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Some Problems About The ^{3}He Superfluid

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Abstract—We discuss the stability problem of the atomic pair in the ^{3}He superfluid. It is similar to the bound-state problem of the hydrogen molecule because both of them are composed of two spin-1/2 nuclei. The difference is the cause for bonding. What is of interest here is whether the nuclear spin coupling between two ^{3}He nuclei is large enough to form a stable atomic pair in the superfluid. According to our calculations, the energy difference between the excited state F=1 and the ground state F=0 caused by the nuclear spin coupling of the two hydrogen nuclei is about 5.6x10^{-9} *eV* when the intermolecular distance is 0.74 Å. When the two ^{3}He nuclei are separated by 2.9 Å, the same energy difference caused by the nuclear spin coupling is 4.8x10^{-10} *eV*. This value is much less than the binding energy of two ^{3}He atoms about 1.0 *eV*, and is also much smaller than the Lennard-Jones potential of 8.74x10^{-4} *eV* between the two helium atoms. Therefore, two ^{3}He atoms cannot form a stable atomic pair due to the nuclear spin coupling even the spin wave.

Index Terms—Superfluid, ^{3}He, atomic pair, I-I nuclear spin coupling, Lennard-Jones potential

1 Introduction

SUPERFLUID is a material state characterized by zero viscosity. When the temperature of the quantum liquid is below critical transition temperature, it becomes superfluid [1-3]. For example, ^{4}He, the most abundant isotope in helium, becomes superfluid below 2.17 K (−270.98 °C) and has very large thermal conductivity. The most striking of these is the thermomechanical effect, or the fountain effect [4]. The theory of superfluid phenomenon of ^{4}He is developed by Lev Landau [1-3], and the phase transition of ^{4}He forming superfluid state is called Lambda phase transition. Another much few isotope, ^{3}He, becomes superfluid at 2.6 *m*K [1-3]. Although the superfluid representations of these two systems are similar, their nature is quite different. ^{4}He is a boson and its superfluidity can be explained by Bose-Einstein statistics [1-3]. However, ^{3}He is a fermion, and its superfluidity is understood by the promotion of the BCS theory in superconductors [1-3]. Among them, the Cooper pairs in superconductors are replaced with the atomic pairs in the superfluid ^{3}He, and the phonons of the attraction mechanism in superconductors are replaced by spin fluctuations in the superfluid ^{3}He [1-3].

We have already discussed the existence of the electron pairs in superconductors, and concluded that two high-speed, antiparallel spin, and inverse momentum electrons cannot form a bound electron-pair by a weak mediated phonon [5]. Another similar concept is the atomic pair in the ^{3}He superfluid. In this article, we will explore whether it is possible to achieve an atomic pair at very low temperature. We first use the ideal Fermi gas to calculate the Fermi velocity, and then calculate the energy difference from the nuclear spin coupling of two ^{3}He nuclei and compare it with the thermal characteristic energy and binding energy. Finally, the Lennard-Jones potential between two helium atoms are also discussed.

2 DISSCUSSIONS

The ^{3}He atom has a total spin of 1/2 and obeys the Fermi-Dirac statistical distribution. Since there is almost no viscosity, it can be treated by the ideal quantum Fermi gas without considering the van der Waals forces between the atoms. In statistical mechanics, the average energy of each ^{3}He atom is

$$\bar{E} \equiv \frac{3E_F}{5} = \frac{3h^2}{10(2m_p + m_n)}\left(\frac{3N}{8\pi V}\right)^{2/3}, \tag{1}$$

where E_F is Fermi energy, N is the total number of the Fermi gas, V is its total volume, and h is the Planck's constant. We can calculate the atomic velocity corresponding to the Fermi energy level in the ^{3}He superfluid. The density of liquid ^{3}He is 0.081 g/cm^3 [6], so its atomic density N/V is similar to the Fermi electron density in the element K, which is 1.626x10^{22} cm^{-3}. Without consideration of the interaction between atoms, its Fermi energy is about 4.253x10^{-4} *eV*, and the corresponding Fermi speed is 5.47x10^{-7} *c*~164.1 *m/s* where *c* is the speed of light in vacuum. The average speed of each atom is 127 *m/s*. Considering the Pauli's exclusion principle, the two ^{3}He atoms do not have the same quantum state. If two ^{3}He atoms with the same spin form an atomic pair, the energy as well as the speed of them must be different. It makes their motions inconsistent. Even only a very small speed deviation, the two atoms will eventually move further and further away. The center-of-mass speed of each ^{3}He atomic pair is inconsistent with each other which causes each atomic pair move apart. Finally, it makes the ^{3}He fluid thinner, or chaotic movements that collide with each other, increasing the viscosity.

Next, we further ask if two ^{3}He atoms form an atomic pair via the *I-I* nuclear spin coupling, is the energy enough to maintain the stability of this atomic pair? If two ^{3}He atoms forms a bond state achieved by the nuclear spin coupling, the situation should be similar to two ^{1}H atoms in the bound state because both cases are consisting of two spin-1/2 nuclei. As we know, the formation of a hydrogen molecules is due to covalent bonding, not nuclear spin coupling. We can estimate this energy produced by the nuclear spin coupling in the discussion of hyperfine structure in the real hydrogen atom [7]. For the *S-I* coupling between an electron and a hydrogen nuclei, the energy difference ΔE between the excited state F=1 and the ground state F=0 is

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$$\Delta E = \frac{4}{3} g_p \frac{m_e}{m_p} (Z\alpha)^4 (m_e c^2) \frac{1}{n^3} \left(\frac{\hat{s} \cdot \widehat{I_p}}{\hbar^2} \right)$$
$$\approx \frac{4}{3} (5.56) \left(\frac{1}{1840} \right) \left(\frac{1}{137} \right)^4 (0.511\ MeV)$$
$$\approx 5.84 \times 10^{-6}\ eV, \tag{2}$$

where g_p=5.56 is the gyromagnetic ration of a proton [7], Z is the atomic number, n is the principal quantum number, s=1/2 is the spin of electron, I_p=1/2 is the spin of proton, and $\hbar = h/2\pi$.

Next, we consider the *I-I* nuclear spin coupling between two hydrogen nuclei in a hydrogen molecule. Before proceeding this estimation, we have to know the wavefunction for two nuclei in the diatomic molecule. This problem has been done in many quantum physics textbooks by introducing the center-of-mass coordinate $\boldsymbol{R}$ and the relative coordinate $\boldsymbol{r}$ [7-10]. This two-particle problem reduces to an one-particle problem, and the fundamental-mode wavefunction of the relative motion is

$$\Psi_{100}(r) = \frac{\sqrt{\sigma}}{\pi^{1/4}} exp\left(-\frac{1}{2} \sigma^2 r^2 \right), \tag{3}$$

where

$$\sigma = \sqrt{\frac{\mu \omega_0}{\hbar}} \tag{4}$$

and the reduced mass

$$\mu = \frac{1}{2} m_p. \tag{5}$$

In Eq. (4), ω_0 is the naturally vibrational frequency. The vibration energy in a diatomic molecule is usually about 10^{-2} times the electronic energy, which is in the range from 0.01 to 0.1 eV [8]. The observed value is 4395 cm^{-1}, or 0.545 eV for the hydrogen molecule [9]. Then the *I-I* nuclear spin coupling in the hydrogen molecule is about [7]

$$H_{I\cdot I} = -\overrightarrow{M_{p1}} \cdot \overrightarrow{B_{p2}}$$
$$= \frac{Z^2 e^2 g_p^2}{4 m_p^2 c^2} \frac{1}{4\pi} \widehat{I_{p1}} \cdot \left[-\widehat{I_{p2}} \nabla^2 \frac{1}{r} + \nabla \left(\widehat{I_{p2}} \cdot \nabla \right) \frac{1}{r} \right], \tag{6}$$

where M_{p1} is the magnetic dipole moment of the proton with spin I_{P1}, B_{p2} is the magnetic field induced by the magnetic dipole moment M_{p2}, I_{p2} is the spin of the other proton, and e is an elementary charge. Because the ^{3}He atomic pair requires two parallel spins, it must be a spin triplet state for this atomic pair. The space wavefunction is antisymmetric and the lowest state is the first excited state for such ^{3}He atomic pair, so the wavefunction of the lowest state is [7]

$$\Phi_{lowest}(\vec{r}) = \frac{1}{\sqrt{2}} [\Psi_{100}(\vec{r}) + \Psi_{2lm}(\vec{r})], \tag{7}$$

where

$$\Psi_{2lm}(\vec{r}) = \frac{\sqrt{2\sigma}}{\pi^{1/4}} \sigma r \cdot exp\left(-\frac{1}{2} \sigma^2 r^2 \right) Y_{lm}(\theta, \phi) \tag{8}$$

and Y_{lm} is the harmonic function with integers l and m in the polar angle θ and azimuth angle φ coordinates. When we take the average of $H_{I\cdot I}$ in the fundamental oscillating mode, it gives

$$\langle H_{I\cdot I} \rangle = -\frac{Z^2 e^2 g_p^2}{24 \pi m_p^2 c^2} \left(\widehat{I_{p1}} \cdot \widehat{I_{p2}} \right) \langle \nabla^2 \frac{1}{r} \rangle$$
$$= \frac{1}{6} (Z^2 \alpha) \frac{g_p^2}{m_p^2 c} \hbar \left(\widehat{I_{p1}} \cdot \widehat{I_{p2}} \right) \int dr^3 \Phi^*_{lowest}(\vec{r})\, \delta(\vec{r}) \Phi_{lowest}(\vec{r})$$
$$= \frac{1}{12} (Z^2 \alpha) \frac{g_p^2}{m_p^2 c} \hbar \left(\widehat{I_{p1}} \cdot \widehat{I_{p2}} \right) |\Psi_0(0)|^2$$
$$= \frac{1}{12} (Z^2 \alpha) \frac{g_p^2}{m_p} \left(\widehat{I_{p1}} \cdot \widehat{I_{p2}} \right) \left(\frac{\hbar \omega_0}{2\pi m_p c^2} \right)^{1/2} \frac{1}{(r_0/2)^2}. \tag{9}$$

When the total spin of the two nuclei is

$$\hat{F} = \widehat{I_{p1}} + \widehat{I_{p2}}, \tag{10}$$

Then

$$\widehat{I_{p1}} \cdot \widehat{I_{p2}} = \frac{\hat{F}^2 - \widehat{I_{p1}}^2 - \widehat{I_{p2}}^2}{2}$$
$$= \frac{\hbar^2}{2} \begin{cases} I_p & F = 1 \\ -I_p - 1 & F = 0 \end{cases}. \tag{11}$$

The energy difference between the excited state F=1 and the ground state F=0 is

$$\Delta E = \frac{1}{3} (5.56)^2 \frac{1}{137} \left[\frac{0.545}{2\pi (9.382 \times 10^8)} \right]^{1/2} \left[\frac{m_e}{m_p} \left(\frac{\hbar}{m_e c} \right) \right]^2 \left(\frac{1}{r_0^2} \right) (m_p c^2)$$
$$= \frac{1}{3} (30.914) \frac{1}{137} (9.6153 \times 10^{-6}) \left(\frac{3.9 \times 10^{-13}}{1836 \times 7.4 \times 10^{-11}} \right)^2 (9.382 \times 10^8)$$
$$= 5.6 \times 10^{-9}\ eV, \tag{12}$$

where r_0=0.74 Å is the equilibrium distance between two hydrogen nuclei [7]. This energy term is much smaller than the binding energy of the hydrogen molecule, which is 4.75 eV [7]. As we know, the bound state of the two hydrogen atoms is due to the covalent bond formed by two 1s electrons, and the *I-I* nuclear spin coupling is too small to be the main cause. Both difference of the energy scale is about 9 order of magnitude.

If both hydrogen nuclei are replaced by two ^{3}He nuclei, then 1836 is replaced by 5508, and g_p is also replaced with g_p=5.56 and $g_n \approx$ -3.81. The energy difference between F=1 and F=0 is

$$\Delta E = \frac{1}{12} (Z^2 \alpha) \frac{\left[\left(2 g_p \widehat{I_p} + g_n \widehat{I_n} \right) \cdot \left(2 g_p \widehat{I_p} + g_n \widehat{I_n} \right) \right]}{(2 m_p + m_n)} \left[\frac{\hbar \omega_0}{2\pi (2 m_p + m_n) c^2} \right]^{1/2} \left(\frac{1}{r_0^2} \right)$$
$$\approx \frac{1}{12} \left(\frac{4}{137} \right) (7.31)^2 \left[\frac{0.545}{2\pi (2.82 \times 10^9)} \right]^{1/2} \left(\frac{3.9 \times 10^{-13}}{5508 \times 2.9 \times 10^{-10}} \right)^2 (2.82 \times 10^9)$$
$$\approx 4.8 \times 10^{-10}\ eV. \tag{13}$$

The equilibrium distance between two ^{3}He nuclei is adopted 2.9 Å [6]. This energy term between two ^{3}He nuclei is almost one order of magnitude smaller than that between two hydrogen nuclei because the distance between two nuclei increases from 0.74 Å to 2.9 Å [6]. However, the binding energy of two He atoms is close to 1.0 eV [11-12]. According to the above discussions, the difference on the two energy scales is about 10 orders of magnitude.

After obtaining the energy difference produced by the nuclear spin coupling, the stability of the atomic pairs directly depend on the thermal fluctuation. The transition temperature of the ^{3}He superfluid is 2.6 mK, and its characteristic thermal energy is about $k_B T$~2.24x10^{-7} eV where k_B is the Boltzman's constant. This characteristic thermal energy is about three order of magnitude larger than the energy difference in Eq. (13).

If an atomic pair is formed by the nuclear spin coupling and the energy difference between F=1 and F=0 states is only 10^{-10} eV, then a slight thermal fluctuation can easily destroy the atomic pair, even it is very close to the absolute zero temperature, which is only 2.6 mK.

For inert gases and even liquids, another consideration is the Lennard-Jones Potential between two atoms, resulting from the Coulomb interaction between the peripheral electrons [6]. For He liquid, the equilibrium distance between two atoms at no pressure is about 2.9 Å, and the Lennard-Jones potential is 8.74x10^{-4} eV lower than that when the two atoms are separated [6]. This value should be for ^{4}He atom whose transition temperature appears superfluidity at 2.17 K and its characteristic thermal energy is k_BT~1.87x10^{-4} eV. Therefore, the ^{4}He superfluid interpretation is based on Bose-Einstein condensation, but the interaction between atoms has to be considered. As for the ^{3}He superfluid, the Lenard-Jones Potential also has to be considered because the two electrons of the atom interact with other electrons from the neighboring atoms, and this effect is much greater than the nuclear spin coupling. Traditionally, the spin wave is used to explain the characteristics of the solid-state structures composed of the ferromagnetic atoms. Each atom is bound to a lattice point and vibrates locally, which can perform the magnetism due to the spin wave. When this model is used in the ^{3}He superfluid, it is necessary to consider the increasing degrees of freedom. The strong effect of the spin wave requires a very strong nuclear spin interaction. However, the previous estimation shows it only about 10^{-10} eV, which is three orders of magnitude smaller than the characteristic thermal energy and six orders of magnitude smaller than the Lenard-Jones Potential. Therefore, the spin wave is not the main cause to form the ^{3}He superfluid.

3 Conclusion

The superfluidity is an attractive field, especially for helium atom. We discuss the stability and possibility of the atomic pair in the ^{3}He superfluid in this research. From the kinetic viewpoint, it tells us that the speed of each ^{3}He atom in an atomic pair is different due to the Pauli's exclusion principle. ^{3}He atom is fermion so each state is occupied by only one atom. In such a situation, two ^{3}He atoms move inconsistent and leave each other even the speed deviation between two atoms is very small.

Then we discuss the nuclear spin coupling to check whether it is possibly stable for the atomic pair by this interaction. This problem is similar to the bound state of two hydrogen atoms because both systems are consisting of two spin-1/2 nuclei. As we know, it is the covalent bond existing in a hydrogen molecule, and the nuclear spin coupling is too small to be the main cause to form a bound state. The calculation reveals that the energy difference between the excited state F=0 and the ground state F=0 caused by the nuclear spin coupling is only 5.6 x10^{-9} eV. When the two hydrogen nuclei are replaced with two ^{3}He nuclei, this energy difference becomes 4.8x10^{-10} eV. This value is 10 order of magnitude smaller than the binding energy between two helium atoms, so the atomic pair cannot stably form by the nuclear spin coupling.

Except for the discussions of kinetic energy and the nuclear spin coupling, we also have to discuss the thermal fluctuation and Lennard-Jones potential. Even at 2.6 mK so close to absolute zero temperature, the characteristic thermal energy is 2.24x10^{-7} eV, three order of magnitude larger than the nuclear spin coupling, so the thermal fluctuation is easy to break the atomic pair. The Lennard-Jones potential is about 8.74x10^{-4} eV at the equilibrium point when two helium atoms are separated by 2.9 Å. It is six order of magnitude larger than the energy difference produced by the nuclear spin coupling. In conclusion, the ^{3}He atomic pair cannot form by the nuclear spin coupling because this coupling results in a very unstable atomic pair.

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