b_n^{-2}))+O(b_n^{-8})\\
=&1+\frac{\sigma^2}{b_n^2}-\frac{\sigma^4}{b_n^4}(2x+1)+\frac{\sigma^6}{b_n^6}(4x^2-2x+1)+O(b_n^{-8}).
\end{aligned} \tag{2.25}
$$

Combining with Lemma 2.1, (2.24) and (2.25), we get

$$
\begin{aligned}
1-F(\beta_n)&=n^{-1}e^{-x}\left[1-\frac{\sigma^4}{b_n^4}\left(x^2+x+\frac{1}{2}\right)+\frac{\sigma^6}{b_n^4}\left(\frac{4}{3}x^3+2x^2-2x+\frac{7}{3}\right)+O(b_n^{-8})\right]\\
&=: n^{-1}e^{-x}\left[1+B_1(t,x)b_n^{-4}+B_2(t,x)b_n^{-6}+O(b_n^{-8})\right].
\end{aligned} \tag{2.26}
$$

The remainder proof is the same as the case of $0<t\neq 2$. We omit it. The proof is complete.

□

Lemma 2.7. *For the normalizing constants c_n and d_n determined by (2.9) and $t=2$, we have*

$$
n\frac{d}{dx}F((c_nx+d_n)^{1/t})=e^{-x}\left\{1-\frac{\sigma^4}{b_n^4}\left(x^2-x-\frac{1}{2}\right)+\frac{\sigma^6}{b_n^6}\left(\frac{4}{3}x^3-2x^2-2x+\frac{1}{3}\right)+O(b_n^{-8})\right\},
$$

as $n\to\infty$.

Proof. By (2.24) and after observing that $c_n=2\sigma^2(1+\sigma^2 b_n^{-2})$, we get

$$
\begin{aligned}
n\frac{d}{dx}F(\beta_n)&=e^{-x}\left(1+\frac{\sigma^2}{b_n^2}\right)\left[1-\frac{\sigma^2}{b_n^2}-\frac{\sigma^4}{b_n^4}\left(x^2-x-\frac{3}{2}\right)+\frac{\sigma^6}{b_n^6}\left(\frac{4x^3}{3}-x^2-3x-\frac{7}{6}\right)+O(b_n^{-8})\right]\\
&=e^{-x}\left\{1-\frac{\sigma^4}{b_n^4}\left(x^2-x-\frac{1}{2}\right)+\frac{\sigma^6}{b_n^6}\left(\frac{4}{3}x^3-2x^2-2x+\frac{1}{3}\right)+O(b_n^{-8})\right\}.
\end{aligned} \tag{2.27}
$$

The proof is complete.

□

As we mentioned in the introduction, Liu and Liu [21] obtained the pointwise convergence rate of distribution of partial maximum to its limiting distribution. Their main results are stated as follows.

Theorem 2.1. *Suppose that $\{X_n, n\geq 1\}$ is a sequence of i.i.d. random variables with cdf MD. Then,*

$$
F^n(\hat{a}_n x+\hat{b}_n)-\Lambda(x)\sim\Lambda(x)e^{-x}\frac{(\log(2\log n))^2}{16\log n}, \tag{2.28}
$$

for large n, where

$$
\hat{a}_n=\frac{\sigma}{(2\log n)^{1/2}}\ \text{and}\ \hat{b}_n=(2\sigma^2\log n)^{1/2}+\frac{\sigma\log(2\log n)+\sigma\log\frac{2}{\pi}}{2(2\log n)^{1/2}}. \tag{2.29}
$$

3 Main result

In this section, we establish the higher-order expansions of the cdf and the pdf of powered maximum from MD sample.

Theorem 3.1.
(i) For $0 < t \neq 2$ and the normalizing constants c_n and d_n given by (2.5), we have

$$\mathbb{P}(|M_n|^t \leq c_n x + d_n) = \Lambda(x) \left\{ 1 - e^{-x} A_1(t,x) b_n^{-2} + e^{-x} \left[\frac{1}{2} e^{-x} A_1^2(t,x) - A_2(t,x) \right] b_n^{-4} + O(b_n^{-6}) \right\}, \tag{3.1}$$

where

$$A_1(t,x) = \sigma^2 \left[1 + x + \frac{1}{2}(t-2)x^2 \right] \tag{3.2}$$

and

$$A_2(t,x) = \sigma^4 \left[\frac{1}{8}(t-2)^2 x^4 + \frac{1}{6}(t-2)(5-2t)x^3 - \frac{x^2}{2} - x - 1 \right]. \tag{3.3}$$

(ii) For $t = 2$ and the normalizing constants c_n and d_n given by (2.9), we have

$$\mathbb{P}(|M_n|^t \leq c_n x + d_n) = \Lambda(x) \left[1 - e^{-x} B_1(t,x) b_n^{-4} - e^{-x} B_2(t,x) b_n^{-6} + O(b_n^{-8}) \right], \tag{3.4}$$

where

$$B_1(t,x) = -\sigma^4 \left(x^2 + x + \frac{1}{2} \right) \tag{3.5}$$

and

$$B_2(t,x) = \sigma^6 \left(\frac{4}{3}x^3 + 2x^2 - 2x + \frac{7}{3} \right). \tag{3.6}$$

Remark 3.1. *From Theorem 3.1, one can easily see that the convergence rates of powered maximum of cdf for MD are proportional to $1/\log n$ and $1/(\log n)^2$ for power index $0 < t \neq 2$ and $t = 2$, respectively, since $1/b_n^2 \sim 2\sigma^2 \log n$ by (1.3).*

Remark 3.2. *From Theorems 2.1 and 3.1 (ii), we can observe that the convergence speed of powered extreme of cdf for MD is better than that of extreme of cdf.*

In the following we provide the higher-order expansions of the pdf of powered maximum.

Theorem 3.2.
(i) For $0 < t \neq 2$ and the normalizing constants c_n and d_n given by (2.5), we have

$$\frac{d}{dx}\mathbb{P}(|M_n|^t \leq c_n x + d_n) = \Lambda'(x) \left[1 + P_1(t,x) b_n^{-2} + P_2(t,x) b_n^{-4} + O(b_n^{-6}) \right], \tag{3.7}$$

where

$$P_1(t,x) = \sigma^2 \left\{ -\left[\frac{(t-2)x^2}{2} + x + 1 \right] e^{-x} + (t-2)x^2 - (t-3)x \right\}$$

and

$$P_2(t,x) = \sigma^4 \left\{ \frac{1}{2} \left[\frac{(t-2)x^2}{2} + x + 1 \right]^2 e^{-2x} \right.$$

$$
- \left[\frac{5(t-2)x^4}{8} - (t-2)\left(\frac{5}{6}t - \frac{10}{3}\right) x^3 + \left(2t + \frac{1}{2}\right) x^2 - 1 \right] e^{-x}
+ \frac{(t-2)^2 x^3}{8} - (t-2)\left(\frac{5}{6}t - \frac{11}{6}\right) x^2 + \frac{(t-3)(2t-3)}{2} x \Big\}.
$$

(ii) For $t = 2$ and the normalizing constants c_n and d_n given by (2.9), we have

$$
\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) = \Lambda'(x)\left[1 + Q_1(t,x) b_n^{-4} + Q_2(t,x) b_n^{-6} + O(b_n^{-8})\right], \tag{3.8}
$$

where

$$
Q_1(t,x) = \sigma^4 \left[\left(x^2 + x + \frac{1}{2} \right) e^{-x} - x^2 + x + \frac{1}{2} \right]
$$

and

$$
Q_2(t,x) = -\sigma^6 \left[\left(\frac{4}{3}x^3 + 2x^2 - 2x + \frac{7}{3} \right) e^{-x} - \frac{4}{3}x^3 + 2x^2 + 2x - \frac{1}{3} \right].
$$

Remark 3.3. *From Theorem 3.2, it is not difficult to observe that the convergence speeds of powered extreme of pdf for MD are the same order of $1/\log n$ and $1/(\log n)^2$ for power index $0 < t \neq 2$ and $t = 2$, respectively, because of $1/b_n^2 \sim 2\sigma^2 \log n$ by (1.3).*

Remark 3.4. *For $t = 2$, the normalizing constants c_n and d_n are not given by (2.9), but we choose them as follows:*

$$
c_n = 2\sigma^2(1 - \sigma^2 b_n^{-2}) \text{ and } d_n = b_n^2 - 2\sigma^4 b_n^{-2}, \tag{3.9}
$$

then we derive

$$
\begin{aligned}
&\mathbb{P}(|M_n|^t \le c_n x + d_n) \\
=&\Lambda(x)\Big\{ 1 - \frac{2e^{-x}\sigma^2}{b_n^2}(x+1) + \frac{e^{-x}\sigma^4}{b_n^4}\left[2e^{-x}(x+1)^2 - x^2 - x - \frac{3}{2} \right] b_n^{-4} \\
&- \frac{e^{-x}\sigma^6}{b_n^6}\left[\frac{4}{3}e^{-2x}(x+1)^3 - 2e^{-x}(x+1)\left(x^2 + x + \frac{3}{2}\right) \right. \\
&\left. + \frac{2}{3}x^3 + 2x^2 + 3x + \frac{14}{3} \right] + O(b_n^{-8}) \Big\}
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
&\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) \\
&= \Lambda'(x)\Big\{ 1 - \frac{2\sigma^2}{b_n^2}[e^{-x}(x+1) - x] + \frac{\sigma^4}{b_n^4}\left[2e^{-2x}(x+1)^2 - \left(5x^2 + 5x + \frac{3}{2}\right) e^{-x} \right. \\
&\left. + x^2 - x + \frac{1}{2} \right] + \frac{\sigma^6}{b_n^6}\left[4x(x+1)^2 e^{-2x} - (4x^3 + 2x^2 + 2x + 1)e^{-x} + \frac{2}{3}x^3 - x - \frac{7}{6} \right] + O(b_n^{-8}) \Big\}.
\end{aligned} \tag{3.11}
$$

Obviously, the convergence rates of the cdf and the pdf of powered extreme given by (3.4) and (3.8), which are proportional to $1/(\log n)^2$, are faster than that given by (3.10) and (3.11). Consequently, the normalizing constants c_n and d_n determined by (2.9) are optimal.

4 Numerical analysis

In this section, we conduct numerical studies to illustrate the accurateness of higher-order expansions for the cdf and the pdf of $|M_n|^t$. Let $T^{(i)}(x)$ and $S^{(i)}(x)$, $i = 1, 2, 3$, respectively represent the first-order, the second-order and the third-order approximations of the cdf and the pdf of $|M_n|^t$. Since the analysis of the case of $t \neq 2$ is similar to that of $t = 2$, we only consider the situation of $t = 2$. By Theorems 3.1 and 3.2, we obtain

$$\begin{aligned} T^{(1)}(x) &= \Lambda(x), \\ T^{(2)}(x) &= \Lambda(x)\left[1 - e^{-x}B_1(t,x)b_n^{-4}\right], \\ T^{(3)}(x) &= \Lambda(x)\left[1 - e^{-x}B_1(t,x)b_n^{-4} - e^{-x}B_2(t,x)b_n^{-6}\right], \end{aligned}$$

and

$$\begin{aligned} S^{(1)}(x) &= \Lambda(x)\exp(-x), \\ S^{(2)}(x) &= \Lambda(x)\exp(-x)\left[1 + Q_1(t,x)b_n^{-4}\right], \\ S^{(3)}(x) &= \Lambda(x)\exp(-x)\left[1 + Q_1(t,x)b_n^{-4} + Q_2(t,x)b_n^{-6}\right]. \end{aligned}$$

Easily observe that the second-order approximation and the third-order relate to the sample size n.

In order to compare the precision of true values with its approximations, let

$$E^{(i)}(x) = \left| F^n(\sqrt{c_n x + d_n}) - T^{(i)}(x) \right|$$

and

$$G^{(i)}(x) = \left| \frac{n c_n}{2\sqrt{c_n x + d_n}} F^{n-1}(\sqrt{c_n x + d_n}) f(\sqrt{c_n x + d_n}) - S^{(i)}(x) \right|$$

respectively stand for the absolute errors of the cdf and the pdf, where $i = 1, 2, 3$. We utilize MATLAB to compute the approximations and the true values of the cdf and the pdf of M_n^2.

First, we estimate the absolute errors of the cdf of M_n^2 at $x = 0.7$, where the sample size n varies from 25 to 1000 with step size 25. For given $x = 0.7$, numerical analysis results of $E^{(i)}(x)$ are recorded in Table 4. The table demonstrates that the precision of all three kinds of approximations of the cdf can be refined as the sample size n increases.

To order to indicate the precision of all approximations more intuitive with the change of the sample size n, the actual values and its approximation of the cdf of M_n^2 are plotted versus the values of n with $x = 1.5$. Figure 2 evidences that the larger n, the better all asymptotics.

Secondly, we estimate the absolute errors of the pdf of M_n^2 at $x = 0.7$, where the value of the sample size n ranges from 375 to 15000 with step length 375. Table 4 lists the numerical analysis results of $G^{(i)}(x)$, where $i = 1, 2, 3$. Table 4 reveals that the precision of all three kinds of approximations of the pdf enhances as the sample size n grows.

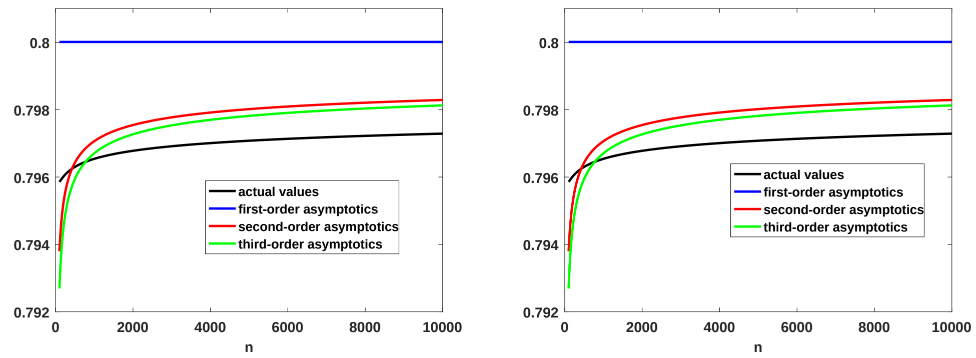
To clear the precision of all approximations more intuitive with n, the actual and its approximations of the pdf of M_n^2 are plotted versus the values of n with $x = 1.5$. Figure 3 indicates that as the sample size n becomes larger, all approximations become better.

n	$E^{(1)}(x)$	$E^{(2)}(x)$	$E^{(3)}(x)$
25	0.0169056391	0.00877452615	0.00733539417
50	0.0143357459	0.00869068028	0.00785819009
75	0.0131346277	0.00843346219	0.00780078242
100	0.0123911158	0.00821865886	0.00768964941
125	0.0118668421	0.00804347997	0.00757945239
150	0.0114683134	0.0078976489	0.00747885611
175	0.0111502039	0.00777354585	0.00738841683
200	0.0108874336	0.00766594114	0.00730705118
225	0.0106648041	0.00757120115	0.00723346892
250	0.0104724714	0.00748673237	0.00716650872
275	0.0103037264	0.00741063197	0.00710519645
300	0.0101538089	0.00734146835	0.00704873143
325	0.0100192298	0.00727814037	0.0069964575
350	0.00989736162	0.00721978455	0.00694783483
375	0.00978618048	0.00716571218	0.00690241634
400	0.00968409693	0.0071153658	0.00685982891
425	0.00958984178	0.00706828819	0.00681975869
450	0.00950238672	0.00702409995	0.00678193964
475	0.00942088789	0.00698248302	0.00674614458
500	0.00934464492	0.00694316822	0.00671217821
525	0.00927307061	0.00690592583	0.00667987148
550	0.00920566819	0.00687055834	0.00664907727
575	0.00914201391	0.00683689469	0.00661966674
600	0.00908174361	0.00680478587	0.00659152652
625	0.00902454227	0.00677410125	0.00656455636
650	0.00897013561	0.00674472575	0.00653866719
675	0.00891828351	0.00671655745	0.00651377957
700	0.00886877463	0.00668950571	0.00648982235
725	0.00882142208	0.00666348955	0.00646673154
750	0.00877605982	0.00663843637	0.00644444946
775	0.00873253972	0.00661428081	0.00642292387
800	0.00869072914	0.00659096382	0.00640210739
825	0.00865050883	0.00656843192	0.00638195685
850	0.00861177127	0.00654663651	0.00636243286
875	0.00857441915	0.00652553326	0.00634349935
900	0.00853836418	0.00650508169	0.00632512325
925	0.008503526	0.0064852447	0.00630727414
950	0.00846983127	0.00646598823	0.00628992398
975	0.0084372129	0.00644728091	0.00627304687
1000	0.00840560939	0.00642909381	0.00625661887

Table 1: Absolute errors between actual values and their asymptotics of the cdf at $x = 0.7$ with $\sigma = 2$

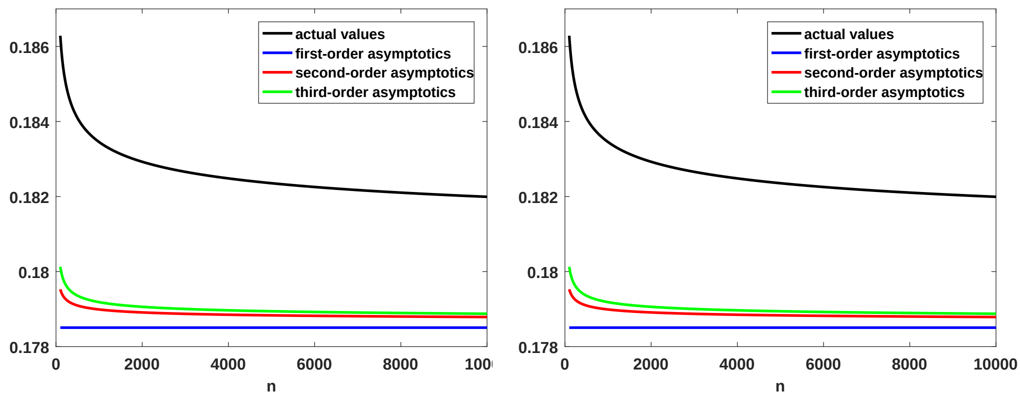
n	$G^{(1)}(x)$	$G^{(2)}(x)$	$G^{(3)}(x)$
375	0.00825613746	0.00585394461	0.00554667797
750	0.00710011928	0.00514055207	0.00491416905
1125	0.0065538405	0.00479753582	0.00460544643
1500	0.00621014157	0.00457905959	0.00440714319
1875	0.00596472382	0.00442157961	0.00426337709
2250	0.00577637198	0.00429979382	0.0041517166
2625	0.00562489953	0.00420122856	0.00406103823
3000	0.00549902795	0.00411887475	0.0039850629
3375	0.00539186326	0.00404842643	0.00391991863
3750	0.00529890627	0.00398706069	0.00386305898
4125	0.00521707021	0.0039328329	0.00381272502
4500	0.0051441522	0.00388435007	0.00376765376
4875	0.00507852912	0.00384058231	0.0037269095
5250	0.00501897306	0.00380074803	0.00368978076
5625	0.00496453429	0.00376424093	0.0036557147
6000	0.00491446415	0.00373058177	0.00362427365
6375	0.00486816283	0.00369938562	0.00359510557
6750	0.00482514285	0.00367033886	0.00356792328
7125	0.00478500308	0.00364318286	0.00354248964
7500	0.00474740971	0.00361770192	0.00351860668
7875	0.00471208225	0.00359371443	0.00349610752
8250	0.00467878285	0.00357106611	0.00347485024
8625	0.00464730824	0.00354962483	0.00345471324
9000	0.0046174834	0.00352927666	0.00343559152
9375	0.00458915666	0.00350992269	0.00341739387
9750	0.00456219575	0.00349147652	0.00340004057
10125	0.00453648471	0.0034738623	0.00338346154
10500	0.00451192133	0.00345701305	0.0033675949
10875	0.00448841509	0.0034408694	0.00335238574
11250	0.00446588547	0.00342537844	0.00333778509
11625	0.00444426057	0.00341049288	0.00332374917
12000	0.00442347592	0.00339617026	0.00331023861
12375	0.00440347351	0.00338237231	0.00329721797
12750	0.00438420096	0.00336906445	0.00328465516
13125	0.00436561084	0.00335621534	0.00327252109
13500	0.00434766007	0.00334379647	0.00326078929
13875	0.00433030941	0.00333178183	0.0032494356
14250	0.00431352301	0.00332014766	0.00323843795
14625	0.00429726807	0.00330887219	0.00322777608
15000	0.00428151447	0.00329793539	0.00321743138

Table 2: Absolute errors between actual values and their asymptotics of the pdf at $x = 0.7$ with $\sigma = 2$



(a) $\sigma = 2$ (b) $\sigma = 0.5$

Figure 2: Actual values and its asymptotics of the cdf of M_n^2 with $x = 1.5$. The actual values drawn in black, the first-order approximations drawn in blue, the second-order approximations drawn in red and the third-order approximation drawn in green.



(a) $\sigma = 2$ (b) $\sigma = 0.5$

Figure 3: Actual values and its asymptotics of the pdf of M_n^2 with $x = 1.5$. The actual values drawn in black, the first-order approximations drawn in blue, the second-order approximations drawn in red and the third-order approximation drawn in green.

5 Proof of main result

Proof of Theorem 3.1. By some fundamental calculations, we get

$$\mathbb{P}(|M_n|^t \leq c_n x + d_n) = F^n((c_n x + d_n)^{1/t}) - (1 - F((c_n x + d_n)^{1/t}))^n. \tag{5.1}$$

First, we consider the case of $0 < t \neq 2$. By (2.16) and similar discussions as for (2.17) and (2.18), we get

$$F^n(\delta_n(x,t)) = \Lambda(x)\left\{1 - A_1(t,x)e^{-x}b_n^{-2} + \left(\frac{1}{2}A_1^2(t,x)e^{-x} - A_2(t,x)\right)e^{-x}b_n^{-4} + O(b_n^{-6})\right\}, \tag{5.2}$$

where $A_1(t,x)$ and $A_2(t,x)$ are determined by (2.11) and (2.12), and

$$(1 - F(\delta_n(x,t)))^n = \left\{\frac{e^{-x}}{n}\left[1 + O(b_n^{-2})\right]\right\}^n = o(b_n^{-\eta}),\ \eta \geq 6. \tag{5.3}$$

A combination of (5.2) and (5.3) implies that (3.1) holds.

For the case of $t = 2$, by similar arguments as for $0 < t \neq 2$, the desired result follows. The proof is complete.

$\square$

Proof of Theorem 3.2. One can easily check that

$$\begin{aligned}\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) =& n\left(\frac{d}{dx}F((c_n x + d_n)^{1/t})\right) \\ &\times\left\{F^{n-1}((c_n x + d_n)^{1/t}) + [1 - F((c_n x + d_n)^{1/t})]^{n-1}\right\}.\end{aligned} \tag{5.4}$$

For $0 < t \neq 2$, combining with Lemmas 2.4 and 2.5, we get

$$\begin{aligned}&\frac{1}{\Lambda'(x)}\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) - 1 = \left\{1 + \frac{\sigma^2 x}{b_n^2}[3 - t - (2-t)x]\right. \\ &\quad\left. + \frac{\sigma^4 x^2}{b_n^4}\left[\frac{1}{2}(3-t)(3-2t) + (t-2)\left(\frac{11}{6} - \frac{5}{6}t\right)x + \frac{1}{8}(t-2)^2 x^2\right] + O(b_n^{-6})\right\} \\ &\quad\times\left\{1 - \sigma^2\left[1 + x + \frac{1}{2}(t-2)x^2\right]e^{-x}b_n^{-2} + \left(\frac{1}{2}\left[1 + x + \frac{1}{2}(t-2)x^2\right]^2 e^{-x}\right.\right. \\ &\quad\left.\left. - \left[\frac{1}{8}(t-2)^2x^4 + \frac{1}{6}(t-2)(5-2t)x^3 - \frac{x^2}{2} - x - 1\right]\right)\sigma^4 e^{-x}b_n^{-4} + O(b_n^{-6})\right\} - 1 \\ &= \frac{\sigma^2}{b_n^2}\left\{-\left[\frac{(t-2)x^2}{2} + x + 1\right]e^{-x} + (t-2)x^2 - (t-3)x\right\} \\ &\quad + \frac{\sigma^4}{b_n^4}\left\{\frac{1}{2}\left[\frac{(t-2)x^2}{2} + x + 1\right]^2 e^{-2x}\right. \\ &\quad - \left[\frac{5(t-2)x^4}{8} - (t-2)\left(\frac{5}{6}t - \frac{10}{3}\right)x^3 + \left(2t + \frac{1}{2}\right)x^2 - 1\right]e^{-x} \\ &\quad\left. + \frac{(t-2)^2x^3}{8} - (t-2)\left(\frac{5}{6}t - \frac{11}{6}\right)x^2 + \frac{(t-3)(2t-3)}{2}x\right\} + O(b_n^{-6}) \\ &= P_1(t,x)b_n^{-2} + P_2(t,x)b_n^{-4} + O(b_n^{-6}),\end{aligned}$$

which deduces (3.7).

The following is for the case of $t = 2$. By (5.4) and Lemmas 2.6 and 2.7, we gain

$$\begin{aligned}&\frac{1}{\Lambda'(x)}\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) - 1 = \left\{1 - \frac{\sigma^4}{b_n^4}\left(x^2 - x - \frac{1}{2}\right)\right. \\ &\quad\left. + \frac{\sigma^6}{b_n^6}\left(\frac{4}{3}x^3 - 2x^2 - 2x + \frac{1}{3}\right) + O(b_n^{-8})\right\} \\ &\quad\times\left[1 + \frac{\sigma^4 e^{-x}}{b_n^4}\left(x^2 + x + \frac{1}{2}\right) - \frac{\sigma^6 e^{-x}}{b_n^6}\left(\frac{4}{3}x^3 + 2x^2 - 2x + \frac{7}{3}\right) + O(b_n^{-8})\right] - 1 \\ &= \frac{\sigma^4}{b_n^4}\left[\left(x^2 + x + \frac{1}{2}\right)e^{-x} - x^2 + x + \frac{1}{2}\right] \\ &\quad - \frac{\sigma^6}{b_n^6}\left[\left(\frac{4}{3}x^3 + 2x^2 - 2x + \frac{7}{3}\right)e^{-x} - \frac{4}{3}x^3 + 2x^2 + 2x - \frac{1}{3}\right] + O(b_n^{-8}) \\ &= Q_1(t,x)b_n^{-4} + Q_2(t,x)b_n^{-6} + O(b_n^{-8}),\end{aligned}$$

which proves (3.8). The proof of Theorem 3.2 is finished.

□

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