Non-commutativity: Unusual View

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Some ambiguities have recently been found in the definition of the partial derivative (in the case of presence of both explicit and implicit dependencies of the function subjected to differentiation). We investigate the possible influence of this subject on quantum mechanics and the classical/quantum field theory. Surprisingly, some commutators of operators of space-time 4-coordinates and those of 4-momenta are *not* equal to zero. We postulate the non-commutativity of 4-momenta and we derive mass splitting in the Dirac equation. Moreover, two iterated limits may not commute each other, in general. Thus, we present an example when the massless limit of the function of $E, \mathbf{p}, m$ does not exist in some calculations within quantum field theory.

1 Introduction

The assumption that the operators of coordinates do *not* commute $[\hat{x}_\mu, \hat{x}_\nu]_- \neq 0$ has been made by H. Snyder [1]. Therefore, the Lorentz symmetry may be broken. This idea [2, 3] received attention in the context of "brane theories". Moreover, the famous Feynman-Dyson proof of Maxwell equations [4] contains intrinsically the non-commutativity of velocities $[\dot{x}_i(t), \dot{x}_j(t)]_- \neq 0$ that also may be considered as a contradiction with the well-accepted theories (while there is no any contradiction therein).

On the other hand, it was recently discovered that the concept of partial derivative is *not* well defined in the case of both explicit and implicit dependence of the corresponding function, which the derivatives act upon [5]. The well-known example of such a situation is the field of an accelerated charge [6].* Škovrlj and Ivezić [7] call this partial derivative as '*complete* partial derivative'; Chubykalo and Vlayev, as '*total* derivative with respect to a given variable'. The terminology suggested by Brownstein [5] is 'the *whole*-partial derivative'.

2 Example 1

Let us study the case when we deal with explicite and implicite dependencies $f(\mathbf{p}, E(\mathbf{p}))$. It is well known that the energy in relativism is related to the 3-momentum as $E = \pm\sqrt{\mathbf{p}^2 + m^2}$; the unit system $c = \hbar = 1$ is used. In other words, we must choose the 3-dimensional mass hyperboloid in the Minkowski space, and the energy is *not* an independent quantity anymore. Let us calculate the commutator of the whole-partial derivatives $\hat{\partial}/\hat{\partial}E$ and $\hat{\partial}/\hat{\partial}p_i$. In order to make distinction between differentiating the explicit function and that which contains both explicit and implicit dependencies, the 'whole partial derivative' may be denoted as $\hat{\partial}$. In the general case one has

$$\frac{\hat{\partial} f(\mathbf{p}, E(\mathbf{p}))}{\hat{\partial} p_i} \equiv \frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial p_i} + \frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial E}\frac{\partial E}{\partial p_i}. \quad (1)$$

Applying this rule, we find surprisingly

$$\left[\frac{\hat{\partial}}{\hat{\partial} p_i}, \frac{\hat{\partial}}{\hat{\partial} E}\right]_- f(\mathbf{p}, E(\mathbf{p})) = \frac{\hat{\partial}}{\hat{\partial} p_i}\frac{\partial f}{\partial E} - \frac{\partial}{\partial E}\left(\frac{\partial f}{\partial p_i} + \frac{\partial f}{\partial E}\frac{\partial E}{\partial p_i}\right) = \frac{\partial^2 f}{\partial E \partial p_i} + \frac{\partial^2 f}{\partial E^2}\frac{\partial E}{\partial p_i} - \frac{\partial^2 f}{\partial p_i \partial E} - \frac{\partial^2 f}{\partial E^2}\frac{\partial E}{\partial p_i} - \frac{\partial f}{\partial E}\frac{\partial}{\partial E}\left(\frac{\partial E}{\partial p_i}\right). \quad (2)$$

So, if $E = \pm\sqrt{m^2 + \mathbf{p}^2}$ and one uses the generally-accepted representation form of $\partial E/\partial p_i = p_i/E$, one has that the expression (2) appears to be equal to $(p_i/E^2)\frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial E}$. Within the choice of the normalization the coefficient may be related to the longitudinal electric field in the helicity basis.† Next, the commutator is

$$\left[\frac{\hat{\partial}}{\hat{\partial} p_i}, \frac{\hat{\partial}}{\hat{\partial} p_j}\right]_- f(\mathbf{p}, E(\mathbf{p})) = \frac{1}{|E|^3}\frac{\partial f(\mathbf{p}, E(\mathbf{p}))}{\partial E}[p_i, p_j]_-. \quad (3)$$

This should also not be zero according to Feynman and Dyson [4]. They postulated that the velocity (or, of course, the 3-momentum) commutator is equal to $[p_i, p_j] \sim i\hbar\epsilon_{ijk}B^k$, i.e., to the magnetic field. In fact, if we put in the corespondence to the momenta their quantum-mechanical operators (of course, with the appropriate clarification $\partial \to \hat{\partial}$), we obtain again that, in general, the derivatives do *not* commute $\left[\frac{\hat{\partial}}{\hat{\partial} x_\mu}, \frac{\hat{\partial}}{\hat{\partial} x_\nu}\right]_- \neq 0$.

Furthermore, since the energy derivative corresponds to the operator of time and the i-component momentum deriva-

*Firstly, Landau and Lifshitz wrote that the functions depended on t', and only through $t' + R(t')/c = t$ they depended implicitly on x, y, z, t. However, later (in calculating the formula (63.7)) they used the explicit dependence of R on the space coordinates of the observation point too. Jackson [8] agrees with [6] that one should find "a contribution to the spatial partial derivative for fixed time t from explicit spatial coordinate dependence (of the observation point)."

†The electric/magnetic fields can be derived from the 4-potentials which have been presented in [9].

tive, to $\hat{x}_i$, we put forward the following anzatz in the momentum representation:

$$[\hat{x}^\mu, \hat{x}^\nu]_- = \omega(\mathbf{p}, E(\mathbf{p}))\, F_\parallel^{\mu\nu}(\mathbf{p}) \frac{\partial}{\partial E}\,, \tag{4}$$

with some weight function ω being different for different choices of the antisymmetric tensor spin basis. The physical dimension of x^μ is $[energy]^{-1}$ in this unit system; $F_\parallel^{\mu\nu}(\mathbf{p})$ has the dimension $[energy]^0$, if we assume the mass shell condition in the definition of the field operators $\delta(p^2 - m^2)$, see [10]. Therefore, the weight function should have the dimension $[energy]^{-1}$. The commutator $[\hat{x}^\mu, \hat{p}^\nu]$ has the same form as in the textbook nonrelativistic quantum mechanics within the presented model.

In the modern literature, the idea of the broken Lorentz invariance by this method concurs with the idea of the *fundamental length*, first introduced by V. G. Kadyshevsky [11] on the basis of old papers by M. Markov. Both ideas and corresponding theories are extensively discussed. In my opinion, the main question is: what is the space scale, when the relativity theory becomes incorrect.

3 Example 2

In the previous Section (see also the paper [12]) we found some intrinsic contradictions related to the mathematical foundations of modern physics. It is well known that the partial derivatives commute in the Minkowski space (as well as in the 4-dimensional momentum space). However, if we consider that energy is an implicit function of the 3-momenta and mass (thus, approaching the mass hyperboloid formalism, $E^2 - \mathbf{p}^2 c^2 = m^2 c^4$) then we may be interested in the commutators of the whole-partial derivatives [5] instead. The whole-partial derivatives do not commute, as you see above. If they are associated with the corresponding physical operators, we would have the uncertainty relations in this case. This is an intrinsic contradiction of the SRT. While we start from the same postulates, on using two different ways of reasoning we arrive at the two different physical conclusions.

In this Section I would like to ask another question. Sometimes, when calculating dynamical invariants (and other physical quantities in quantum field theory), and when studying the corresponding massless limits we need to calculate iterated limits. We may encounter a rare situation when two iterated limits are not equal each other in physics. See, for example, Ref. [10]. We were puzzled calculating the iterated limits of the aggregate $\frac{E^2-\mathbf{p}^2}{m^2}$ (or the inverse one, $\frac{m^2}{E^2-\mathbf{p}^2}$, $c = \hbar = 1$):

$$\lim_{m\to 0} \lim_{E\to\pm\sqrt{\mathbf{p}^2+m^2}} \left(\frac{m^2}{E^2-\mathbf{p}^2}\right) = 1\,, \tag{5}$$

$$\lim_{E\to\pm\sqrt{\mathbf{p}^2+m^2}} \lim_{m\to 0} \left(\frac{m^2}{E^2-\mathbf{p}^2}\right) = 0\,. \tag{6}$$

Similar mathematical examples are presented in [13]. Physics should have well-defined dynamical invariants. Which iterated limit should be applied in the study of massless limits? The question of the iterated limits was studied in [14]. However, the answers leave room for misunderstandings and contradictions with the experiments. One can say: "The two limits are of very different sorts: the limit of $E \to \pm\sqrt{\mathbf{p}^2 + m^2}$ is a limit that subsumes the statement under the theory of Special Relativity. Such limits should be done first." However, cases exist when the limit $E \to \pm\sqrt{\mathbf{p}^2 + m^2}$ cannot be applied (or its application leads to the loss of the information). For example, we have for the causal Green's function used in the scalar field theory and in the $m \to 0$ quantum electrodynamics (QED), Ref. [15]:

$$D^c(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ip\cdot x}}{m^2 - p^2 - i\epsilon} \tag{7}$$
$$= \frac{1}{4\pi}\delta(\lambda) - \frac{m}{8\pi\sqrt{\lambda}}\theta(\lambda)\left[J_1(m\sqrt{\lambda}) - iN_1(m\sqrt{\lambda})\right]$$
$$+ \frac{im}{4\pi^2\sqrt{-\lambda}}\theta(-\lambda)K_1(m\sqrt{-\lambda}),$$

$\lambda = (x^0)^2 - \mathbf{x}^2$; J_1, N_1, K_1 are the Bessel functions of the first order. The application of $E \to \pm\sqrt{\mathbf{p}^2 + m^2 - i\delta}$ results in non-existence of the integral. Meanwhile, the massless limit is made in the integrand in the Feynman gauge with no problems. Please remember that integrals are also the limits of the Riemann integral sums. The $m \to 0$ limits are made first sometimes.

Next, the application of the mass shell condition in the Weinberg-Tucker-Hammer $2(2S+1)$-formalism leads to the fact that we would not be able to write the dynamical equation in the covariant form $[\gamma^{\mu\nu}\partial_\mu\partial_\nu - m^2]\Psi_{(6)}(x) = 0$. Apart, the information about the propagation of longitudinal modes would be lost (cf. formulas (19,20,27,28) of the first paper [10]). Moreover, the Weinberg equation and the mapping of the Tucker-Hammer equation to the antisymmetric tensor formalism have different physical contents on the interaction level [16, 17].*

Next, if we would always apply the mass shell condition first then we come to the derivative paradox of the previous Section. Finally, the condition $E^2 - \mathbf{p}^2 = m^2$ does not always imply the generally-accepted special relativity only. For instance, see the Kapuscik work, Ref. [18], who showed that similar expressions for energy and momentum exist for particles with $V > c$ and $m_\infty \in \Re e$.

Meanwhile, the case $m = 0$ appears to be equivalent to the light cone condition $r = ct$, which can be taken even without

*I take this opportunity to note that problems (frequently forgotten) may also exist with the direct application of $m \to 0$ in relativistic quantum equations. The case is: when the solutions are constructed on using the relativistic boosts in the momentum space the mass may appear in the denominator, $\sim 1/m^n$, which cancels the mass terms of the equation giving the non-zero corresponding result.

the mass shell condition to study the theories extending the special relativity. Not everybody realizes that it can be used to deduce the Lorentz transformations between two different reference frames. Just take squares and use the lineality: $r_1^2 - c^2t_1^2 = 0 = r_2^2 - c^2t_2^2$. Hence, in $d = 1 + 1$ we have $x_2 = \gamma(x_1 - vt_1)$, $\quad t_2 = \alpha(t_1 - \frac{\beta}{c}x_1)$ with $\alpha = \gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$, the Lorentz factor; $\beta = v/c$.

4 Example 3

The problem of explaining mass splitting of leptons (e, μ, τ) and quarks has a long history. See, for instance, a method suggested in Refs. [19], and some new insights in [20]. Non-commutativity [1] also exhibits interesting peculiarities in the Dirac case. Recently, we analyzed the Sakurai-van der Waerden method of deriving the Dirac (and higher-spin) equation [21]. We can start from

$$(EI^{(2)} - \boldsymbol{\sigma}\cdot\mathbf{p})(EI^{(2)} + \boldsymbol{\sigma}\cdot\mathbf{p})\Psi_{(2)} = m^2\Psi_{(2)}\,, \tag{8}$$

or

$$(EI^{(4)} + \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta)(EI^{(4)} - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\Psi_{(4)} = 0\,. \tag{9}$$

E and $\mathbf{p}$ form the Lorentz 4-momentum. Obviously, the inverse operators of the Dirac operators exist in the non- commutative case. As in the original Dirac work, we have $\beta^2 = 1$, $\alpha^i\beta + \beta\alpha^i = 0$, $\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}$.

We also postulate non-commutativity relations for the components of 4-momenta:

$$[E, \mathbf{p}^i]_- = \Theta^{0i} = \theta^i\,, \tag{10}$$

as usual. Therefore the equation (9) will *not* lead to the well-known equation $E^2 - \mathbf{p}^2 = m^2$. Instead, we have

$$\{E^2 - E(\boldsymbol{\alpha}\cdot\mathbf{p}) + (\boldsymbol{\alpha}\cdot\mathbf{p})E - \mathbf{p}^2 - m^2 - i(\boldsymbol{\sigma}\otimes I_{(2)})[\mathbf{p}\times\mathbf{p}]\}\Psi_{(4)} = 0\,.$$

For the sake of simplicity, we may assume the last term to be zero. Thus, we arrive at

$$\left\{E^2 - \mathbf{p}^2 - m^2 - (\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\right\}\Psi_{(4)} = 0\,. \tag{11}$$

We can apply the unitary transformation. It is known [22, 23] that one can* $U_1(\boldsymbol{\sigma}\cdot\mathbf{a})U_1^{-1} = \sigma_3|\mathbf{a}|$. For $\boldsymbol{\alpha}$ matrices we re-write as

$$\mathcal{U}_1(\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\mathcal{U}_1^{-1} = |\boldsymbol{\theta}|\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \alpha_3|\boldsymbol{\theta}|\,. \tag{12}$$

The explicit form of the U_1 matrix is ($a_{r,l} = a_1 \pm ia_2$):

$$U_1 = \frac{1}{\sqrt{2a(a+a_3)}}\begin{pmatrix}a + a_3 & a_l\\ -a_r & a + a_3\end{pmatrix} = \tag{13}$$

$$= \frac{1}{\sqrt{2a(a+a_3)}}[a + a_3 + i\sigma_2 a_1 - i\sigma_1 a_2]\,,$$

$$\mathcal{U}_1 = \begin{pmatrix}U_1 & 0\\ 0 & U_1\end{pmatrix}. \tag{14}$$

We now apply the second unitary transformation:

$$\mathcal{U}_2\alpha_3\mathcal{U}_2^\dagger = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\alpha_3\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} =$$

$$= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}. \tag{15}$$

The final equation is then

$$\left[E^2 - \mathbf{p}^2 - m^2 - \gamma^5_{chiral}|\boldsymbol{\theta}|\right]\Psi'_{(4)} = 0\,. \tag{16}$$

In physical terms this implies mass splitting for a Dirac particle over the non-commutative space, $m_{1,2} = \pm\sqrt{m^2 \pm \theta}$. This procedure may be attractive as explanation of mass creation and mass splitting in fermions.

5 Conclusions

We found that the commutator of two derivatives may be *not* equal to zero. As a consequence, for instance, the question arises, if the derivative $\hat{\partial}^2 f/\hat{\partial}p^\nu\hat{\partial}p^\mu$ is equal to the derivative $\hat{\partial}^2 f/\hat{\partial}p^\mu\hat{\partial}p^\nu$ in all cases?† The presented consideration permits us to provide some bases for non-commutative field theories and induces us to look for further development of the classical analysis in order to provide a rigorous mathematical basis for operations with functions which have both explicit and implicit dependencies. Several other examples are presented. Thus, while for physicists everything is obvious in the solutions of the paradoxes, this is not so for mathematicians.

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*Some relations for the components $\mathbf{a}$ must be assumed. Moreover, in our case $\boldsymbol{\theta}$ must not depend on E and $\mathbf{p}$. Otherwise, we must take the non-commutativity $[E, \mathbf{p}^i]_-$ into account again.

†The same question can be put forward when we have differentiation with respect to the coordinates too, that may have impact on the correct calculations of the problem of accelerated charge in classical electrodynamics.

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