A Solution of the Laplacian Using Geodetic Coordinates

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ABSTRACT: Using the geodetic coordinates (φ, λ, h), we give the expression of the laplacian $\Delta V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}$ in these coordinates. A solution of $\Delta V = 0$ of type $V = f(\lambda).g(\varphi, h)$ is given. The partial differential equation satisfied by $g(\varphi, h)$ is transformed in an ordinary differential equation of a new variable $u = u(\varphi, h)$.

To the memory of my teachers and my professors

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KEYWORDS: geodetic coordinates, laplacian, ordinary differential equations of second order.

[1]This paper was written when I was a student at ENSG (Ecole Nationale des Sciences Géographiques, Paris) in 1978-1981.

Contents

1 Introduction

The Laplace equation plays an important role in physical geodesy. Into geodetic literature, this equation was resolved in spherical and ellipsoidal coordinates to determine the expression of the potential. In this paper, we use the geodetic coordinates (φ, λ, h) and we will give the expression of $\Delta V = 0$ in these coordinates. We study the solutions of the expression of $\Delta V = 0$ of the type $V = f(\lambda).g(\varphi, h)$. A choice of a new variable $u = u(\varphi, h)$ transforms the partial differential equation satisfied by $g(\varphi, h)$ in an ordinary differential equation of second order.

2 The expression of ΔV in geodetic coordinates

In three dimensional euclidean space (O, x, y, z), the expression of the laplacian of a function $V(x, y, z)$ enough differentiable is:

$$\Delta = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{2.1}$$

Let a and e respectively the semi-major axis and the first eccentricity of an ellipsoid of revolution, to $M(x, y, z)$ we associate the triplet (φ, λ, h) determined by the well known relations:

$$\begin{cases} x = (N+h)cos\varphi cos\lambda \\ y = (N+h)cos\varphi sin\lambda \\ z = (N(1-e^2)+h)sin\varphi \end{cases} \tag{2.2}$$

where:

$$N = \frac{a}{\sqrt{1 - e^2 sin^2\varphi}} \tag{2.3}$$

Differentiating (2.2), we obtain:

$$\begin{cases} dx = -(\rho+h)sin\varphi cos\lambda d\varphi - (N+h)cos\varphi sin\lambda d\lambda + cos\varphi cos\lambda dh \\ dy = -(\rho+h)sin\varphi sin\lambda d\varphi + (N+h)cos\varphi cos\lambda d\lambda + cos\varphi sin\lambda dh \\ dz = (\rho+h)cos\varphi d\varphi + sin\varphi dh \end{cases} \tag{2.4}$$

where:

$$\rho = \frac{a(1-e^2)}{(1-e^2sin^2\varphi)^{-3/2}} \tag{2.5}$$

We will use often the relation :

$$d(Ncos\varphi) = -\rho sin\varphi d\varphi \tag{2.6}$$

Then we can write:

$$\begin{aligned} ds^2 = dx^2+dy^2+dz^2 &= (\rho+h)^2d\varphi^2 + (N+h)^2cos^2\varphi d\lambda^2 + dh^2 \quad (2.7)\\ &= h_1^2dq_1^2 + h_2^2dq_2^2 + h_3^2dq_3^2 \quad with \end{aligned}$$

$$\begin{cases} q_1 = \varphi, & h_1 = (\rho+h) \\ q_2 = \lambda, & h_2 = (N+h)cos\varphi \\ q_3 = h, & h_3 = +1 \end{cases} \tag{2.8}$$

The expression of the laplacian becomes using the geodetic coordinates (q_1, q_2, q_3) (Heiskanen and Moritz, [1]):

$$\Delta V = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial V}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial V}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial V}{\partial q_3}\right)\right] \tag{2.9}$$

Using (2.8), we find:

$$\begin{aligned} \Delta V = \frac{1}{(\rho+h)(N+h)cos\varphi}\left[\frac{\partial}{\partial\varphi}\left(\frac{(N+h)cos\varphi}{\rho+h}\frac{\partial V}{\partial\varphi}\right) + \right.\\ \left.\frac{\partial}{\partial\lambda}\left(\frac{\rho+h}{(N+h)cos\varphi}\frac{\partial V}{\partial\lambda}\right) + \frac{\partial}{\partial h}\left((\rho+h)(N+h)cos\varphi\frac{\partial V}{\partial h}\right)\right] \end{aligned} \tag{2.10}$$

3 The Resolution of $\Delta V = 0$

$\Delta V = 0$ gives the equation:

$$\frac{\partial}{\partial\varphi}\left(\frac{(N+h)cos\varphi}{\rho+h}\frac{\partial V}{\partial\varphi}\right) + \frac{\partial}{\partial\lambda}\left(\frac{\rho+h}{(N+h)cos\varphi}\frac{\partial V}{\partial\lambda}\right) + \frac{\partial}{\partial h}\left((\rho+h)(N+h)cos\varphi\frac{\partial V}{\partial h}\right) = 0 \tag{3.1}$$

To solve (3.1), let us consider solutions of type:

$$\Delta V = f(\lambda).g(\varphi, h) \tag{3.2}$$

Substituting equation (3.2) in equation (3.1) gives:

$$f(\lambda)\left[\frac{\partial}{\partial\varphi}\left(\frac{(N+h)cos\varphi}{\rho+h}\frac{\partial g}{\partial\varphi}\right) + \frac{\partial}{\partial h}\left((\rho+h)(N+h)cos\varphi\frac{\partial g}{\partial h}\right)\right] + \frac{\rho+h}{(N+h)cos\varphi}g(\varphi,h).\frac{d^2f}{d\lambda^2} = 0 \tag{3.3}$$

By separating the variables, we obtain:

$$\frac{(N+h)cos\varphi}{(\rho+h)g(\varphi,h)}\left[\frac{\partial}{\partial\varphi}\left(\frac{(N+h)cos\varphi}{\rho+h}\frac{\partial g}{\partial\varphi}\right)+\frac{\partial}{\partial h}\left((\rho+h)(N+h)cos\varphi\frac{\partial g}{\partial h}\right)\right]=-\frac{1}{f(\lambda)}\frac{d^2f}{d\lambda^2} \quad (3.4)$$

The term on the left of (3.4) depends only of φ and h, the term on right is a function of the variable λ, it follows that:

$$-\frac{d^2f}{d\lambda^2}=m^2f(\lambda) \quad (3.5)$$

and :

$$\frac{(N+h)cos\varphi}{\rho+h}\frac{\partial}{\partial\varphi}\left(\frac{(N+h)cos\varphi}{\rho+h}\frac{\partial g}{\partial\varphi}\right)+\frac{(N+h)cos\varphi}{\rho+h}\frac{\partial}{\partial h}\left((\rho+h)(N+h)cos\varphi\frac{\partial g}{\partial h}\right)=m^2g(\varphi,\lambda) \quad (3.6)$$

where m is scalar (real or complex). The solutions of (3.5) are:

$$f_m(\lambda)=\alpha_m e^{im\lambda}+\beta_m e^{-im\lambda} \quad (3.7)$$

with α_m,β_m being constants. To simplify the equation (3.6), we put:

$$A(\varphi,h)=\frac{(N+h)cos\varphi}{\rho+h},\quad B(\varphi,h)=(\rho+h)(N+h)cos\varphi \quad (3.8)$$

The equation (3.6) becomes:

$$A(\varphi,h)\frac{\partial}{\partial\varphi}\left(A(\varphi,h)\frac{\partial g}{\partial\varphi}\right)+A(\varphi,h)\frac{\partial}{\partial h}\left(B(\varphi,h)\frac{\partial g}{\partial h}\right)=m^2g(\varphi,\lambda) \quad (3.9)$$

4 The change of variables

To solve the equation (3.9), we try to find a change of variables so that the equation (3.9) becomes more simple. Our idea is to transform (3.9) in an ordinary differential equation depending of a new variable $u=u(\varphi,h)$. We also write:

$$g(\varphi,h)=G(u(\varphi,h))=G(u) \quad (4.1)$$

Using the last equation, we obtain the derivatives of g in the equation (3.9):

$$\begin{cases}\dfrac{\partial g}{\partial\varphi}=\dfrac{dG}{du}.\dfrac{\partial u}{\partial\varphi}\\[2ex] \dfrac{\partial g}{\partial h}=\dfrac{dG}{du}.\dfrac{\partial u}{\partial h}\\[2ex] \dfrac{\partial^2 g}{\partial\varphi^2}=\dfrac{d^2G}{du^2}\left(\dfrac{\partial u}{\partial\varphi}\right)^2+\dfrac{dG}{du}.\dfrac{\partial^2 u}{\partial\varphi^2}\\[2ex] \dfrac{\partial^2 g}{\partial h^2}=\dfrac{d^2G}{du^2}\left(\dfrac{\partial u}{\partial h}\right)^2+\dfrac{dG}{du}.\dfrac{\partial^2 u}{\partial h^2}\end{cases} \quad (4.2)$$

Substituting the equations (4.2) in the equation (3.9) gives:

$$A^2\left(\left(\frac{\partial u}{\partial \varphi}\right)^2+\frac{B}{A}\left(\frac{\partial u}{\partial h}\right)^2\right)\frac{d^2G}{du^2}+A\frac{dG}{du}\left(A\frac{\partial^2 u}{\partial \varphi^2}+B\frac{\partial^2 u}{\partial h^2}+\frac{\partial A}{\partial \varphi}\frac{\partial u}{\partial \varphi}+\frac{\partial B}{\partial h}\frac{\partial u}{\partial h}\right)=m^2G(u) \tag{4.3}$$

The coefficient of $\dfrac{d^2G}{du^2}$ in (4.3), using the equations (3.8) is:

$$K(\varphi,h)=A^2\left(\left(\frac{\partial u}{\partial \varphi}\right)^2+(\rho+h)^2\left(\frac{\partial u}{\partial h}\right)^2\right) \tag{4.4}$$

The resolution of (4.3) would been simple if the coefficients of $\dfrac{d^2G}{du^2}$ and $\dfrac{dG}{du}$ were polynomial functions and:

$$\left(\frac{\partial u}{\partial \varphi}\right)^2=(\rho+h)^2 sin^2\varphi u^{2p}$$

$$\left(\frac{\partial u}{\partial h}\right)^2=cos^2\varphi u^{2p}$$

where p is an integer, so we should have:

$$K(\varphi,h)=\frac{(N+h)^2cos^2\varphi}{(\rho+h)^2}\left[(\rho+h)^2sin^2\varphi u^{2p}+(\rho+h)^2cos^2\varphi u^{2p}\right]=(N+h)^2cos^2\varphi u^{2p} \tag{4.5}$$

Using the above equation, our choice of the new variable is:

$$u(\varphi,h)=u=(N+h)cos\varphi \tag{4.6}$$

Then we obtain the following relations:

$$\begin{gathered}\frac{\partial u}{\partial \varphi}=-(\rho+h)sin\varphi,\quad \frac{\partial u}{\partial h}=cos\varphi\\ \frac{\partial^2 u}{\partial u^2}=-\rho'(\varphi)sin\varphi-(\rho+h)cos\varphi,\quad \frac{\partial^2 u}{\partial h^2}=0,\quad \rho'=\frac{d\rho}{d\varphi}\end{gathered} \tag{4.7}$$

The equation (4.4) becomes:

$$K(\varphi,h)=u^2 \tag{4.8}$$

From the definition of the A and B given by (3.8), we obtain the expressions of $\dfrac{\partial A}{\partial \varphi}$ and $\dfrac{\partial B}{\partial h}$ as:

$$\begin{gathered}\frac{\partial A}{\partial \varphi}=-\left(sin\varphi+\frac{\rho' u}{(\rho+h)^2}\right)\\ \frac{\partial B}{\partial h}=u+(\rho+h)cos\varphi\end{gathered} \tag{4.9}$$

Substituting the equations (4.7-4.8-4.9) in (4.3) gives:

$$u^2\frac{d^2G}{du^2}+u\frac{dG}{du}-m^2G(u)=0 \tag{4.10}$$

5 The resolution of the ordinary differential equation (4.10)

We shall now find the conditions to obtain a particular solution of the equation (4.10) of the type $G(u) = u^n$. We obtain:

$$n(n-1)u^n + nu^n - m^2u^n = 0 \Longrightarrow (n^2 - m^2)u^n = 0$$

For no trivial solutions, the conditions are:

$$n = \pm m \tag{5.1}$$

Now, we suppose that $n = \pm m$, general solutions of (4.10) are obtained by putting $G(u) = u^n\psi(u)$ where ψ is an unknown function.

The equation (4.10) becomes:

$$u^{n+1}\left[u\frac{d^2\psi}{du^2} + (2n+1)\frac{d\psi}{du}\right] = 0 \tag{5.2}$$

After two integrations, we obtain:

$$\psi(u) = C_1u^{-2n} + C_2 \tag{5.3}$$

C_1, C_2 two constants. General solutions of (4.10) are:

$$G(u) = u^n\psi(u) = C_1u^{-2n} + C_2 \tag{5.4}$$

where $n = \pm m$. Returning to the function $g(\varphi, h)$, we have:

$$g_m(\varphi, h) = C_m(N+h)^m cos^m\varphi + \frac{C_{-m}}{(N+h)^m cos^m\varphi} \tag{5.5}$$

C_m, C_{-m} two constants. We note if we use the variable $u = \dfrac{1}{(N+h)cos\varphi}$ in the change of variables in the equation (4.3), we obtain the same solutions.

6 Expression of V the solutions in geodetic coordinates (φ, λ, h)

To write a solution of $\Delta V = 0$ on the type $V = f(\lambda).g(\varphi, h)$, we combine the equations (3.7) and (5.5), we obtain for m a scalar:

$$V = \left[C_m(N+h)^m cos^m\varphi + \frac{C_{-m}}{(N+h)^m cos^m\varphi}\right]\left(\alpha_m e^{im\lambda} + \beta_m e^{-im\lambda}\right) \tag{6.1}$$

If $m \geq 0$ is an integer, we consider two expressions of the solutions of $\Delta V = 0$, by writing:

$$Vm = Vm(\varphi, \lambda, h) = (N+h)^m cos^m \varphi (A_m cosm\lambda + B_m sinm\lambda) \tag{6.2}$$

$$V_m = V_m(\varphi, \lambda, h) = \frac{1}{(N+h)^m cos^m \varphi}(A'_m cosm\lambda + B'_m sinm\lambda) \tag{6.3}$$

A_m, B_m, A'_m, B'_m being constants.

We can write that:

$$\begin{aligned} \mathcal{V}_1 &= \sum_{m=0}^{+\infty} (N+h)^m cos^m \varphi (A_m cosm\lambda + B_m sinm\lambda) \\ \mathcal{V}_2 &= \sum_{m=0}^{+\infty} \frac{1}{(N+h)^m cos^m \varphi}(A'_m cosm\lambda + B'_m sinm\lambda) \end{aligned} \tag{6.4}$$

are also solutions of Laplace's equation $\Delta V = 0$.

References

[1] **Weiko A. Heiskanen and Helmut Moritz**. 1967. Physical Geodesy. Edition W.H. Freeman and Compagny. 364 pages.