

Quantum Mechanics Where Spin is $SO(3)$

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Abstract

In 2014 Steven Weinberg noted that quantum mechanics can avoid various difficulties, such as the many worlds hypothesis, by taking the quantum states to be density matrices without reference to state vectors. An immediate consequence of Weinberg's idea is that electron spin can be taken to follow $SO(3)$ instead of $SU(2)$. This radical departure from present understanding motivates our exploration of density matrices as a method of going beyond the Standard Model.

An important tool for Standard Model calculations is the Feynman path integral formulation of quantum field theory. When the path integral is Wick rotated from time to imaginary time or temperature it becomes a method of cooling down density matrices. While this does not show that one goes beyond the Standard Model by Wick rotation, it does show a close relation between this method of cooling density matrices and quantum field theory of the Standard Model. We explore these ideas and exhibit toy models with particle content and symmetry similar to the Standard Model.

Keywords

Quantum Mechanics; Electron Spin; Density Matrix; Symmetry

1. Introduction

Weinberg's analysis of density matrices[1] suggests a look at density matrices as the origin of the particle content and symmetries seen in the Standard Model of elementary particles. The Standard Model's weak $SU(2)$ symmetry is broken at low temperatures and becomes unbroken above the electroweak or Fermi scale of around 250 GeV. This is a change in symmetry that depends on temperature. To model it with density matrices, suppose a grand canonical ensemble is pure neutrino at low temperatures but the ensemble becomes mixed neutrinos and electrons at temperatures where the heat bath can provide W^\pm s. With a mass of about 90 GeV, it requires 180 GeV to produce a W^\pm particle pair, comparable to the energy where $SU(2)$ symmetry breaks.

Weinberg's observation [1] is that several of the unexplained behaviors of quantum mechanics are eliminated when one switches from state vectors to density matrices. State vectors famously imply that measurements cause the instantaneous modification of the mathematical representation of distant isolated systems when they are entangled. While such measurements cannot be used to transmit information, their effect on state vectors casts doubt on their being a true description of reality. Unitary evolution of state vectors cause an ontological problem when measurements are considered. A solution is the many-worlds interpretation which would create a ridiculously large number of branches of history. Weinberg notes that these problems can all be solved by an interpretation of quantum mechanics where the density matrix, rather than the state vector or wave function is the description of reality.

A difficulty in understanding the application of symmetry principles to density matrices is that the physics community is well educated in the application of symmetry to state vectors and this knowledge can make Weinberg's version of density matrix symmetry confusing. Section 2.0 covers the symmetry of density matrix models of spin-1/2. The section is elementary and a readers might consider skipping or skimming the section if the following paragraph already makes sense to them:

The state-vector model of spin-1/2 is a doublet or 2 representation of SU(2). A density matrix model of spin-1/2 is given by a $\bar{2} \times 2$ representation of SU(2). The rules of SU(2) symmetry are that $\bar{2} = 2$ and $2 \times 2 = 3 + 1$ so a density matrix model must follow the 3 + 1 representation of SU(2). The 1 is the arbitrary complex phase that is cancelled in a density matrix so we have that a density matrix model of spin-1/2 follows the 3 representation of SU(2) which is the same as the 3 representation of SO(3). What representation of SU(2) or SO(3) does a density matrix model of spin-3/2 follow? ¹

Since density matrices and state vectors are equivalent formulations of quantum mechanics,[2] we lose no experimental predictions by moving to density matrices. However, conversion between different formulations of quantum mechanics can be difficult. Quantum field theory is a version of state vector quantum mechanics where the vector components are occupation numbers for each possible energy. To use density matrices as the source of the SU(3)xSU(2)xU(1) symmetry of the Standard Model we need to relate density matrices to a quantum field theory. Our approach is to analyze an idealized Stern-Gerlach experiment from the two approaches and to compare their symmetry as a component of the SU(3)xSU(2)xU(1) Standard Model symmetry. This is the subject of Section 3.0 which ends with a derivation, using generalized Stern-Gerlach apparatus, of the well known fact that when fermions follow the defining representation of an SU(n) symmetry, their vector bosons follow the adjoint representation.

Section 4.0 discusses the derivation and symmetry consequences of density matrix cooling. In the example of weak SU(2) symmetry breaking the low temperature limit are two particles with different charges. This is typical of density matrix symmetry breaking. The word "algebra" in

¹Hint: Coincidentally, the number of Frieze group symmetries.

mathematics defines a set of objects that can be multiplied and added (and some other stuff). As an example the 3x3 complex matrices form an algebra, called a matrix algebra. Generalized Stern-Gerlach experiments in a fundamental representation of $SU(n)$ can be represented by the primitive idempotents of $n \times n$ complex matrices. We show that combinations of these experiments provide all the elements of the matrix algebra so that examining algebras is a natural way of explaining the Standard Model.

If we are to take density matrices as the fundamental object, then the industry's preferential use of state vectors has been due only to their mathematical convenience. A spinor is the square root of a vector in the same way that a Dirac operator is a square root of a Laplacian. That is the solution of a Dirac equation gives four solutions of a Laplace wave equation. The reverse is not true, that is, four arbitrary solutions to the Laplace wave equation cannot always be assembled into a solution of the Dirac equation. This suggests that the Dirac equation arises from couplings between Laplace wave equations. These couplings are encoded in the gamma matrices, so we propose that they are physical and their symmetry explains the Standard Model particle content. The details are in Section 5.0

Section 6.0 is a conclusion.

2. $SO(3)$ Spin Symmetry

Elementary particle theory is typically taught using gamma matrices and bispinors (i.e. "Dirac spinors" or "4-component spinors"). This paper uses spinors (or "2-component spinors") and the Pauli spin matrices, a method that is sometimes used in supersymmetry but is not commonly taught. If the reader is unfamiliar with this application of spinors, an excellent review paper that shows how to convert between spinor and bispinor calculations as well as a complete set of spinor Feynman diagrams for the Standard Model is [3]. Since we will be using spinors instead of bispinors in this paper, spin-1/2 is a natural choice for familiarizing the reader with the Weinberg method of density matrix symmetry.

Weinberg pointed out[1] that some of the bizarre behavior of quantum mechanics disappears when density matrices (DM) are considered without state vectors (SV). This allows symmetry operations more general than with state vectors. When density matrices are considered in the traditional way, as a sum over products of kets and bras, a symmetry operator U is applied twice, once to the bra side and once to the ket side, for example:

$$U(|a\rangle\langle a|) = U|a\rangle\langle a|U^\dagger \quad (1)$$

With n -dimensional state vectors, U is an $n \times n$ unitary matrix, while $|a\rangle\langle a|$ is an Hermitian $n \times n$ matrix.

For most of this section we will specialize to spin-1/2 so our state vectors are 2-dimensional complex vectors, our density matrices are 2×2 Hermitian matrices, and the symmetry operator U is a 2×2 unitary matrix. The density matrix has a total of $2 \times 2 = 4$ complex components. Weinberg's observation is that the four complex components of a density matrix allow symmetry operators appropriate to four complex components, that is, our symmetry operators can be 4×4 matrices. Another way of describing this is that Weinberg is allowing the density matrices to have two distinct types of multiplication; for the multiplication of density matrices the traditional

2×2 matrix multiplication is used. But for symmetry operations the density matrix is rewritten as a 4 element vector and symmetry operators are 4×4 matrices (with additional restrictions beyond being unitary).

Having two different multiplications defined for the density matrix can be confusing. So to give the reader a hint of which type of multiplication we are assuming we will use “SV” and “DM” for the traditional state vector type of multiplication and the new, density matrix only method of multiplication discussed in this section. Then the usual and Weinberg methods of symmetry operations on a density matrices ρ are:

$$\begin{aligned} U^{SV}(\rho) &= U^{SV} \rho U^{SV\dagger}, && \text{Usual} \\ U^{DM}(\rho) &= U^{DM} \rho, && \text{Weinberg.} \end{aligned} \tag{2}$$

The matrix multiplication in the “Usual” line of the above is 2×2 matrices while the “Weinberg” line consists of a 4×4 matrix U multiplying ρ considered as a 4-dimensional vector.

Weinberg illustrated his idea with a 3x3 Hermitian matrix:

$$\rho_{ab} = \begin{pmatrix} a_1 & b_3 & b_2^* \\ b_3^* & a_2 & b_1 \\ b_2 & b_1^* & a_3 \end{pmatrix} \tag{3}$$

where the a_k transform as $SU(3)^{SV}$ singlets subject to $a_1 + a_2 + a_3 = 1$ and the b_k transforms as an $SU(3)^{SV}$ triplet. This gives $3 \times \bar{3} = 3 + \bar{3} + 1 + 1 + 1$ rather than the expected result of $3 \times \bar{3} = 8 + 1$.

Weinberg’s example has state vectors packed into a density matrix. This paper will consider a slight modification; we will pack density matrices into density matrices. This seems more compatible with the assumption that density matrices are the fundamental formulation of quantum mechanics.

We will find it useful to have a closed form result for the pure density matrix corresponding to spin-1/2 in an arbitrary direction $\vec{u} = (u_x, u_y, u_z)$ a real 3-vector of length 1. A derivation with $SU(2)$ state vectors might use Euler angles to rotate the spin-up state vector. This is a calculation which is much longer than the density matrix calculation, see Appendix C.1, pp 167-172, of [3]. We will now give the $SU(2)^{DM}$ density matrix calculation.

The spin operator for spin in the \vec{u} direction is given by the dot product $\vec{u} \cdot \vec{\sigma}$:

$$\sigma_u = \vec{u} \cdot \vec{\sigma} = \begin{pmatrix} +u_z & u_x - iu_y \\ u_x + iu_y & -u_z \end{pmatrix}. \tag{4}$$

The reader can verify that, like the Pauli spin matrices, σ_u squares to unity and has trace zero. We seek an eigenvector $|u\rangle$ with $\sigma_u |u\rangle = |u\rangle$. We can then make the pure density matrix ρ_u by

$$\rho_u = |u\rangle\langle u|. \tag{5}$$

We see that ρ_u will be a double-sided eigenvector of σ_u with eigenvalue 1:

$$\sigma_u \rho_u = \rho_u \sigma_u = \rho_u \tag{6}$$

But since

$$\sigma_u (1 + \sigma_u) = \sigma_u + \sigma_u^2 = \sigma_u + 1 = 1 + \sigma_u, \tag{7}$$

we have that $(1 + \sigma_u)$ is a double sided eigenvector of σ_u and so ρ_u is proportional to $(1 + \sigma_u)$. The proportionality constant is defined by requiring that the trace be unity giving:

$$\rho_u = 0.5(1 + \sigma_u) = \frac{1}{2} \begin{pmatrix} 1 + u_z & u_x - iu_y \\ u_x + iu_y & 1 - u_z \end{pmatrix}. \tag{8}$$

To convert ρ_u to a spinor, take any non-zero column and normalize.

We could also derive ρ_u from the SO(3) symmetry of spin-1/2 density matrices. First, parameterize the Hermitian 2x2 matrices using the Pauli spin matrices and the unit matrix.² So an arbitrary Hermitian 2x2 matrix can be written using four real parameters α_χ :

$$\begin{aligned} H(\alpha) &= \alpha_1 1 + \alpha_x \sigma_x + \alpha_y \sigma_y + \alpha_z \sigma_z, \\ &= \begin{pmatrix} \alpha_1 + \alpha_z & \alpha_x - i\alpha_y \\ \alpha_x + i\alpha_y & \alpha_1 - \alpha_z \end{pmatrix}. \end{aligned} \tag{9}$$

Since in SO(3) vectors transform as vectors, we deduce that $(\alpha_x, \alpha_y, \alpha_z)$ is proportional to \vec{u} and the requirements of idempotency and trace =1 gives the result up to the sign of $\pm\vec{u}$.

We've shown that the spin-1/2 density matrix transforms under the SO(3) fundamental representation. This is the same representation as spin-1 SU(2). Thus Weinberg's method could be described as having kept the symmetry the same as with state vectors (i.e. SU(2)) but the representation has changed from the SU(2) fundamental representation of the usual spin-1/2 state vectors to the SU(2) adjoint representation. This is a general property of Weinberg's method; a state vector that transforms under the SU(n) fundamental representation corresponds to a density matrix that transforms under the SU(n) adjoint representation. Thus the up quarks, which transform under color SU(3) as a triplet in state vector form, correspond to density matrices that transform under the SU(3) adjoint representation.

That Weinberg's method can be thought of as changing the representation (from fundamental to adjoint) but leaving the symmetry alone suggests we could have titled this paper as "Electron Spin Transforms as SU(2) Adjoint Representation." We've not done so for two good reasons. The first is that Weinberg's idea is that density matrices are to be considered without respect to state vectors. The second is that we will show in Section 5 that color SU(3) can be derived from SO(3) combined with a discrete symmetry implied by the gamma matrices. A further reason is that the paper title would be slightly less shocking.

As an exercise for the reader, consider a general Hermitian 3x3 matrix. Just as we parameterized the 2x2 Hermitian matrices with the three Pauli spin matrices we can parameterize the 3x3 Hermitian matrices using the eight Gell-Mann matrices λ_k . Then any 3x3 pure density matrix ρ_a can be written with eight real numbers a_k :

$$\rho_a = \frac{1}{3}(1 + \sum_{k=1}^8 a_k \lambda_k). \tag{10}$$

²Physicists are likely to find this confusing because the Pauli spin matrices are the infinitesimal generators for SU(2) rotations. We are not using them in any sort of Lie algebra way. Instead, we are using them as three elements of a basis set for a real vector space. The vector space consists of the Hermitian 2x2 matrices. Any 2x2 Hermitian matrix can be written using a basis consisting of the Pauli spin matrices and the unit matrix. No infinitesimal rotations involved.

Find a quadratic equation that the a_k must satisfy. The quadratic equation for the 2x2 case was $u_x^2 + u_y^2 + u_z^2 = 1$ and any solution to this equation defined a pure 2x2 density matrix. In the 3x3 case additional restrictions are needed.

3. Density matrices and QFT

To apply Weinberg's density matrix symmetry methods to the symmetry of the Standard Model fermions we need a method of relating density matrix symmetry to QFT symmetry. We will use generalized Stern-Gerlach experiments. These experiments do not change the number of particles so they can be modeled with density matrices and the matrices can be finite dimensional which is compatible with our concerns with spin and internal symmetries.

A method of relating density matrices to QFT is implied by the "measurement algebra" that Julian Schwinger developed in the 1950s and published in a series of four papers: [4, 5, 6, 7]. In creating the measurement algebra, Schwinger's objective was to create a foundation for QFT but he did not realize that goal. He did use the measurement algebra as an introduction to quantum mechanics when teaching and his class notes were compiled into two textbooks [8, 9] that were used by some of his students when they eventually taught introductory quantum mechanics. These texts develop the usual subjects of introductory quantum mechanics with the measurement algebra as the foundation and the reader interested in how quantum mechanics can be developed from the measurement algebra are referred to them. These textbooks quickly move from the measurement algebra to state vectors. Also see this author's paper [10] on the application of the measurement algebra to classical and quantum binary measurements.

Hints of how Schwinger saw his measurement algebra as a foundation for QFT are seen in his definition of a vacuum state (called the "null state") and creation and annihilation operators in the first pages of [5]. The annihilation operator fits into the measurement algebra the way that a bra is part of a density matrix while the creation operator is the ket. This makes a density matrix where the created particle of type a replaces an annihilated particle of type b :

$$\rho_{ab} = |a\rangle\langle b| \quad (11)$$

In the above, $\langle b|$, called $M(0, b') = \Phi(b')$ in Schwinger's paper, is the annihilator and $|a\rangle$ which he labels $M(a', 0) = \Psi(a')$ is the creator. The annihilation operator eliminates a particle in the state b leaving what Schwinger calls the "nonphysical state", the null state 0 or vacuum. The creation operator adds the state a to the vacuum. The overall effect of ρ_{ab} , which Schwinger calls $M(a', b') = M(a', 0) M(0, b')$, is to annihilate a particle in state b and replace it with a particle in state a . These hints of how the measurement algebra could work as a foundation for QFT are from Schwinger's PNAS journal articles and do not appear in the later measurement algebra textbooks, perhaps indicating that Schwinger abandoned it. In relating density matrices to QFT, it's possible we will illuminate some of the problems that Schwinger faced.

Since our interest is in density matrices, we will not split Schwinger's measurements into bras and kets, but instead will use only measurements

of the “first type” i.e. $|a\rangle\langle a|$ which he labels $M(a')$ in [4]. Our approach avoids the introduction of arbitrary complex phases.

Schwinger’s measurements of the “second type” $M(a', b')$ convert states from a state b to a state a . While we will not be using these type measurements, they are worth discussing as they are important in his derivation of quantum mechanics and they illustrate the meaning of the quantum vacuum from Schwinger’s point of view or density matrices. Note that we will be concerned with extensions of the Standard Model where all the fermions are related by a continuous symmetry. For example, the left-handed electron and left-handed neutrinos of the Standard Model transform as a doublet according to weak $SU(2)$ but appear as two separate particles at the low energy limit. We assume that at sufficiently high energies all the Standard Model fermions (of the same handedness) are related by continuous symmetries. For these situations, it is always possible to define a measurement of the second type as a real multiple of products of measurements of the first type. If $M(a') = \rho_a$ and $M(b') = \rho_b$ do not annihilate, then $M(a', b')$ can be defined as a real multiple of the product $M(a') M(b') = \rho_a \rho_b$. In terms of density matrices made from bras and kets, this is:

$$M(a', b') \equiv \rho_a \rho_b / \sqrt{\text{tr}(\rho_a \rho_b)} = \frac{|a\rangle\langle a| |b\rangle\langle b|}{\sqrt{\langle b|a\rangle\langle a|b\rangle}}, \quad (12)$$

which depends on $\langle b|a\rangle \neq 0$. To make this sort of definition work when the b and a state annihilate, consider $b = +z$ as spin-up and $a = -z$ as spin-down so that $M(+z, -z)$ is the raising operator for spin-1/2. Since these states annihilate, that is, $M(+z) M(-z) = 0$, we cannot use the product. For these cases we insert an intermediate state(s) so that the products do not annihilate. For example, we can use spin in the $+y$ direction to define

$$M(+z, -z) \propto \frac{1 + \sigma_z}{2} \frac{1 + \sigma_y}{2} \frac{1 - \sigma_z}{2} = \begin{pmatrix} 0 & -i/2 \\ 0 & 0 \end{pmatrix}. \quad (13)$$

The complex phase $-i$ is a matter of choice of the intermediate state. The usual value 1 is obtained by using $+x$.

Our interest is in the Standard Model symmetry $SU(3) \times SU(2) \times U(1)$, the symmetry of the forces experienced by the elementary fermions. In QFT, forces are accomplished by the emission and absorption of bosons. Density matrices preserve the number of particles and so must accomplish forces another way, typically by potentials. In the usual Stern-Gerlach experiment for spin, the force is due to a magnetic field so the corresponding QFT bosons are photons, but the potential is not explicitly defined. This makes the Stern-Gerlach experiment more compatible with quantum field theory than situations that explicitly use potentials such as the energy levels of the hydrogen atom. In addition, while QFT is used for corrections to the energy levels of hydrogen it is quite difficult to use it to find the energy levels themselves. The Stern-Gerlach experiment can be approached with both QFT and density matrices.

Given a particle type defined by a pure density matrix ρ , there is a generalized Stern Gerlach experiment that selects for that particle. Such an experiment keeps the particles of type defined by ρ and throws all the others away (into the other branch of the beam splitter, and then to a beam dump or other safe beam absorption). We can naturally describe such a Stern-Gerlach experiment by the type of particle it selects for, i.e. ρ . Note

that ρ is an Hermitian idempotent. That is $\rho^\dagger = \rho$ and $\rho^2 = \rho$. If we have two such experiments, say ρ_1 and ρ_2 , we can connect them together so that the particle first enters ρ_1 , and if it survives continues into ρ_2 . This is a compound Stern-Gerlach experiment, it is represented by the product of the two Hermitian idempotents: $\rho_{12} = \rho_2 \rho_1$. This is a matter of matrix multiplication of the usual type used in density matrices (rather than the new method for density matrix symmetry). We will associate the propagators of QFT with the pure density matrices. Propagators can handle arbitrary spin directions while density matrices restrict to a single case but our concern will be in the symmetry of the vector bosons so we will not discuss this in detail. To understand how density matrices work as models of Stern-Gerlach machines we will consider products of pure density matrices.

Note that the matrix ρ_{12} representing the compound measurement ρ_1 followed by ρ_2 is not, in general, Hermitian, idempotent or with trace 1. We can illustrate this fact by looking at the spin-1/2 case and letting $\rho_1 = \rho_{+z}$ and $\rho_2 = \rho_{+x}$. Then we have

$$\begin{aligned} \rho_{+x}\rho_{+z} &= | + x \rangle \langle + x | | + z \rangle \langle + z |, \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}. \end{aligned} \tag{14}$$

The trace isn't unity because half the particles have been lost between the two measurements. The lack of Hermiticity corresponds to the fact that the measurements do not commute. That is, a compound measurement of spin in the $+x$ direction followed by $+z$ is not the same as the compound measurement of spin in the $+z$ direction followed by the $+x$ direction. Finally, half the particles are lost in the transition from $+z$ to $+x$. This breaks idempotency but only by a factor of two. That is, $2\rho_{+x}\rho_{+z}$ is idempotent:

$$(2\rho_{+x}\rho_{+z})(2\rho_{+x}\rho_{+z}) = (2\rho_{+x}\rho_{+z}), \tag{15}$$

with trace 1 but it is not Hermitian.

In general, a product of two pure density matrices does not have trace 1 and is not idempotent but so long as the product is not zero, we can repair these through multiplication with a real number. Hermiticity can also be repaired; to do this we need to multiply by another pure density matrix so that the product begins and ends with the same pure density matrix. In this case, so long as the product is not zero, we can multiply by a complex number to make the product a pure density matrix, i.e. Hermitian, with trace 1 and idempotent. As an example with spin-1/2, we have

$$\rho_{+z}\rho_{+y}\rho_{+x}\rho_{+z} = \frac{1-i}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{16}$$

so $2(1+i)$ multiplies $\rho_{+z}\rho_{+y}\rho_{+x}\rho_{+z}$ to give the pure density matrix ρ_{+z} . That such a complex number can be found for non zero products of pure density matrices that begin and end with the same matrix is generally true in all finite dimensions. For the above, the phase of $(1-i)/\sqrt{2}$ is $-\pi/4$. This is for an octant. If the path encloses half the sphere the phase will be four times as much so that there will be a factor of -1 as is appropriate

for swapping two spin-1/2 fermion state vectors. Spin-1 bosons pick up twice the phase so they will have $-\pi/2$ for the octant and -2π for half the sphere; swapping two boson state vectors does not change the sign.

A force boson QFT interaction, for example, where an electron emits a photon, is a term in a perturbation calculation where the most basic term is electron propagation without boson emission. A Stern-Gerlach apparatus combines these two terms. In addition, our density matrix model of Stern-Gerlach machines includes a beam dump that is not modeled in QFT. To bring these into alignment, we will consider Stern-Gerlach apparatus with very small angles; with these, the electron has a vanishingly small probability of failing to make the transition.

To obtain Stern-Gerlach experiments with no particle loss, consider an experiment that changes the spin direction by an angle θ between two directions \vec{u} and \vec{v} with $\vec{u} \cdot \vec{v} = \cos(\theta)$. The probability of making the transition is $\text{tr}(\rho_u \rho_v)$ so the probability of failing to make the transition for small angles is

$$1 - \text{tr}(\rho_u \rho_v) = 1 - (1 + \cos(\theta))/2 \approx \theta^2/4. \quad (17)$$

Replacing the single Stern-Gerlach apparatus of angle θ with N of them each with angle θ/N we see that the loss decreases from $\theta^2/4$ to $N(\theta/N)^2/4 = \theta^2/4N$ so we can eliminate the particle loss by letting N increase arbitrarily. As an exercise for the student, show that the procedure of replacing θ with θ/N does not change the quantum phase.

The pure density matrix ρ_u has several interpretations. Mathematically, it is a projection operator with trace 1. It's symmetry is SO(3) so the \vec{u} specifies a direction in space. That direction is the spin orientation of the particle and in a Stern-Gerlach apparatus it is the direction of the inhomogeneity of the magnetic field. Thus it simultaneously describes the symmetry of the selected particle spin and also the symmetry of the device that selects it. Thus the product $\rho_v \rho_u$ has the symmetry of a device that alters the spin direction from \vec{u} to \vec{v} . That is, in some way this product is a description of the vector boson that causes the spin direction to change from \vec{u} to \vec{v} . But it also includes various other things; to extract just the symmetry of the vector boson we can consider the reverse product $\rho_v \rho_u$. The reverse product corresponds to a vector boson with the opposite effect. We can think of these as describing the same vector boson if one emits while the other absorbs. So to obtain the symmetry of that vector boson we subtract:

$$\gamma_{uv} = (\rho_v \rho_u - \rho_u \rho_v)/2. \quad (18)$$

Writing $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and similarly with \vec{u} and \vec{v} so we can use the summation convention we use Eq. 8 to write the above in terms of Pauli matrices. All the terms cancel but the products of Pauli matrices leaving:

$$\begin{aligned} \gamma_{uv} &= (v_j \sigma_j u_k \sigma_k - u_k \sigma_k v_j \sigma_j)/2, \\ &= v_j u_k (\sigma_j \sigma_k - \sigma_k \sigma_j)/2, \\ &= v_j u_k [\sigma_j, \sigma_k]/2. \end{aligned} \quad (19)$$

Note that in the above, we have not been using the Pauli spin matrices as the infinitesimal generators of SU(2). Instead, we've been using them as part of a basis set for the Hermitian 2x2 matrices. Nevertheless, we can

reduce the commutator $[\sigma_j, \sigma_k]$ by using the structure constants of SU(2), i.e. $f^{jkl} = 2i\epsilon^{jkl}$:

$$[\sigma_j, \sigma_k] = f^{jkl} \sigma_l, \tag{20}$$

so that γ_{uv} is

$$\gamma_{uv} = v_j u_k f^{jkl} \sigma_l, \tag{21}$$

and thus γ_{uv} has the symmetry of the structure constants and we have that the vector bosons follow the adjoint or spin-1 representation of SU(2). This is the same as the spin-1 representation of SO(3). The calculation works similarly with SU(3) so that, under color rotations, gluons follow the adjoint representation of SU(3).

4. Charge, Temperature and Symmetry Breaking

Our interest here is in the symmetry of single particles rather than particle interactions so we will follow the lecture notes of Michael Cross’s Physics 127c class “Statistical Mechanics” at Caltech. The evolution operator for a single particle moving under a Hamiltonian H is

$$U(x, x'; t) = \langle x | \exp(-iHt/\hbar) | x' \rangle. \tag{22}$$

Ignoring normalization, so we have only a proportionality instead of an equality, the formula for an unnormalized density matrix is similar:

$$\rho(x, x', \beta) \propto \langle x | \exp(-\beta H) | x' \rangle, \tag{23}$$

where $\beta = 1/(k_B T)$ is the thermodynamic beta function that is proportional to inverse temperature and has units of inverse energy. The density matrix can be obtained by rotating the time t of the evolution operator to imaginary time / temperature by a rotation in the complex time plane:

$$t \rightarrow -i\beta\hbar. \tag{24}$$

This is known as a “Wick rotation”.

We can use the fact that $\int |x\rangle\langle x| = 1$ to multiply evolution operators:

$$U(x, x'; t_1+t_2) = \int dx'' \langle x | \exp(-iHt_1/\hbar) | x'' \rangle \langle x'' | \exp(-iHt_2/\hbar) | x' \rangle. \tag{25}$$

Repeating this process, and using infinitesimal time steps, gives the Feynman path integral. The same can be done with density matrices. Our interest in density matrices does not include spatial dependence giving a familiar statistical mechanical equation:

$$\rho(\beta) \propto \exp(-\beta H). \tag{26}$$

The integral over dx'' in the evolution operators will become matrix multiplication between two density matrices giving an equation for multiplying density matrices:

$$\rho(\beta_1 + \beta_2) = \rho(\beta_1) \rho(\beta_2) \tag{27}$$

Let us illustrate this with a concrete example in 2x2 matrices. Let β_0 be small compared to the Hamiltonian energy which we leave implied but leave its determination as an exercise for the reader:

$$\rho_0(\beta_0) = (1 + 0.400\sigma_y)/2. \tag{28}$$

In the above, the 0.400 gives the amount that ρ_0 differs from the high temperature limit which is the unit matrix divided by two. Using the density matrix multiplication equation to compute $\rho_0(2\beta_0)$ we find:

$$\begin{aligned}\rho_0(2\beta_0) &\propto (\rho_0(\beta_0))^2, \\ &\propto (1 + 2(0.400)\sigma_y + (0.4)^2)/4, \\ &= (1 + 0.690\sigma_y)/2\end{aligned}\quad (29)$$

so that reducing the temperature by half has the effect of increasing the 0.400 factor to approximately 0.690. If we continue the process, in the limit of infinite β (or zero temperature), this factor will approach 1 and $\rho_0(\infty) = (1 + \sigma_y)/2$. Replacing the 0.400 with a real number f between 0 and 1, and generalizing from σ_y to σ_u we find

$$\begin{aligned}\rho(f) &= (1 + f\sigma_u)/2, \text{ and squaring leads to} \\ \rho(2f) &= (1 + (2f)\sigma/(1 + f^2))/2.\end{aligned}\quad (30)$$

Seeing $f \rightarrow 2f/(1 + f^2)$ reminds us of the formula for $\tanh(2\theta)$ so we derive a general equation for any 2x2 density matrix that depends on temperature T :

$$\rho_u(T_0/T) = (1 + \tanh(T_0/T)\sigma_u)/2 \quad (31)$$

where T_0 is a positive real constant giving the temperature scale. This defines the density matrices $\rho(T)$ as a path in the 2x2 complex matrices.

In the Standard Model, the left handed electron and neutrino form an SU(2) doublet. This is a natural system to explore symmetry breaking and charge in density matrices. Putting the neutrino in the top position and the electron in the bottom position of an SU(2) doublet ket we have the density matrices for these particles as

$$\rho_{\nu L} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_{eL} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (32)$$

Their weak isospin I_3 values are $\pm 1/2$:

$$\begin{array}{c|c} & I_3 \\ \hline \nu_L & +1/2 \\ e_L & -1/2 \end{array}, \quad (33)$$

so the weak isospin charge operator is

$$I_3 = 0.5 \sigma_z = \begin{pmatrix} +1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \quad (34)$$

We will use the signs of the weak isospin charges to distinguish the two particles. Then we can write their temperature dependant density matrices as:

$$\rho_{\pm}(T_0/T) = (1 \pm \tanh(T_0/T)\sigma_z)/2 \quad (35)$$

Note that these density matrices have no σ_x or σ_y portion; they are made out of the unit matrix and σ_z only. Consequently, the Hamiltonian that is implied by them is a multiple of σ_z . And the weak isospin operator I_3 is also a multiple of σ_z and so commutes with the Hamiltonian. This is a general property of charge operators, they commute with any Hamiltonian. In addition, the non zero elements of the above 2x2 matrix consist of two

1x1 blocks on the diagonal. The block diagonal matrices are important in the Standard Model as we discuss below.

Since the density matrices in Equation 35 do not have any σ_x or σ_y component, they describe statistical mixtures of electrons and neutrinos only. That is, no linear superpositions are included. If we take a random particle from such an ensemble, it will either be a pure neutrino or a pure electron and cannot be a superposition of these states. This is compatible with the idea of “superselection sectors” which limits quantum states to those that commute with the charge operators. The charge operators divide the Hilbert space into superselection sectors which are each characterized by a specific collection of charges. Superselection sectors apply to the cold temperature limit of quantum states; the subject was explored in the years before symmetry breaking became popular.

For a theory that describes full SU(2) symmetry breaking the absence of superpositions between the electron and neutrino is a bit of a problem. It arises from our using a constant Hamiltonian. If we allowed an initial high temperature density matrix that included σ_x or σ_y , it would cool to a state that would be a superposition between electron and neutrino and so would have mixed charge. To fix this, we need to have the Hamiltonian depend on temperature. Well above the mass of the W^\pm the Hamiltonian can include σ_x and σ_y but at temperatures well below the σ_x and σ_y components are forced to zero. We can use a function $\Theta(T)$ which is zero when T is well below the W^\pm mass and one when T is well above and define $H(T)$ as

$$H(T) = \Theta(T)(H_x \sigma_x + H_y \sigma_y) + H_z \sigma_z \quad (36)$$

which at temperatures well above the W^\pm mass will on cooling, rapidly converge to $(1 + (H_x \sigma_x + H_y \sigma_y + H_z \sigma_z) / \sqrt{H_x^2 + H_y^2 + H_z^2}) / 2$ but on further cooling to well below the W^\pm mass will converge to $(1 + H_z \sigma_z / \sqrt{H_z^2}) / 2$. It should be clear that so long as the function $\Theta(T)$ is zero near the low temperature limit and one near the high temperature limit it's choice does not effect the symmetry of the low temperature states.

This paper's objective is an explanation for the symmetry of the low temperature states so we will use a $\Theta(T)$ that is zero except at temperatures above our range. In other words, we will require our Hamiltonians to commute with the charge operators and so our low temperature limits will avoid superpositions between particles with different charges. In this context, it's important to note that corresponding particles from different generations share the same charges and so it is possible, for example, to form superpositions between an electron and a muon or tau. These superpositions of quarks and leptons are seen in the CKM and MNS matrices.

The quantum states that form the low temperature limit of a set of Hamiltonians (that commute with the charge operators) are contained in a “subalgebra” of the matrix algebra. When we use the word “algebra” in this paper we mean it in the mathematical sense of an “algebra over the field of complex numbers”. This is defined as a vector space over the complex numbers, that also has a bilinear product. The NxN complex matrices are an example of an algebra; the bilinear product is matrix multiplication. The vector space addition is matrix addition and the vector space multiplication by a complex constant is given by multiplying the matrix by a complex constant.

For the neutrino and electron example using the 2x2 matrices, the subalgebra that contains the low temperature limit states are the 2x2 matrices with zeros in their off diagonal positions. Such matrices are closed under matrix multiplication and vector (matrix) addition, and are closed under multiplication by complex constants and so form a subalgebra of the matrix algebra. The low temperature limit density matrix states are exactly those elements ρ of this subalgebra that are Hermitian, have $\rho^2 = \rho$ and have trace = 1. The mathematicians call these the Hermitian “primitive idempotents”. They are the projection operators (idempotents) that cannot be written as the sum of non zero projection operators (are primitive).

As far as defining the low temperature limit (pure) density matrices, Hermitian “primitive idempotents” works for defining pure density matrices but we need to give a physical explanation for why we are considering subalgebras, as most of the elements of a subalgebra are not pure density matrices.

In Section 3.0, we briefly introduced Schwinger’s Measurement Algebra. In considering the paths $\rho(T)$ of thermal density matrices over temperature, the space where they live are these algebras described by Schwinger (expanded to include temperature). If we have two such measurements, we can add them by combining the beams, which is then described by the matrix sum of the two measurements. This assumes that we do not normalize the sum so that the trace is 1, but instead follow Schwinger and let the trace indicate the intensity. The matrix product corresponds to taking the output beam from one measurement and using it as the input to the next measurement. Thus matrix addition and multiplication correspond to physical experiments. As we saw in Section 3.0, the products of Hermitian matrices are not, in general, Hermitian but can introduce complex phases.

Introducing a phase of -1 makes subtraction a physical operation. Then we can define a basis for the complex NxN matrices entirely using pure density matrices. For example, consider:

$$\begin{aligned} (\rho_{+x} - \rho_{-x})\rho_{+z} &= ((1 + \sigma_x)/2 - (1 - \sigma_x)/2)(1 + \sigma_z)/2, \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (37)$$

The above construct generalizes to any basis element of a complex NxN matrix and therefore the entire matrix algebra. Thus if a subalgebra of a matrix algebra can be written in block diagonal form, the individual blocks can be physically realized as complicated Stern-Gerlach experiments and we have a reason for assuming that the cold temperature limit of thermal density matrices can be found in block diagonal matrix subalgebras. But it is precisely the block diagonal matrix subalgebras that are of interest in the Standard Model as we now show.

In terms of density matrices, charge operators force the Hamiltonian into block diagonal form. The full set of first generation left handed fermions adds the up and down quarks to the electron and neutrino. The up and down quarks have an internal SU(3) triplet color symmetry and so in our model (which at this point ignores SU(2) spin-1/2) are represented by 3x3 blocks. The up and down quarks have the same $\pm 1/2$ weak isospin quantum numbers of the electron and neutrino so we need another quantum number to distinguish them; we will use weak hypercharge where the left handed leptons take -1 while the left handed quarks use $+1/3$. Then

elements and the vector space is 6-dimensional. An example of vector addition in the vector space is:

$$\begin{aligned} [1.3(\) + 2.1i(23)] + [-1.8(123) + (3.0 + 2.0i)(23)] \\ = [1.3(\) - 1.8(123) + (3.0 + 4.1i)(23)] \end{aligned} \quad (41)$$

In this vector space, multiplication by a complex constant simply means multiplying the coefficients as in any vector space.

To raise this vector space to an algebra, we need to define how to multiply two elements. This is done by multiplying terms as one multiplies polynomials, and then using the finite group multiplication rule to reduce products such as $(12)(13) = (132)$. For example, $0.5(\) + 0.5(12)$ is idempotent:

$$\begin{aligned} [0.5(\) + 0.5(12)] [0.5(\) + 0.5(12)], \\ = 0.25(\)(\) + 0.25(\)(12) + 0.25(12)(\) + 0.25(12)(12), \\ = 0.25(\) + 0.25(12) + 0.25(12) + 0.25(\), \\ = 0.5(\) + 0.5(12), \end{aligned} \quad (42)$$

but it is not primitive.

As Hammermesh proves, and demonstrates with the example of $\mathbf{C}[S_3]$, a complex group algebra can be put into block diagonal form. An $n \times n$ block has n^2 complex degrees of freedom so the six complex degrees of freedom in $\mathbf{C}[S_3]$ can be written as a sum of squares. There are two possibilities:

$$\begin{aligned} 6 &= 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2, \text{ or} \\ 6 &= 2^2 + 1^2 + 1^2. \end{aligned} \quad (43)$$

But S_3 is not Abelian so $\mathbf{C}[S_3]$ cannot be diagonal. This eliminates the first possibility and we are left with $6 = 2^2 + 1^2$. The block diagonal form will have a 2×2 block and two 1×1 blocks and the Hermitian elements of $\mathbf{C}[S_3]$ can be put into this form:

$$\begin{pmatrix} a & & & \\ & b & & \\ & & c + c_z & c_x - ic_y \\ & & c_x + ic_y & c - c_z \end{pmatrix} \quad (44)$$

where a, b, c, c_x, c_y, c_z are six real numbers. The three real numbers a, b, c define multiples of the unit matrix in the three diagonal blocks. The elements of the algebra that commute with everything in the algebra (mathematicians call the center, physicists call the charges) are written with only the a, b, c non zero. They are related to the character table of the finite group.

With state vectors, the character table for a finite group defines the possible representations of the symmetry of that group. Contemporary readers of this paper should be very familiar with the process. Character tables are also of use in describing the blocks of the block diagonal form of a complex group algebra. This is an application of character tables to mixed density matrices and of course it is completely different from the application of character tables to state vector symmetries. Accordingly, the reader is advised to forget their understanding of state vector uses of character tables when reading these next few paragraphs. Instead, they

should try to read them as if they had never previously considered the symmetry of mixed density matrices defined by a group algebra.

Hammermesh shows how we can read the block diagonal form of a complex group algebra from the character table of the group. We will illustrate this with S_3 ; if our demonstration is confusing, try Hammermesh for the same story but with more emphasis on mathematical rigor. Each line in the character table corresponds to a $n \times n$ diagonal block. For S_3 we have $6 = 1^2 + 1^2 + 2^2$ so there are three lines in the character table. Labeling the lines with the size of the block they define we have the character table for S_3 :

$$\begin{array}{c|ccc}
 \chi(g) & () & (123) + (132) & (12) + (13) + (23) \\
 \hline
 a, 1 \times 1 & 1 & 1 & 1 \\
 b, 1 \times 1 & 1 & 1 & -1 \\
 c, 2 \times 2 & 2 & -1 & 0
 \end{array} \tag{45}$$

The three rows are labeled with a, b, c according as they correspond to the three diagonal blocks of Equation 44.

The six group elements fall into three classes, the identity, the 3-cycles and the swaps. If we sum over one of these classes, we get a charge and these charges are used as the heads of the columns of the character table. For example, the three swaps $(12), (13), (23)$ form an equivalence class so their sum $(12) + (13) + (23)$ is a charge of the algebra. That is, any element of the algebra commutes with $(12) + (13) + (23)$.

The three diagonal blocks of Equation 44 have unit matrices given by the a, b, c parameters. There the parameters are defined in terms of where they appear in the matrix. The lines of the character table show how to define these terms using the finite group basis. For example, b is the coefficient for the second 1×1 diagonal block of Equation 44. That block is defined in the second row of the character table with charge coefficients $1, 1, -1$. This means that when we put $\mathbf{C}[S_3]$ into block diagonal form, the second diagonal or b block will be defined by

$$\begin{aligned}
 & b[(\) + (123) + (132) - (12) - (13) - (23)]/6 \\
 & = \begin{pmatrix} 0 & & & \\ & b & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}.
 \end{aligned} \tag{46}$$

In the above, the character table coefficients have been divided by 6, the size of the group. The general rule is that we multiply by the block size and divide by the group size. The block size for block “ c ” is 2×2 so that unit matrix takes a $2/6$ factor and is given by $[2(\) - (123) - (132)]/6$. The unit matrices for the three blocks add up to $(\)$, the algebra’s identity.

5. Gamma Matrices and the Standard Model

In the previous section we’ve seen that the states available to a generalized Stern-Gerlach experiment and combinations of those experiments define an algebra and that algebras are in block diagonal form. Each block defines a particle with generally distinct charges and the $n \times n$ blocks are particles with internal symmetry such as the quarks which have color $SU(3)$. And the

block diagonal structure can be read off of the character table of the finite group. With this information, we could look through the finite groups and find those that are compatible with the observed structure of the Standard Model fermions. Rather than do that, we will instead expand Weinberg's idea on density matrices to the gamma matrices and deduce the finite group that defines the group algebra.

The difference between density matrices and state vectors is that, essentially, a state vector is the square root of a density matrix. State vectors are easier to use because they are linear. So we can generalize Weinberg's observation to one where Nature is more naturally described by bilinear equations but our mathematics works best on linear problems so part of our difficulty in understanding Nature is that we try too hard to describe her bilinear simplicity with complicated linear mathematics.

This concept of Nature being simpler with bilinear equations has an echo in the Dirac equation. The Dirac operator is rather complicated. For the Dirac basis:

$$\gamma^\mu \partial_\mu = \begin{pmatrix} \partial_t & 0 & \partial_z & \partial_x - i\partial_y \\ 0 & \partial_z & \partial_x + i\partial_y & -\partial_z \\ -\partial_z & -\partial_x + i\partial_y & -\partial_t & 0 \\ -\partial_x - i\partial_y & \partial_z & 0 & -\partial_t \end{pmatrix}. \quad (47)$$

When the above matrix is squared, it becomes much simpler:

$$(\gamma^\mu \partial_\mu)^2 = (\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (48)$$

In this paper we are working in the spinor formalism so our Dirac equation is massless:

$$(\gamma^\mu \partial_\mu) \vec{\psi}(x, y, z, t) = 0, \quad (49)$$

and therefore $\vec{\psi}$ is a solution to the square of the Dirac operator:

$$(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = 0. \quad (50)$$

This gives four independent wave equations:

$$(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2) \psi_j = 0 \quad (51)$$

for $j = 1, 2, 3, 4$.

The four independent wave equations given in Equation 51 are much simpler; why don't we use them instead of the Dirac equation? Of course the answer is that while the Dirac equation gives four solutions to the simple wave equation the reverse is not the case. If we have four independent solutions to the simple wave equation they cannot, in general be put together to create a solution to the Dirac equation. Apparently we need the gamma matrices to provide coupling between these simple wave equations.

Dirac's original reason for postulating gamma matrices was to obtain a relativistic wave equation. In our modern inclination to treat mass as

a coupling we have lost our virginity on the subject; why not explore the rest of the whore house and treat the gamma matrices also as a type of coupling?

Since we are using spinors instead of bispinors we will leave the coefficients of ∂_t alone and consider only the spatial gamma matrices. Under this restriction, the spatial gamma matrices are equivalent to the Pauli spin matrices, so we will deal with them instead. If these matrices are couplings, then they are not simply a mathematical representation of a symmetry and we need to treat them in a way appropriate to physical objects.

Our objective is to detect hidden symmetries in solutions to the Dirac wave equation. It's clear that we can make Lie group transformation on the Pauli spin matrices but our interest here is not in the infinitesimal transformations. That is, an infinitesimal transformation changes the couplings to a set of new couplings that are physically different. Instead, we're interested in transformations that lead to couplings that are equivalent.

There are three Pauli spin matrices, $\sigma_x, \sigma_y, \sigma_z$ and these multiply the three partial derivatives $\partial_x, \partial_y, \partial_z$ but in truth we do not know which goes where. The tradition is to combine them as

$$\nabla = \sigma_x \partial_x + \sigma_y \partial_y + \sigma_z \partial_z, \tag{52}$$

but it works just as well to cycle them and instead use:

$$\nabla = \sigma_y \partial_x + \sigma_z \partial_y + \sigma_x \partial_z. \tag{53}$$

These two choices give the same physics; they cannot be distinguished. The rules for quantum mechanics are clear. We must symmetrize over the various possible methods of using the Pauli spin matrices.

Cycling the Pauli spin matrices amounts to performing a 120 deg rotation about the (1, 1, 1) axis. Rotating them by 90 deg around the z-axis gives

$$\nabla = \sigma_y \partial_x - \sigma_x \partial_y + \sigma_z \partial_z. \tag{54}$$

This again gives physics that we cannot distinguish, here with a minus sign introduced. More general transformations are restricted by the requirement that $[\sigma_x, \sigma_y] = i\sigma_z$ which allows only the proper rotations.

Excluding the improper rotations, there are 6 places to send σ_x , i.e. $\pm x, \pm y, \pm z$. Fixing σ_x leaves four choices for σ_y and our choice for σ_y determines σ_z . Thus the finite group of sigma-matrix transformations has $6 \times 4 = 24$ members. It is a point group with Hermann-Mauguin notation 432. In crystallography it is called gyroidal and the character table is:

432	<i>E</i>	3	2_z	2_d	4_z	(55)
A_1	1	1	1	1	1	
A_2	1	1	1	-1	-1	
<i>E</i>	2	-1	2	0	0	
T_1	3	0	-1	-1	1	
T_2	3	0	-1	1	-1	

In terms of density matrices, there are 5 particles. There are two SU(3) triplets that we can assign to the up and down quarks, and two singlets for the leptons. The remaining particle is an SU(2) doublet and is unique in that it has zero charges in the last two columns. We can assign it as dark matter.

The point group 432 is a finite subgroup of $SO(3)$. Together with our theme that electron spin should follow $SO(3)$, this suggests that the gauge principle of the Standard Model should follow from expanding the 432 symmetry to $SO(3)$. That is, the large finite transformations that define the density matrices follow the 432 subset of $SO(3)$ while infinitesimal transformations near unity define the gauge transformations. To explore this, we will consider the internal $SU(3)$ color of the quarks.

To be definite, we'll look at the down quarks. A complete basis for them consists of red, green and blue. In terms of state vectors they will be:

$$d_R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d_G = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad d_B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (56)$$

State vectors include an arbitrary complex phase but we assume density matrix states so the phases do not matter.

Infinitesimal gauge transformations do not change the color much but we can consider a series of them that change the colors, for example from d_r to d_g . Such a series of transformations will be a product of $SU(3)$ Lie group elements and so will be in the $SU(3)$ Lie group. The transformations that permute colors form a group. For instance if one transformation swaps d_R and d_G and we follow it with a transformation that swaps d_g and d_b the product transformation will cycle the three states. Therefore the $SU(3)$ fundamental Lie group elements that permute colors form a subgroup of the Lie group. By symmetry, we will require the subgroup to treat the colors equivalently (which will force these groups to be finite).

To be in the color permutation finite subgroup a 3×3 complex matrix needs to have 6 zeros and 3 complex phases, with the phases distributed so that there is one in each row and one in each column. To be in the $SU(3)$ Lie group it must also have determinant 1. So for a permutation that swaps green and blue two arbitrary complex phases determine the third, and such a matrix is of the form:

$$(GB)_{\alpha,\beta} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\alpha^* \beta^* & 0 \end{pmatrix} \quad (57)$$

Squaring this matrix gives a matrix that is the identity as far as permutations go:

$$(GB)_{\alpha,\beta}^2 = ()_{\alpha} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & -\alpha^* & 0 \\ 0 & 0 & -\alpha^* \end{pmatrix} \quad (58)$$

In addition to the above, we also have the usual identity = the unit matrix, and treating colors equally, there are two more cases of the above:

$$\begin{pmatrix} -\alpha^* & 0 & 0 \\ 0 & -\alpha^* & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}, \quad \begin{pmatrix} -\alpha^* & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & -\alpha^* \end{pmatrix}. \quad (59)$$

This gives a total of four copies of the identity. Requiring that these four representatives of the identity be closed under multiplication implies that

$\alpha = 1$ and the four representations of the permutation identity are:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{60}$$

It remains to determine β .

The other permutations can be multiplied by these four, so they will also have multiplicity 4. Since there are 6 permutations, the total size of the finite group of SU(3) color permutations will be 24. It should not be a surprise that the group will be the same as the 432 we derived earlier in this section. Thus considering the effect of finite gauge transformations on the color of quarks reveals the same hidden symmetry group in the Standard Model.

Putting $\alpha = 1$ in Equation 57 and multiplying by the four identities gives the four versions of the green blue swap:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta^* & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & -\beta^* & 0 \end{pmatrix}, \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta^* & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & \beta^* & 0 \end{pmatrix}. \tag{61}$$

Requiring that the colors be treated equally gives the four red green swaps and four red blue swaps. Multiplying these together eventually leads to a requirement that $\beta^3 = 1$ so that there are three solutions $\beta_g = \exp(2ig\pi/3)$ for $g = 0, 1, 2$. If we take $\beta_0 = 1$ our permutations are precisely the proper SO(3) rotation matrices that take axes to axes and so the group is the 432 seen earlier in this section.

The other two solutions to $\beta^3 = 1$ give two more copies of the 432 group. We associate these with the other two generations. This raises the question, “why didn’t we get three generations in the derivation of 432 in the first half of this section?” Of course the reason is that this is a density matrix paper and we used the state vector version of the Pauli spin matrices. Moving from SU(2) to SO(3) we have three “Pauli density spin” matrices and they do come in three generations:

$$\begin{pmatrix} 0 & -i\beta_g^* & 0 \\ i\beta_g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i\beta_g \\ 0 & 0 & 0 \\ -i\beta_g^* & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\beta_g^* \\ 0 & i\beta_g & 0 \end{pmatrix}, \tag{62}$$

6. Conclusion

We’ve explored density matrix methods for understanding the symmetry of the Standard Model fermions. We’ve shown the existence of a hidden point group symmetry using two methods; the assumption that the gamma matrices are couplings and the extension of the color gauge symmetry to transformations that leave the colors unchanged. Either way we obtain the

same symmetry, the point group 432, and when interpreted as a complex algebra, it contains room for the leptons and quarks in three generations.

What we are missing is a description of the mass interaction. We need this to split the particles into left and right handed particles that interact by mass. The generations are defined by mass matrices so an understanding of mass should lead to derivations of the CKM and MNS matrices. The author and collaborators [12] have been working on this. A method of putting electron spin into SO(4) while keeping mass interactions is given by de Vries.[13]

The method we've used here has been to explore Steven Weinberg's observation that making density matrices fundamental leads to fewer cases of bizarre behavior in quantum mechanics.[1] Our use of density matrices has been according to Julian Schwinger's measurement algebra. The measurement algebra suggests an explanation for a few more oddities of quantum mechanics. Since it is based on Stern-Gerlach experiments, using it as the foundation of quantum mechanics implies that our understanding of measurements should follow our understanding of those experiments

And Stern-Gerlach experiments could be the most thoroughly understood, from an intuitive point of view of all quantum experiments. The description of a particular experiment depends on whether the experiment has already taken place or is taking place in the future of the observer. It is only the future experiments that are represented by waves; the past experiments have particle results. This adds the significance of the present in the interpretation of quantum mechanics; the measurement process is determined by the passage of time for the observer. This makes the passage of time in the Schroedinger equation simply a method of extrapolating wave equations into the future; it cannot be used to model the passage of time we experience as humans. The result is that the "block time" assumption of quantum mechanics (and relativity) is invalid and need no longer conflict with the evidence of our own human observations of reality. Instead, the confusion of quantum mechanics arises from the mathematics one uses to define the time dependence of quantum mechanics from the simplicity of the Stern-Gerlach experiment.

It may not take an extreme amount of effort to add gravitation to the particle forces. We might take the Weinberg prescription to be "when one has several different formulations of physics, one should prefer the one that is the most boring, from the point of view of science fiction." Applying this to general relativity, it's evident that we can avoid worm holes and the like by using the formulations that have a flat background metric. Such formulations are also more compatible with astronomical observations.

This author's favorite flat metric general relativity formulation is the "Gauge Theory Gravity" described by Lasenby, Doran and Gull in 1998.[14] It is written in the "geometric algebra"[15] which essentially consists of the usual gamma matrices but carefully done without reference to any particular representation. The astute reader might note that this paper was written with this in mind; for example, we generally use $(1 + \sigma_x)/2$ as the projection operator for spin-1/2 in the $+x$ direction rather than writing it out in matrices that depend on the choice of the Pauli spin matrix representation.

With Gauge Theory Gravity, the gravitational field is defined by, at each point in spacetime, using a gamma matrix to define a rotation and

a boost. Gravity then appears as a result of the difference in gamma matrices between nearby spacetime points. (Please forgive the author for being unable to describe this succinctly and correctly.)

General relativity allows a wide variety of metrics to describe a rotating black hole. In describing how gravitons act, it's natural to look at the metrics that are implied by Gauge Theory Gravity. The mathematics are well described in an intuitive paper by Hamilton and Lisle's called "The River Model of Black Holes".[16] The boost at each point in spacetime modifies the speed of light at that point. So at the event horizon of a black hole the boost has magnitude c and the speed of light is zero going out and $2c$ going into the black hole.

This modification of the speed of light suggests that the gravitons emitted from the black hole should be altering the speed of photons and the massless fermions we discuss in this paper. As far as reducing the speed of light, that's easy, it's arranged by absorbing a fermion and then reemitting it with some delay. For particles to exceed the speed of light we assume that they are emitted at a point and time faster than their track would have been.

The author plans to continue working on these problems; he hopes that the reader has been as well entertained as he has.

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