Fibonacci Motives

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Abstract

Motives are well connected to graphical techniques in quantum field theory. In motivic quantum gravity we consider categorical axioms, starting with the ribbon category of the Fibonacci anyon. Quantum logic dictates that the cardinality of a classical set is replaced by the dimension of a space. In this note we look at the geometry underlying Fibonacci numbers and apply it to the algebra of multiple zeta values and cyclic particle braids.

1 The Fibonacci numbers

One often considers a positive integer $N \in \mathbb{N}$ as a product $e_1^{r_1} e_2^{r_2} \cdots e_k^{r_k}$ of k prime power factors. It is less well known (Zeckendorff's theorem) that N is uniquely a sum of nonconsecutive Fibonacci numbers. With the exception of $F_6 = 8$ and $F_{12} = 144$, every F_n greater than 1 contains a prime factor that is not in any lower F_m (Carmichael's theorem). The basic recursion rule is

$$F_{n+2} = F_{n+1} + F_n, \tag{1}$$

recovering the Fibonacci numbers for $F_1 = F_2 = 1$. The Lucas numbers have the same recursion rule, starting with $L_1 = 1$ and $L_2 = 3$. Then $L_{n-1} = F_n + F_{n-2}$. Observe that

$$F_{n+k} = F_{k+1} F_n + F_k F_{n-1}, \tag{2}$$

from which the double limit gives

$$\frac{F_{n+1}}{F_{n-1}} \to \phi + 1 \tag{3}$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio, so that F_n / F_{n-1} tends to ϕ. The Lucas numbers satisfy $L_n = \phi^n + (-1)^n \phi^{-n}$, and the limit of L_{n+2}/F_{n-1} is $\phi + 2$.

Lemma 1: For primes p equal to 1 or 4 mod 5, p divides F_{p-1}. For 2 or 3 mod 5, p divides F_{p+1}. Lastly, 5 divides F_5.

Proposition 1: For every $n \in \mathbb{N}$, the sequence of F_i mod n is a cycle of length $L(n)$ (see Table 1) such that $p + 1$ divides $L(n)$ when p is a factor of $n = 2, 3$ mod 5. Similarly, $p - 1$ divides $L(n)$ when $n = 1, 4$ mod 5.

Table 1: F_n cycle lengths

n	2	3	5	6	7	8	9	10	12	13
$L(n)$	3	8	20	24	16	12	24	60	24	28

Proof: A cycle requires a string $1, 0, 1$ in the sequence F_i mod n. By the lemma, for any prime p there exists an $F_m \equiv 0$ mod p, and the same is true for any n. If n divides F_m then it divides all F_{km} for $k \geq 2$, since the greatest common divisor of F_{km} and $F_{(k+1)m}$ is F_m, containing n. It remains to find the first $F_{t-2} \equiv 1$ mod n next to a zero. Cassini's rule

$$F_{t-1}F_{t+1} - F_t^2 = (-1)^t \tag{4}$$

has a right hand side equal to ± 1. In general, write $F_{t-1} = ln$ and $F_{t-2} = jn + 1 + x$ for $0 \leq x \leq n-2$. Then the left hand side of (4) mod n is a square $(x+1)^2$. We can consider either a small x or a small $y = n - x$. When the right hand side equals -1, x cannot equal 0. When the right hand side equals 1, $x = 0$ as required. Since the zero places occur at all multiples of $t-1$, there must be even values for t giving $x = 0$. $\square$

Up to mod 12, all cycles fit into one of length 240, increasing to 1680 at mod 13. The Cassini rule comes from the determinant of the tth power of

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

the composition of the modular group matrix TS with a reflection Z. This matrix is closely related to the following representation of $S_3 = PSL(2,2)$. We want to permute three roots for a cubic leading to a function substitution, under the correspondence

$$z = (1), \quad 1 - \frac{1}{z} = (312), \quad \frac{1}{1-z} = (231), \tag{6}$$

$$\frac{1}{z} = (31), \quad \frac{z}{z-1} = (32), \quad 1 - z = (21).$$

Now S_3 is represented by the matrices

$$(1) = I, \quad (312) = TS, \quad (231) = ST^{-1}, \tag{7}$$

$$(31) = -SZ, \quad (32) = -ST^{-1}SZ, \quad (21) = -TZ.$$

A second representation is obtained by multiplying on the right by Z, giving the Fibonacci number generator $(312) = TSZ$, but only so long as $T^2 = I$. That is, we need a morphism $\mathbb{F} \to \mathbb{F}_2$ sending all even numbers to zero, giving precisely $PSL(2,2)$. Then for a mod n cycle, $(TSZ)^{L(n)} = I$ and $T^n = I$, giving a sequence of finite Fibonacci groups. In projective geometry, the zero at $F_{p\pm 1}$ gives the identity, implying $(TSZ)^{p\pm 1} = I$. So at mod 3, $(TSZ)^4 = I$ and S_4 is generated by TSZ, SZ and T.

Our interest in the Fibonacci numbers comes from the importance of the ring of integers in $\mathbb{Q}(\sqrt{5})$ and other number fields in quantum gravity. Here the half integers in dimension 8 are embedded densely [1] in $\mathbb{C}$ under the map $\mathbb{Z}^4/2 \to \mathbb{R}$

$$(a, b, c, d) \mapsto a + b\phi + c\rho + d\phi\rho, \tag{8}$$

where ϕ is the golden ratio and $\rho = \sqrt{\phi + 2} = 5^{1/4}\sqrt{\phi}$. Note that $\mathbb{Q}(\rho)$ is sufficient. In the necessarily categorical axioms for gravity, we cannot begin with $\mathbb{R}$ or $\mathbb{C}$, which are awkward in higher dimensional analogues of a topos [2][3] and do not correspond to motivic periods. What is the geometric information behind the Fibonacci numbers?

2 Golden word spaces

The ribbon category of the Fibonacci anyon [4][5] is universal [6] for quantum computation [7]. It's $SU(2)$ representation for the braid group B_3[8] is defined in terms of quaternion units J and K. Let

$$g = \exp^{7\pi J/10}, \quad f = J\phi + K\sqrt{\phi}, \quad h = fgf^{-1}. \tag{9}$$

Then $ghg = hgh$ is the braid relation. This is a rotation, by a 9° mixing angle[1], of the quaternion braid generators

$$\frac{1}{\sqrt{2}}(1 + J), \quad \frac{1}{\sqrt{2}}(1 + K). \tag{10}$$

A cyclic set I, J, K of three generators is associated to the $\mathbb{C} \otimes \mathbb{O}$ ideal algebra [9][10][11] for Standard model leptons and quarks in the Bilson-Thompson [12][13] ribbon scheme. Each ribbon strand on a particle diagram carries either a zero or 1/3 electric charge. Collecting the diagrams for a lepton pair and six quarks of positive charge, we obtain a parity cube

$$000, 00+, 0+0, +00, 0++, +0+, ++0, +++ \tag{11}$$

of charges on the vertices. The underlying permutation $(231) \in S_3$ is associated to the left handed neutrino.

In this section we will obtain the parity cube in each dimension using the Fibonacci anyons. This qubit state space is extended to general qudits by subdividing the edges of a cube. For example, a square with halved edges carries three labels along each edge, defining all pure two qutrit states.

The integers $\mathbb{Z}$ should be infinite dimensional, because each discrete dimension is naturally labeled by a prime power path sequence (1, e, e^2 etc. in the positive direction). Then the positive quadrant in the infinite dimensional cubic

[1] Note that $(\phi\rho)^{-1} = \tan 18^\circ$ and that $\phi = 2\cos 2\pi/10$, where $2\pi/10 - 2\pi/12 = 72^\circ$, the pentagon angle. The basic phase $2\pi/12$ is the difference between ν and $\overline{\nu}$ phases in the Koide formalism. The difference between $7\pi/10$ and $\pi/4$ is 81°, which is 9° away from both 90° and 72°.

lattice represents exactly $\mathbb{Z}$, following Pythagoras. Little cubes near the origin are the squarefree numbers, and an auxilliary infinite string of 1s allows for affine words (inhomogeneous monomials) on the diagonal simplices. Now a point in Spec($\mathbb{Z}$) is the infinite discrete cone without the orthogonal axis hyperplane at the boundary, just like a Zariski set. The whole infinite space is a vector space union (span) of all these axis boundaries. An infinite parity cube picks out the square free number $\prod_i p_i$ over all primes.

Along with cube and permutohedra tiles we work with the associahedra. The fusion map for the anyon category is an associator arrow on the pentagon. Let $F(abcd)^y_x$ be a fusion coefficient for an internal edge y on the input tree and internal edge x in the set of allowed trees, with d labelling the root of a three leaved tree. Our anyon objects are 1 and τ, such that $\tau \circ \tau = 1 + \tau$. Following [4], the interesting coefficients satisfy the pentagon relation

$$\begin{aligned} F(\tau\tau c\tau)^d_a F(a\tau\tau\tau)^c_b = F(\tau\tau\tau d)^c_1 F(\tau 1\tau\tau)^d_b F(\tau\tau\tau b)^1_a \\ + F(\tau\tau\tau\tau)^c_\tau F(\tau\tau\tau\tau)^d_b F(\tau\tau\tau b)^\tau_a. \end{aligned} \tag{12}$$

When $(abcd)$ contains a 1, the coefficients are 0 or 1. At $(abcd) = (\tau 11\tau)$, we obtain $F(\tau\tau\tau\tau)^1_1 = (F(\tau\tau\tau\tau)^1_\tau)^2$. From $(abcd) = (1\tau 1\tau)$ it follows that $F(\tau\tau\tau\tau)^1_1 = -F(\tau\tau\tau\tau)^1_1$. Let $F(\tau\tau\tau\tau)^1_1 = -A$ and $F(\tau\tau\tau\tau)^1_\tau = i\sqrt{A}$. Then $(abcd) = (\tau\tau\tau\tau)$ gives $A^2 - A - 1 = 0$ with solution $A = -1/\phi$. In summary, the all-τ coefficients are

$$\begin{pmatrix} F^1_1 & F^\tau_1 \\ F^1_\tau & F^\tau_\tau \end{pmatrix} = \begin{pmatrix} \frac{1}{\phi} & \frac{i}{\sqrt{\phi}} \\ \frac{i}{\sqrt{\phi}} & \frac{-1}{\phi} \end{pmatrix}. \tag{13}$$

These appear in the B_3 representation

$$\sigma_1 = \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \frac{e^{4\pi i/5}}{\phi} & \frac{e^{-3\pi i/5}}{\sqrt{\phi}} \\ \frac{e^{-3\pi i/5}}{\sqrt{\phi}} & \frac{-1}{\phi} \end{pmatrix}, \tag{14}$$

with phases from the hexagon rule. The phase in (9) comes from the difference of these phases.

Consider the number of fusion diagrams on d leaves when all inputs are set to τ and the bracketing is nested to the left. We write words in 1 and τ by following the internal edges from a leaf down to the root. Since all words start with τ, we omit this letter, leaving words of length $d - 1$. For three leaves the words are $\tau 1$, 1τ and $\tau\tau$, counted by the Fibonacci number F_{d+1}, as given in Table 2. Figure 1 shows how these words are allocated to vertices of a parity cube in dimension t equal to the number of τ letters. The + parity marks the placement of a 1.

The number F_{d+1} is graded across cubes of different dimension,

$$F_{d+1} = \sum_{n=0}^{f(d/2)} \binom{d-n}{n}, \tag{15}$$

Table 2: internal edges for fusion

n	words
2	1, τ
3	1τ, $\tau 1$, $\tau\tau$
4	$\tau 1\tau$, $\tau\tau 1$, $1\tau\tau$, $\tau\tau\tau$, $1\tau 1$
5	$1\tau 1\tau$, $1\tau\tau 1$, $1\tau\tau\tau$, $\tau 1\tau 1$, $\tau\tau\tau\tau$, $\tau\tau\tau 1$, $\tau\tau 1\tau$, $\tau 1\tau\tau$

where $f(i)$ is the integer part. Consider the subset labels for cubic vertices [14], which appear in the octonion basis. The $+++$ target in Figure 1 is the word $e_1^3 e_2^3 e_3^3$, where the superscript denotes the dimension. The vertex $+00$ is $e_1^3 = 1\tau\tau$ and so on. Denote a source by e_0. Then F_5 counts the set

$$e_0^4, e_1^3, e_2^3, e_3^3, e_1^2 e_2^2. \tag{16}$$

Such words are often interpreted as differential forms, but we might think of them as numbers with prime factors e_i. The XOR operation on subsets (in the cube) defines addition in a Boolean ring, giving the Fano plane basis for $\mathbb{O}$. Since the number of tree diagrams on d leaves is the Catalan number

$$C_d = \frac{1}{d}\binom{2(d-1)}{d-1}, \tag{17}$$

the total number of fusion trees is

$$N(d) = F_{d+1} C_d \quad \in 1, 2, 6, 25, 112, 546, \cdots \tag{18}$$

The number of strands d in a generic braid diagram for anyons is closely related to the M theory dimension in a higher algebra approach [15][16]. For example, in ten dimensions there is a 3-cube of electric charges and a 7-cube for magnetic information.

Let us now recall the connection [17][18] between knots and multiple zeta values [19]. From our perspective, the appearance of the golden ratio in a fusion map is reminiscent of the Drinfeld associator, with its infinite series of multiple zeta values. In the iterated integral form, a zeta argument is a word in two letters such that one letter only occurs as a singlet, much like the 1 in our fusion words.

3 Knots and motivic numbers

A multiple zeta value (MZV) is the unsigned case of the signed Euler sum

$$\zeta(n_1, n_2, n_3, \cdots, n_l; \sigma_1, \cdots, \sigma_l) = \sum_{k_i > k_{i+1} > 0} \frac{\sigma_1^{k_1} \sigma_2^{k_2} \cdots \sigma_l^{k_l}}{k_1^{n_1} k_2^{n_2} \cdots k_l^{n_l}} \tag{19}$$

of *depth* l and *weight* $n = \sum_i n_i$, with $\sigma_i \in \pm 1$. Recall that the Mobius function $\mu(n)$ on $\mathbb{N}$ is zero on non square free n and $(-1)^r$ for r prime factors. The square

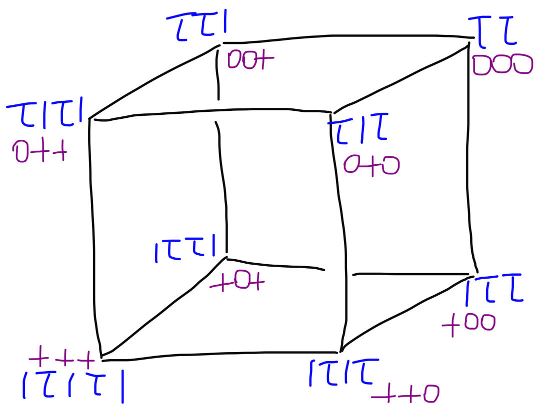

Figure 1: Fusion words

free $n \in \mathbb{N}$ are the targets of undivided cubes in the infinite dimensional $\mathbb{Z}$. An MZV is irreducible if not expressed as a $\mathbb{Q}$ linear combination of other MZVs of the same weight. The number E_n of irreducible signed Euler sums of weight n is [17][18]

$$E_n = \frac{1}{n} \sum_{D|n} \mu(n/D) L_D = \frac{1}{n} \sum_{D|n} \mu(n/D)(F_{D-1} + F_{D-3}), \tag{20}$$

where L_D is the Lucas number. The number M_n of irreducible MZVs of weight n is the number of knots with n positive crossings (and no negative crossings). It's value replaces L_D by P_D, the Perrin number, satisfying the recursion

$$P_D = P_{D-2} + P_{D-3} \tag{21}$$

for $P_1 = 0$, $P_1 = 2$ and $P_3 = 3$.

An argument $(n_1, \cdots, n_l)$ of an MZV, such that only n_l may equal 1, is expressed as a word in two letters A and B, such that all words start with A and end with B, and B only occurs as a singleton. First reduce the argument to the ordinals $(n_1 - 1, n_2 - 1, \cdots, n_l - 1)$. The corresponding word is $A^{n_1-1} B A^{n_2-1} B \cdots A^{n_l-1} B$. Each copy of A is assigned the form dz/z and each B the form $dz/(1-z)$ in the iterated integral expression for the MZV. For example

$$\zeta(3,1) = \int_0^1 \int_0^1 \int_0^1 \int_{0, z_4 > \cdots > z_1}^1 \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{1-z_3} \frac{dz_4}{1-z_4}. \tag{22}$$

Compare this to our golden words, with the additional τ at the start of every word, and a 1 occuring only as a singleton. Add a 1 at the end of every allowed word, to obtain precisely a set of MZV words. This extra 1 adds a bigon piece to the root edge of the polygon that is being chorded by a dual tree. The length of the internal word is essentially the weight,

$$\sum n_i - 2 = n - 2 = d - 1, \tag{23}$$

where d is the number of leaves on the fusion tree. Thus the weight $d+1$ is associated to braids in the category on d strands, but fewer than d strands may be used to draw a knot.

An example of a positive knot with n crossings and $n-1$ strands is the trefoil knot σ_1^3 in B_2. It corresponds to $\zeta(3)$, from the internal word τ on two leaves. The word 1 on two leaves gives $\zeta(2,1)$. Other torus knots of type $(2k+1,2)$ define the zeta values $\zeta(2k+1)$ [17]. At three leaves, the MZVs are $\zeta(3,1)$, $\zeta(4)$ and $\zeta(2,2)$. The total number of MZVs of weight $d+1$ is F_{d+1}, and the recursion $F_d + F_{d-1}$ splits F_{d+1} into words ending in either τ or 1 respectively.

Now recall that renormalisation in quantum field theory relies on the symmetric Hopf algebra of labeled rooted trees [21][22]. Our internal fusion words label a generic corolla tree with d leaves, which is a building block for symmetric trees. By restricting to the F_d words that end in τ, we ensure that the grafting of little corollas onto other trees is always possible. This suggests the F_{D-1} term in (20). F_{D-3} counts the number of internal words ending in $\tau 1$ and starting with τ. Thus L_D only excludes words that begin and end with 1, the so called vacuum words.

The square free divisors D of $d+1$ pick out a parity cube at the origin in dimension k inside $\mathbb{Z}$, where k is the number of prime factors in $d+1$. This cube has 2^k vertices, each marked with a Lucas number L_D in (20), for D a word in the e_i and a target word $e_1 e_2 \cdots e_k$. The signs of μ alternate on parity for the differential forms in e_i.

Values of M_n correspond to full words with even clusters of τ letters, corresponding to odd arguments for MZVs, as proved in [20]. This theorem requires an auxiliary five crossings to obtain the correct weight. For example, $M_{13} = 3$ [17] comes from fusion words on 7 strands, where the irreducibles are $\zeta(3,5)$, $\zeta(5,3)$ and $\zeta(7,1)$.

Consider the shuffle algebra for MZVs. The shuffle unit is the empty letter. The recursion law on A and B words is

$$\begin{aligned} l_1 l_2 \cdots l_u \cup k_1 k_2 \cdots k_v = {} & l_1 (l_2 \cdots l_u \cup k_1 \cdots k_v) \\ & + k_1 (l_1 \cdots l_u \cup k_2 \cdots k_v). \end{aligned} \tag{24}$$

The minimum zeta shuffle is

$$\zeta(2) \cup \zeta(2) = AB \cup AB = 2ABAB + 4AABB = 2\zeta(2,2) + 4\zeta(3,1). \tag{25}$$

Since this is a weight 4 rule, the trivalent vertex for $\zeta(3)$ comes from non MZV words. In particular, $\tau \cup \tau 1$ gives $2\tau\tau 1 + \tau 1 \tau$, which is $2\zeta(3)$ plus the word $\tau 1 \tau$. This is a cyclic representation of single 00+ charges. The fusion vertex $\tau \circ \tau$ corresponds to $\tau\tau 1 + \tau 11$, giving also $2\zeta(3)$. Thus a trivalent fusion graph resembles the Tutte graph [23] for the trefoil knot. For our fusion letter 1 we have $1 \cup 1 = 2 \cdot 11$ and $1^{\cup n} = 2^n \cdot 1 \cdots 1$. The single leaf tree gives $\zeta(2)$. Note that only 1/4 of the vertices on a cube are in the MZV algebra.

The cyclic B_3 representation appearing in $\tau \cup \tau 1$ is used for positively charged particles [11], with every particle carrying a ν braid $\sigma_2^{-1}\sigma_1$ and the charge correspondence

$$+00 = \sigma_1\sigma_2, \quad 0+0 = \sigma_2\sigma_3, \quad 00+ = \sigma_3\sigma_1, \tag{26}$$

so that ribbon twists may be removed by additional (positive) crossings in B_3. Category theoretically however, the Fibonacci anyons have a ribbon structure and general braids carry twists. Charge conjugation is complex conjugation, with i defined in B_2, giving the additional $2\pi i$ of MZV algebras.

In quantum gravity, universal cohomology does not begin with a consideration of classical geometries, which emerge from quantum computation. For instance, the compactified Minkowski space component $SU(2)$ emerges from the B_3 ribbons [6]. As category theorists, we are looking here at a functor from the category of finite sets (parity cubes) to a Fibonacci category. The intersection of two ordinals gives their greatest common divisor, which translates to the functorality of gcd on F_n. Primes e_i are in **Sets** (albeit not in the usual way) while the additive decomposition is in **F**.

Prime powers act as a tensor coarse graining of cubic diagrams. We usually truncate an axis by qudits, where anyons use only F_d dits. Each edge on the torus homology has $p+1$ points. The next section considers cubes that are divided by qudit points, and the connection to quantum information and the associahedra.

4 Parking functions, associahedra and mass

MZVs are computed using the associahedra of dimension equal to the weight [24][25]. Our two basic fusion vertices appear with the 14 vertex associahedron, defined by chorded hexagons. That is, two triangles fitted together create a hexagon. In general, F_n copies of an n-gon (of leaves or roots on a tree) fit together on a polygon with nF_n sides. In [26] it is shown that an ellipse with axes ratio $\sqrt{5}$ has a product of chord lengths equal to nF_n.

Now we consider cubes with subdivided edges. This extends the anyonic $SU(2)_3$ for parity to higher levels. A qutrit cube, as opposed to a parity cube, has three points along each edge, one for each digit 0, 1 or 2. In three dimensions, a qutrit cube has 27 vertices, namely all words of length 3 in the digits. These 27 words are also viewed as the 27 paths of length 3 on a 4-dit cube that end on the diagonal discrete (tetractys) simplex. In this case we use the path letters X, Y and Z. Now instead of anyon words in τ and 1, we consider words with extra letters.

The associahedra are naturally embedded in an n-dit cube of dimension n inside an $(n+1)$-dit space [27] as follows. Look at the diagonal simplex defined by paths of length n. The parking function words fit onto a subset of vertices, defining the associahedron. In particular, the pentagon sits inside the tetractys simplex, giving the 16 words of Figure 2, while the associator edge carries 3 words on the diagonal of a square.

For qutrits, we might extend the Fibonacci algebra to words in τ, 1 and 2. For general qudits we would require letters $1, 2, \cdots, n-1$. A restriction to words

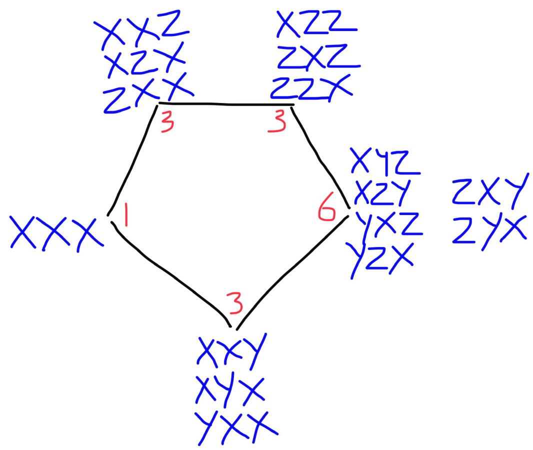

Figure 2: 16 parking functions on a pentagon

starting in τ and ending in $n-1$ then reduces a cube by a factor n^2, leaving n^{n-2} allowed words on a divided cube, which equals the number of parking functions. So the parking functions list the noncommutative monomials attached to points on the commutative simplex.

Focus on the 27 length 3 paths in X, Y and Z. Our letters might represent powers of the three matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

where ω is the primitive cubed root of unity. The centre spatial matrix generates cycles in S_3, underlying B_3, while the momentum operator is obtained from the quantum Fourier transform. These 27 matrices define a discrete phase space, and are used to define the 27 dimensional exceptional Jordan algebra over $\mathbb{O}$ using $\mathbb{F}_3 \times {\mathbb{F}_3}^3$, as shown in [28]. These four trits appear as follows.

To properly separate the 27 components of the Jordan algebra, the tetractys needs to be replaced by a simplex carrying 81 paths, as shown in Figure 3. These paths live in a cube with $5^3 = 125$ vertices. An 8-dimensional spinor or vector component appears when one selects one out of four letters. For instance, choosing $ZXXX$ out of four possible permutations marks the first letter for deletion, leaving the word XXX, so that a parity cube is labelled

$$ZXXX, \{XXXZ, XXZX, XZXX\}, \{ZXZZ, ZZXZ, ZZZX\}, XZZZ. \tag{28}$$

On this 81 path simplex, the corners $XXXX$, $YYYY$ and $ZZZZ$ define the diagonal of the 3×3 matrix. Observe that the central 54 paths, which are not included in our 27, then reduce either to existing paths on parity cubes or to the six XYZ paths of S_3. In this way the spinor splitting $27 = 16 + 11$, which ignores the shadow 54, reduces on the tetractys tile to $12 + 9$, so that the tetractys centre S_3 is sourced from the shadow paths. These 6 out of 27 tetractys paths define a Koide phase for rest mass triplets [14] in $\mathbb{C}S_3$.

The target vertex of an associahedron always carries $n!$ paths in S_n, which are the paths on a parity cube. There are 125 parking functions on the 14 vertex associahedron.

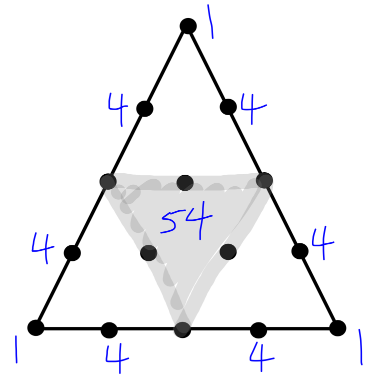

Figure 3: Paths on a four trit simplex

References

[1] F. Battaglia and E. Prato, Commun. Math. Phys. 299, 577, 2010

[2] M. D. Sheppeard, PhD thesis, University of Canterbury, 2007

[3] S. Mac Lane and I. Moerdijk, *Sheaves and geometry in logic*, Springer, 2000

[4] B. Field and T. Simula, arXiv:1802.06176, 2018

[5] A. Sagodi, BSc thesis, University of Amsterdam, 2017

[6] M. Freedman, M. Larsen and Z. Wang, Commun. Math. Phys. 227, 605-622, 2002

[7] L. H. Kauffman and S. L. Lomonaco Jr, arXiv:0804.4304, 2008

[8] L. H. Kauffman, arXiv:1710.04650, 2017

[9] C. Furey, Phys. Lett. B742, 195-199, 2015

[10] C. Furey, Phys. Rev. D86, 025024, 2012

[11] N. G. Gresnigt, Phys. Lett. B783, 212, 2018

[12] S. O. Bilson-Thompson, arXiv:hep-ph/0503213, 2005

[13] M. D. Sheppeard, viXra:1304.0003, 2013

[14] M. D. Sheppeard, J. Phys.: Conf. Ser. 1194, 012097, 2019

[15] P. Truini, Pacific J. Math. 260, 227, 2012

[16] P. Truini, M. Rios and A. Marrani, arXiv:1711.07881, 2017

[17] D. J. Broadhurst and D. Kreimer, arXiv:hep-ph/9504352, 1995

[18] D. J. Broadhurst and D. Kreimer, arXiv:hep-th/9609128, 1996

[19] F. Brown, arXiv:1102.1312, 2011

[20] C. Glanois, PhD thesis, University of Pierre et Marie Curie, 2016

[21] J. C. Butcher, Math. Comput. 26, 117, 79-106, 1972

[22] A. Connes and D. Kreimer, Commun. Math. Phys. 210, 249-273, 2000

[23] B. Bollobas, *Modern Graph Theory*, Springer, 1998

[24] F. Brown, arXiv:math/0606419, 2006

[25] F. Brown, arXiv:1512.06410, 2015

[26] B. Blum-Smith and J. Wood, arXiv:1810.00492, 2018

[27] A. Postnikov, Int. Math. Res. Not. 2009, 6, 1026-1106, 2009; arXiv:math/0507163

[28] R. L. Griess Jr, J. Alg. 131, 281-293, 1990