Evidential divergence measures in Dempster–Shafer theory

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***Abstract*—The Dempster–Shafer evidence (DSE) theory, as a generalization of the Bayes probability theory, has more capability to handle the uncertainty in the decision-making problems. In the DSE theory, however, how to measure the divergence between basic belief assignments (BBAs) is still an open issue which has attracted many attentions. On account of this point, in this paper, new evidential divergence measures are developed to measure the difference between BBAs in the DSE theory, called as EDMs. The EDMs consider both of the correlations between BBAs and the subset of set of BBAs, respectively. Consequently, they can provide a much more convincing and effective way to measure the discrepancy between BBAs. In a word, the EDMs as the generalization of the divergence measures in the Bayes probability theory have the universal applicabilities. Additionally, a new Belief–Jensen–Shannon divergence measure is derived based on the EDMs, in which different weights can be assigned to the BBAs involved, so that it provides a promising solution to be applied in solving the problems of decision-making. Finally, numerical examples are illustrated that the proposed methods are more feasible and reasonable to measure the divergence between BBAs in the DSE theory.**

***Index Terms*—Dempster–Shafer theory, Basic belief assignments, Belief divergence measure**

I. NEW EVIDENTIAL DIVERGENCE MEASURES

A. Correlation between BBAs

Definition 1: (Inverse correlation coefficient between the hypotheses of BBAs).

Let m_1 and m_2 be two BBAs in the frame of discernment $\mathbb{E}$, consisting of H mutually exclusive and collectively exhaustive events, where $\mathcal{A}_i$ is a hypothesis of m_1 and $\mathcal{A}_j$ is a hypothesis of m_2 $(i,j = 1,\ldots,2^H)$. An inverse correlation coefficient between the hypotheses of BBAs is defined as

$$\mathfrak{R}(\mathcal{A}_i,\mathcal{A}_j) = 1 - \frac{|\mathcal{A}_i \cap \mathcal{A}_j|}{|\mathcal{A}_i \cup \mathcal{A}_j|}, \tag{1}$$

in which $\mathcal{A}_i \cap \mathcal{A}_j$ and $\mathcal{A}_i \cup \mathcal{A}_j$ denote the intersection and union of $\mathcal{A}_i$ and $\mathcal{A}_j$, respectively; $|\mathcal{A}_i \cap \mathcal{A}_j|$ and $|\mathcal{A}_i \cup \mathcal{A}_j|$ represent the cardinalities of $\mathcal{A}_i \cap \mathcal{A}_j$ and $\mathcal{A}_i \cup \mathcal{A}_j$, respectively.

B. Divergence measures for BBAs

On the basis of the definition of inverse correlation coefficient between BBAs, several evidential divergence measures are defined as follows.

Let $\mathbb{E}$ be the frame of discernment which has H mutually exclusive and collectively exhaustive events. Let m_1 and m_2 be two BBAs on $\mathbb{E}$, where $\mathcal{A}_i$ is a hypothesis of BBA m_1 and $\mathcal{A}_j$ is a hypothesis of BBA m_2 $(i,j = 1,\ldots,2^H)$.

1) Belief–$\mathcal{I}$ divergence measure:

Definition 2: (Belief–$\mathcal{I}$ divergence measure).

The Belief–$\mathcal{I}$ divergence, denoted as $\mathcal{BI}$ between the two BBAs m_1 and m_2 is defined by

$$\mathcal{BI}(m_1,m_2) = \sum_{i=1}^{2^H}\sum_{j=1}^{2^H} m_1(\mathcal{A}_i)\log\frac{m_1(\mathcal{A}_i)}{m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j), \tag{2}$$

with

$$\Delta(\mathcal{A}_i,\mathcal{A}_j) = \begin{cases} 1, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|} = 1, \\ 0, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|} = 0, \\ \mathfrak{R}(\mathcal{A}_i,\mathcal{A}_j), & \text{otherwise.} \end{cases} \tag{3}$$

When the hypotheses of BBAs are made of singleton set, the $\mathcal{BI}$ divergence degrades into the $\mathcal{I}$ divergence.

$$\mathcal{BI}(m_1,m_2) = \sum_{i=1}^{H} m_1(\mathcal{A}_i)\log\frac{m_1(\mathcal{A}_i)}{m_2(\mathcal{A}_i)}. \tag{4}$$

2) Belief–$\mathcal{J}$ divergence measure:

Definition 3: (Belief–$\mathcal{J}$ divergence measure).

The Belief–$\mathcal{J}$ divergence, denoted as $\mathcal{BJ}$ between the two BBAs m_1 and m_2 is defined by

$$\begin{aligned}\mathcal{BJ}(m_1,m_2) &= \mathcal{BI}(m_1,m_2) + \mathcal{BI}(m_2,m_1) \\ &= \sum_{i=1}^{2^H}\sum_{j=1}^{2^H}(m_1(\mathcal{A}_i) - m_2(\mathcal{A}_j))\log\frac{m_1(\mathcal{A}_i)}{m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j),\end{aligned} \tag{5}$$

with

$$\Delta(\mathcal{A}_i,\mathcal{A}_j) = \begin{cases} 1, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|} = 1, \\ 0, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|} = 0, \\ \mathfrak{R}(\mathcal{A}_i,\mathcal{A}_j), & \text{otherwise.} \end{cases} \tag{6}$$

When the hypotheses of BBAs are made of singleton set, the $\mathcal{BJ}$ divergence degrades into the $\mathcal{J}$ divergence.

$$\mathcal{BJ}(m_1,m_2) = \sum_{i=1}^{H}(m_1(\mathcal{A}_i) - m_2(\mathcal{A}_i))\log\frac{m_1(\mathcal{A}_i)}{m_2(\mathcal{A}_i)}. \tag{7}$$

3) Belief–$\mathcal{K}$ divergence measure:

Definition 4: (Belief–$\mathcal{K}$ divergence measure).

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The Belief–$\mathcal{K}$ divergence, denoted as $\mathcal{BK}$ between the two BBAs m_1 and m_2 is defined by

$$\mathcal{BK}(m_1, m_2) = \sum_{i=1}^{2^H}\sum_{j=1}^{2^H} m_1(\mathcal{A}_i)\log\frac{m_1(\mathcal{A}_i)}{\frac{1}{2}m_1(\mathcal{A}_i)+\frac{1}{2}m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j), \tag{8}$$

with

$$\Delta(\mathcal{A}_i,\mathcal{A}_j) = \begin{cases} 1, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|}=1,\\ 0, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|}=0,\\ \Re(\mathcal{A}_i,\mathcal{A}_j), & \text{otherwise.}\end{cases} \tag{9}$$

When the hypotheses of BBAs are made of singleton set, the $\mathcal{BK}$ divergence degrades into the $\mathcal{K}$ divergence.

$$\mathcal{BK}(m_1, m_2) = \sum_{i=1}^{H} m_1(\mathcal{A}_i)\log\frac{m_1(\mathcal{A}_i)}{\frac{1}{2}m_1(\mathcal{A}_i)+\frac{1}{2}m_2(\mathcal{A}_i)}. \tag{10}$$

4) Belief–$\mathcal{L}$ divergence measure:

Definition 5: (Belief–$\mathcal{L}$ divergence measure).

The Belief–$\mathcal{L}$ divergence, denoted as $\mathcal{BL}$ between the two BBAs m_1 and m_2 is defined by

$$\begin{aligned}\mathcal{BL}(m_1, m_2) &= \mathcal{BK}(m_1, m_2) + \mathcal{BK}(m_2, m_1)\\ &= \sum_{i=1}^{2^H}\sum_{j=1}^{2^H} m_1(\mathcal{A}_i)\log\frac{m_1(\mathcal{A}_i)}{\frac{1}{2}m_1(\mathcal{A}_i)+\frac{1}{2}m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j)+\\ &\quad \sum_{i=1}^{2^H}\sum_{j=1}^{2^H} m_2(\mathcal{A}_i)\log\frac{m_2(\mathcal{A}_i)}{\frac{1}{2}m_1(\mathcal{A}_i)+\frac{1}{2}m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j),\end{aligned} \tag{11}$$

with

$$\Delta(\mathcal{A}_i,\mathcal{A}_j) = \begin{cases} 1, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|}=1,\\ 0, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|}=0,\\ \Re(\mathcal{A}_i,\mathcal{A}_j), & \text{otherwise.}\end{cases} \tag{12}$$

When the hypotheses of BBAs are made of singleton set, the $\mathcal{BL}$ divergence degrades into the $\mathcal{L}$ divergence.

$$\begin{aligned}\mathcal{BL}(m_1, m_2) = &\sum_{i=1}^{H} m_1(\mathcal{A}_i)\log\frac{m_1(\mathcal{A}_i)}{\frac{1}{2}m_1(\mathcal{A}_i)+\frac{1}{2}m_2(\mathcal{A}_i)}+\\ &\sum_{i=1}^{H} m_2(\mathcal{A}_i)\log\frac{m_2(\mathcal{A}_i)}{\frac{1}{2}m_1(\mathcal{A}_i)+\frac{1}{2}m_2(\mathcal{A}_i)}.\end{aligned} \tag{13}$$

II. A NEW BELIEF–JENSEN–SHANNON DIVERGENCE MEASURE

In order to make the belief divergence measure $\mathcal{BL}$ to be applicable in the study of decision-making problems, the Belief–Jensen–Shannon divergence is derived in this section. The main contribution of the Belief–Jensen–Shannon divergence is that different weights can be assigned to the BBAs involved according to the requirement of the decision-making problems.

Definition 6: (Belief–Jensen–Shannon divergence measure)

Let w_1, w_2 be the weights of BBAs m_1 and m_2 on the frame of discernment $\mathbb{E}$, respectively, where $w_1, w_2 \geq 0$ and $\sum_j w_j = 1$, $(j = 1, 2)$. Let $\mathcal{A}_i$ be a hypothesis of BBA m_1 and $\mathcal{A}_j$ be a hypothesis of BBA m_2 $(i, j = 1, \ldots, 2^H)$. The Belief–Jensen–Shannon divergence, denoted as $\mathcal{BJS}_w$ between the two BBAs m_1 and m_2 is defined by

$$\begin{aligned}\mathcal{BJS}_w(m_1, m_2) = &\, w_1\sum_{i=1}^{2^H}\sum_{j=1}^{2^H} m_1(\mathcal{A}_i)\log\frac{m_1(\mathcal{A}_i)}{w_1 m_1(\mathcal{A}_i)+w_2 m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j)+\\ &\, w_2\sum_{i=1}^{2^H}\sum_{j=1}^{2^H} m_2(\mathcal{A}_i)\log\frac{m_2(\mathcal{A}_i)}{w_1 m_1(\mathcal{A}_i)+w_2 m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j),\end{aligned} \tag{14}$$

with

$$\Delta(\mathcal{A}_i,\mathcal{A}_j) = \begin{cases} 1, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|}=1,\\ 0, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|}=0,\\ \Re(\mathcal{A}_i,\mathcal{A}_j), & \text{otherwise.}\end{cases} \tag{15}$$

For the $\mathcal{BJS}_w$ divergence, one of the major features is that different weights can be assigned to the BBAs according to their reliability, importance or other factors, which is especially useful in the research of decision problems.

When $w_j = \frac{1}{2}$ for $1 \leq j \leq 2$ that indicates the BBAs m_1 and m_2 have the same weights, $\mathcal{BJS}_{\frac{1}{2}}(m_1, m_2)$, as a special case of the $\mathcal{BJS}_w$ divergence, is defined by

$$\begin{aligned}\mathcal{BJS}_{\frac{1}{2}}(m_1, m_2) = &\, \frac{1}{2}\sum_{i=1}^{2^H}\sum_{j=1}^{2^H} m_1(\mathcal{A}_i)\log\frac{m_1(\mathcal{A}_i)}{\frac{1}{2}m_1(\mathcal{A}_i)+\frac{1}{2}m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j)+\\ &\, \frac{1}{2}\sum_{i=1}^{2^H}\sum_{j=1}^{2^H} m_2(\mathcal{A}_i)\log\frac{m_2(\mathcal{A}_i)}{\frac{1}{2}m_1(\mathcal{A}_i)+\frac{1}{2}m_2(\mathcal{A}_j)}\Delta(\mathcal{A}_i,\mathcal{A}_j),\end{aligned} \tag{16}$$

with

$$\Delta(\mathcal{A}_i,\mathcal{A}_j) = \begin{cases} 1, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|}=1,\\ 0, & \frac{|\mathcal{A}_i\cap\mathcal{A}_j|}{|\mathcal{A}_i\cup\mathcal{A}_j|}=0,\\ \Re(\mathcal{A}_i,\mathcal{A}_j), & \text{otherwise.}\end{cases} \tag{17}$$

The relationship between $\mathcal{BJS}_{\frac{1}{2}}(m_1, m_2)$ and $\mathcal{BL}(m_1, m_2)$ divergence measures can be derived as

$$\mathcal{BJS}_{\frac{1}{2}}(m_1, m_2) = \frac{1}{2}\mathcal{BL}(m_1, m_2). \tag{18}$$

When the hypotheses of BBAs are made of singleton sets, the $\mathcal{BJS}_w$ divergence degrades into the $\mathcal{JS}$ divergence.

$$\mathcal{BJS}_w(m_1, m_2) = w_1 \sum_{i=1}^{H} m_1(\mathcal{A}_i) \log \frac{m_1(\mathcal{A}_i)}{w_1 m_1(\mathcal{A}_i) + w_2 m_2(\mathcal{A}_i)} + w_2 \sum_{i=1}^{H} m_2(\mathcal{A}_i) \log \frac{m_2(\mathcal{A}_i)}{w_1 m_1(\mathcal{A}_i) + w_2 m_2(\mathcal{A}_i)}. \tag{19}$$

III. Conclusion

In this paper, the evidential divergence measures, called as EDMs were proposed in the Dempster–Shafer evidence (DSE) theory, which included $\mathcal{BI}$, $\mathcal{BJ}$, $\mathcal{BK}$ and $\mathcal{BL}$ divergence measures. The main contribution of this study was that the EDMs, as the generalization of the divergence measures in the Bayes probability theory had more powerful ability to measure divergences, not only for the probability distributions, but also for the BBAs in the DSE theory. In addition, based on the proposed EDMs, a new Belief–Jensen–Shannon divergence measure was derived, in which different weights could be assigned to the BBAs involved that was more suitable for the study of the decision-making problems. Consequently, the new EDMs solved the problem of divergence measure between BBAs in the DSE theory. It provided a promising solution to measure the difference between BBAs in the DSE theory by considering both of the correlations between BBAs and the subset of set of BBAs, respectively.

Conflict of Interest

The author states that there are no conflicts of interest.

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