A Class of Non-autonomous and Nonlinear Singular Liénard Equations

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Abstract

A class of non-autonomous and nonlinear singular Liénard equation is developed by nonlocal transformation of the linear harmonic oscillator equation. It is shown that it includes some Kamke equations and a nonlinear equation of the general relativity as special cases.

Keywords: Nonlocal transformation, linear harmonic oscillator, non-autonomous equations, quadratic Liénard equations, Kamke equations, general relativity.

1. Introduction

One of the most interesting topics in mathematics and mathematical physics is the generalization of existing theories or model equations for more adequate and satisfactory description of the Nature. For example the newtonian classical mechanics is known to be a limiting case of the special relativity in physics. In this way the linear harmonic oscillator, that is the prototype of the second order ordinary differential equation, is well known to be for instance a particular case of the cubic Duffing equation widely used to model nonlinear phenomena. The linear harmonic oscillator may also be considered as a limiting case of other ordinary differential equations [1]. As such, consider the Kamke equation 2.79 [2]

$$u'(x) - l\frac{g'(x)}{g(x)}u'(x) + cg(x)^{2l}u(x) = 0 \qquad (1)$$

which reduces to the equation of the linear oscillator when $l = 0$, and x is the time. In this context it is appropriate to investigate the generalized form of equation (1). Unfortunately the nonlinear and generalized Kamke equation 2.79 [2] which can be mapped into the linear harmonic oscillator is not clearly documented in the literature. This is a drawback in the mathematical theory of differential equations as no one can answer the question: Does equation (1) can be generalized to nonlinear differential equation linearizable into the linear harmonic oscillator ? The present work assumes the existence of such a nonlinear differential equation. The predicted equation is mathematically interesting since its exact and explicit general solution may be expressed in terms of the linear harmonic oscillator equation and used to model nonlinear phenomena in physics and engineering applications. To demonstrate this prediction the general theory is first established (section 2) and secondly illustrative examples are studied (section 3). A conclusion for the work is finally addressed.

2. General theory

This section is devoted to the establishment of the nonlinear and generalized KamKe equation 2.79 [2] by nonlocal transformation of the linear harmonic oscillator equation. Let

$$y''(\tau) + cy(\tau) = 0 \qquad (2)$$

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be the equation of the linear harmonic oscillator where $c \succ 0$ is an arbitrary parameter and prime stands for the derivative with respect to argument. Let also [1, 3, 4]

$$y(\tau) = f(x)e^{\gamma\varphi(u)}, \qquad \tau = \int g(x)^l\, dx \tag{3}$$

be a nonlocal transformation of variables τ and $y(\tau)$, where γ and l are arbitrary parameters, and $\varphi(u)$, $f(x) \neq 0$, and $g(x) \neq 0$, are arbitrary functions of u and x respectively. Then the following theorem may be proved.

***Theorem 1**. Consider equation (2). Then by application of (3), equation (2) may reduce to*

$$\begin{aligned}&\frac{d^2u(x)}{dx^2} + \left[\gamma\varphi'(u) + \frac{\varphi''(u)}{\varphi'(u)}\right]\left(\frac{du}{dx}\right)^2 + \left[2\frac{f'(x)}{f(x)} - l\frac{g'(x)}{g(x)}\right]\frac{du}{dx} + \\ &\frac{1}{\gamma\varphi'(u)}\left[\frac{f''(x)}{f(x)} - l\frac{f'(x)}{f(x)}\frac{g'(x)}{g(x)} + cg(x)^{2l}\right] = 0\end{aligned} \tag{4}$$

where $\varphi'(u) \neq 0.$

Proof. Under the application of (3) the first derivative of $y(\tau)$ may be immediately written as

$$\frac{dy}{d\tau} = \gamma f(x)\varphi'(u)e^{\gamma\varphi(u)}g(x)^{-l}u'(x) + f'(x)e^{\gamma\varphi(u)}g(x)^{-l} \tag{5}$$

so that one may easily find

$$\begin{aligned}\frac{d^2y}{d\tau^2} &= \gamma f(x)\varphi'(u)e^{\gamma\varphi(u)}g(x)^{-2l}u''(x) + \gamma\left(\frac{du}{dx}\right)^2\left[\gamma\varphi'(u)^2 + \varphi''(u)\right]e^{\gamma\varphi(u)}g(x)^{-2l} + \\ &\gamma\varphi'(u)u'(x)\left[2f'(x) - lf(x)\frac{g'(x)}{g(x)}\right]e^{\gamma\varphi(u)}g(x)^{-2l} + \left[f''(x) - lf'(x)\frac{g'(x)}{g(x)}\right]e^{\gamma\varphi(u)}g(x)^{-2l}\end{aligned} \tag{6}$$

Introducing (6) into equation (2), and taking into consideration $y(\tau) = f(x)e^{\gamma\varphi(u)}$, yields (4). Equation (4) is the desired nonlinear and generalized Kamke equation 2.79. To observe that one may consider the following theorem.

Theorem 2. *If* $\gamma = \frac{1}{2}$, $f(x) = 1$, *and* $\varphi(u) = \ln(u^2)$, *then equation (4) transforms into equation (1).*

Proof. If $\varphi(u) = \ln(u^2)$, then $\varphi'(u) = \frac{2}{u}$, and $\varphi''(u) = -\frac{2}{u^2}$. In this context equation (4) becomes

$$u''(x) + (2\gamma - 1)\frac{u'(x)^2}{u(x)} - l\frac{g'(x)}{g(x)}u'(x) + \frac{c}{2\gamma}g(x)^{2l}u(x) = 0 \tag{7}$$

for $f(x) = 1.$

For $\gamma = \frac{1}{2}$, the term in parenthesis, $2\gamma - 1 = 0$, and equation (4) reduces to equation (1). Due to the presence of the term $\left(\frac{du}{dx}\right)^2$, equation (4) is a nonlinear differential equation.

Now some examples may be considered to illustrate the theory.

3. Examples

The objective in this section is to give some well known examples of differential equations to illustrate the usefulness of the developed theory.

3.1 A nonlinear equation of general relativity

Consider the following theorem

Theorem 3. *Let* $\gamma = -1,\ l = 1, c = 0$*, and* $g(x) = x$*. Then equation (7) reduces to*

$$u''(x) - 3\frac{u'(x)^2}{u(x)} - \frac{1}{x}u'(x) = 0 \qquad (8)$$

Proof. The proof is immediate. It suffices to note that (8) is a special case of (7) when $\gamma = -1,\ l = 1, c = 0$, and $g(x) = x$.

Equation (8) is obtained in study of general relativity in [5]. This equation is later investigated in [6, 7] by means of the Prelle-Singer method but with no exact and explicit solutions. The problem now is to solve equation (8) exactly and explicitly. For $c = 0$, the solution to (2) take the form

$$y(\tau) = A\tau + B \qquad (9)$$

where A and B are integration constants. By application of the Theorem 3, one may compute from (3) the solution

$$u(x) = \frac{1}{\sqrt{A\tau + B}} \qquad (10)$$

and

$$\tau = \frac{1}{2}x^2 \qquad (11)$$

Therefore the solution $u(x)$ may take the expression

$$u(x) = \frac{1}{\sqrt{\frac{A}{2}x^2 + B}} \qquad (12)$$

which may be definitively written in the form given in [5] as

$$u(x) = K_1(1 + Kx^2)^{\frac{-1}{2}} \qquad (13)$$

where $K_1 = \frac{\sqrt{B}}{B}$, et $K = \frac{A}{2B}$

3.2 Kamke equation 6.169

The following theorem can be established.

Theorem 4. *Let* $\gamma = 1,\ c = 0,\ l = 1$ *and* $g(x) = x$. *Then equation (7) reduces to*

$$u''(x) + \frac{u'(x)^2}{u(x)} - \frac{u'(x)}{x} = 0 \tag{14}$$

Proof. One may easily observe that equation (14) is a special case of (7) when $\gamma = 1,\ c = 0,\ l = 1$, and $g(x) = x$. Equation (14) is the Kamke equation 6.169. The explicit and exact solution may be computed as follows. Using the solution $y(\tau) = A\tau + B$, one may compute, by application of (3), the solution $u(x)$ as

$$u(x)^2 = A\tau + B \tag{15}$$

and

$$\tau = \frac{1}{2}x^2 \tag{16}$$

such that

$$u(x)^2 = \frac{A}{2}x^2 + B \tag{17}$$

Making $C_1 = \frac{A}{2}$, and $C_2 = B$, one may arrive at the solution given in [2]

$$u(x)^2 = C_1 x^2 + C_2 \tag{18}$$

Conclusion

A nonlocal transformation, that is a non-point change of variables has been developed to study second order non-autonomous and nonlinear differential equations. In this way, the nonlocal transformation of the linear harmonic oscillator equation has generated a nonlinear and generalized Kamke equation 2.79. This nonlinear and generalized equation has not been mentioned in Kamke book [2]. The generated equation has been used to solve some well known interesting nonlinear equations of the literature. In this regard the usefullness of the developed theory has been shown. It is finally worth to note that for special choice of functions and parameters, the established equation may reduce to the purely mixed or quadratic Liénard type nonlinear differential equations which occur often in physics and engineering applications.

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