Generalized Fibonacci Numbers and $4k+1$-fold symmetric Quasicrystals

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Abstract

Given that the two-parameter p, q quantum-calculus deformations of the integers $[n]_{p,q} = (p^n - q^n)/(p-q) = F_n$ coincide precisely with the Fibonacci numbers (integers), as a result of Binet's formula when $p = \tau = \frac{1+\sqrt{5}}{2}$, $q = \tilde{\tau} = \frac{1-\sqrt{5}}{2}$ (Galois-conjugate pairs), we extend this result to the *generalized* Binet's formula (corresponding to generalized Fibonacci sequences) studied by Whitford. Consequently, the Galois-conjugate pairs $(p, q = \tilde{p}) = \frac{1}{2}(1 \pm \sqrt{m})$, in the very special case when $m = 4k+1$ and square-free, generalize Binet's formula $[n]_{p,q} = G_n$ generating integer-values for the generalized Fibonacci numbers G_n's. For these reasons, we expect that the two-parameter $(p, q = \tilde{p})$ quantum calculus should play an important role in the physics of quasicrystals with $4k+1$-fold rotational symmetry.

Keywords : Quantum Calculus; Golden Mean; Quasicrystals; Noncommutative Geometry.

Recently [1] we reviewed the two-parameter quantum calculus used in the construction of Fibonacci oscillators and presented the (p, q)-deformed Lorentz transformations which (still) leave invariant the (undeformed) Minkowski spacetime interval $t^2 - x^2 - y^2 - z^2$. The (p, q) number is defined for any number n as

$$[n]_{p,q} = [n]_{q,p} \equiv \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + \ldots + pq^{n-2} + q^{n-1} \tag{1}$$

which is a natural generalization of the q-number

$$[n]_q \equiv \frac{1 - q^n}{1 - q} = 1 + q + \ldots + q^{n-2} + q^{n-1} \tag{2}$$

In [1] we remarked that when p, q are given by the Golden Mean, and its Galois conjugate, respectively

$$p = \tau = \frac{1+\sqrt{5}}{2}, \quad q = \tilde{\tau} = -1/\tau = \frac{1-\sqrt{5}}{2} \tag{3}$$

the p, q *deformations* of the integers $[n]_{p,q}$ *coincide* precisely with the Fibonacci numbers (also integers) as a result of Binet's formula

$$[n]_{p,q} = [n]_{q,p} \equiv \frac{\tau^n - (-1)^n \tau^{-n}}{\sqrt{5}} = F_n, \quad F_1 = F_2 = 1. \tag{4}$$

To continue, we may ask what other Galois-conjugate pairs $(p, q = \tilde{p})$, besides $p = \tau; q = \tilde{\tau}$, yield integer values for $[n]_{q,p} = N$ in eq-(1) ?; are $p = \tau; q = \tilde{\tau}$ special or are there an infinite number of Galois-conjugate pairs of solutions ?

Given the Galois-conjugate pairs $\frac{1}{2}(1 \pm \sqrt{m})$, where m is a square-free integer, the Binet formula (6) can be *generalized* to [2]

$$\frac{(\frac{1+\sqrt{m}}{2})^n - (\frac{1-\sqrt{m}}{2})^n}{\sqrt{m}} = G_n \tag{5}$$

where $\frac{1}{2}(1 \pm \sqrt{m})$ are the roots to the quadratic equation $x^2 - x - (\frac{m-1}{4}) = 0$. However we must emphasize that *not* all values of G_n corresponding to the square-free integers $m = 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, \cdots$ are *integers*. The numbers G_n belong to a generalized Fibonacci sequence given by [2]

$$G_{n+2} = G_{n+1} + (\frac{m-1}{4})\, G_n, \quad G_1 = G_2 = 1 \tag{6}$$

Therefore, by forcing all the G_n to be integers leads to $m = 4k+1, k = 1, 2, \cdots$ subject to the condition that m must be square-free. Two specific examples for the values of m where all the G_n are integers are $m = 13, 17$, their Galois-conjugate pairs are then given by $\frac{1}{2}(1 \pm \sqrt{13})$ and $\frac{1}{2}(1 \pm \sqrt{17})$, and whose integer sequences (6) are respectively

$$\{ 1,\ 4,\ 7,\ 19,\ \cdots \} \quad \{ 1,\ 5,\ 9,\ 29,\ \cdots \} \tag{7}$$

The Golden mean is associated with the 5-fold symmetry of the Penrose tiling (quasicrystal) of the two-dim plane, and can be obtained via the cut-and-projection method of the cubic lattice in $5D$ onto the two-dim plane. Quasicrystals with $10, 8, 12, 18$-fold rotational symmetry are well known (to the experts). A search in the literature to find examples of quasicrystals with an N-fold symmetry, for $N = 1, 2, 3, \cdots, 23$ brought us to the work of [3]. These authors fabricated several $2D$ Penrose and Thue-Morse quasicrystals in the mesoscale range, using a novel single-beam technique based on the spatial modulation of the optical beam by means of Spatial Light Modulator (SLM) and ComputerGenerated Hologram (CGH). They were able to produce, with single-beam optical setup, $2D$ Penrose patterns of rotational symmetry as high as 23-fold, never reached before.

The Golden mean seems to be very special in the sense that $\tau = 2\ cos(\frac{\pi}{5})$. Can one find other cases where

$$\frac{1+\sqrt{4k+1}}{2} \;=\; N\; cos(\frac{\pi}{4k+1}), \quad k = 1,2,\cdots, \quad N = integer \tag{8}$$

with $N =$ integer ? Concluding, the Galois-conjugate pairs $\frac{1}{2}(1 \pm \sqrt{m})$, with $m = 4k+1$, and square-free, which generalize Binet's formula (4) and which generate integer-values for *all* the G_n's (eqs-(5,6)) deserves further scrutiny. The work of Connes [4] revealed that the Penrose quasicrystal is a flower in the garden of Noncommutative geometry. Therefore, the two-parameter $(p, q = \tilde{p})$ quantum calculus should play a key role in the physics of quasicrystals with $4k+1$-fold rotational symmetry.

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References

[1] C. Castro Perelman, "Two-parameter Deformed Special Relativity and Fibonacci Oscillators" Physics and Astronomy International Journal, vol **2**, issue 4 (2018) 341-347.

[2] A.K. Whitford, "Binet's formula Generalized". Fib. Quart., **15** (1977), pp. 21, 24, 29.

A. G. Shannon, J. V. Leyendekkers, "The Golden Ratio family and the Binet equation, Notes on Number Theory and Discrete Mathematics, Vol. **21**, (2015), No. 2, 35-42.

Sloane, N.J.A., Plouffe, S. (1995) *The Encyclopedia of Integer Sequences.* San Diego,CA: Academic Press; Sloane, N.J.A. 1964. https://oeis.org/.

[3] G. Zito,B. Piccirillo, E. Santamato, A. Marino, V. Tkachenko, and G. Abbate"Two-dimensional photonic quasicrystals by single beam computer-generated holography" Optics Express **16**, issue 8 (2008) 5164.

[4] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives* (American Mathematical Society, Colloquium Publications vol. 55, 2007).