A Note on Jordan Algebras, Three Generations and Exceptional Periodicity*

Carlos Castro Perelman
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314
Ronin Institute, 127 Haddon Pl., NJ. 07043
perelmanc@hotmail.com

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Abstract

It is shown that the algebra $\mathbf{J}_3[\mathbf{C} \otimes \mathbf{O}] \otimes \mathbf{Cl}(\mathbf{4}, \mathbf{C})$ based on the complexified Exceptional Jordan, and the complex Clifford algebra in **4D**, is rich enough to describe all the spinorial degrees of freedom of three generations of fermions in **4D**, and include additional fermionic dark matter candidates. Furthermore, the model described in this letter can account also for the Standard Model gauge symmetries. We extend these results to the Magic Star algebras of Exceptional Periodicity developed by Marrani-Rios-Truini and based on the Vinberg cubic **T** algebras which are generalizations of exceptional Jordan algebras. It is found that there is a one-to-one correspondence among the real spinorial degrees of freedom of **4** generations of fermions in **4D** with the off-diagonal entries of the spinorial elements of the *pair* $\mathbf{T}_3^{8,n}, (\bar{\mathbf{T}}_3^{8,n})$ of Vinberg matrices at level $n = 2$. These results can be generalized to higher levels $n > 2$ leading to a higher number of generations beyond **4**. Three *pairs* of **T** algebras and their conjugates $\bar{\mathbf{T}}$ were essential in the Magic Star construction of Exceptional Periodicity [20] that extends the $\mathbf{e}_8$ algebra to $\mathbf{e}_8^{(n)}$ with n integer.

Keywords: Exceptional Periodicity; Jordan algebras; Vinberg cubic algebras; Clifford algebras, Division Algebras; Standard Model.

*Dedicated to the loving memory of Chris Baker

1 Introduction

In a recent paper [6] the basis states of the minimal left ideals of the complex Clifford algebra $Cl(8, C)$ were shown to contain three generations of Standard Model fermion states, with full Lorentzian, right and left chiral, weak isospin, spin, and electro-color degrees of freedom. The left adjoint action algebra of $Cl(8, C) \cong C(16)$ on its minimal left ideals contains the Dirac algebra, weak isopin and spin transformations. The right adjoint action algebra on the other hand encodes the electro-color $U(3)$ symmetries.

These results extend earlier work in the literature that shows that the eight minimal left ideals of $Cl(6, C) \cong C(8)$ contain the quark and lepton states of one generation of fixed spin. The key behind the construction of [6] was the *triality* automorphism of the $Cl(8)$ algebra which allows the extension from a single generation of fermions to exactly three generations. This triality automorphism of $Spin(8)$ permutes the two spinor and fundamental vector representations, all three of which are eight-dimensional.

Dixon [7] many years ago proposed an algebraic design of the fundamental particles in Physics showing the key role that the composition algebra (the Dixon algebra) $\mathbf{T} = \mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ had in the architecture of the Standard Model. More recently, it has been shown by Furey how the $\mathbf{C} \otimes \mathbf{O}$ algebra acting on itself allows to find the Standard Model particle representations [8]. A geometric approach to the physics of the standard model and gravity based on Noncommutative Geometry can be found in [9]. Other original work on the role of Octonions, Exceptional Jordan algebras in Physics can be found in [1], [2], [3], [5], [13], [18], [22], [14].

A geometric basis for the Standard Model gauge group based on the real $Cl(7, R)$ algebra was found earlier on by [10]. Since dim $Cl(7, R) = 128$ coincides with the real dimension of the complex $Cl(6, C)$ algebra, it was shown later on by [8], [11] that the eight minimal left ideals of the algebra of 8×8 complex matrices $C(8) \cong Cl(6, C)$ contains the 64 elementary fermion states of one generation of fixed spin, including their antiparticles.

The authors [6] displayed in detail how the sixteen minimal left ideals (spinors) of the 2^8 complex-dimensional Clifford algebra $Cl(8, C)$ can be represented by sixteen *column* vectors in the isomorphic matrix algebra $C(16)$ of 16×16 matrices over the complex numbers C. They showed how the action of left and right matrix multiplication differs. Left multiplication of a minimal left ideal column-matrix (a square matrix with one non-zero column) interchanges rows (along the column), and hence produces transformations within the minimal left ideals themselves. In contrast, right multiplication of a minimal left ideal by an arbitrary matrix in $C(16)$ interchanges columns, and hence transforms between different minimal left ideals.

The $Cl(8, C)$ spinor index A in $\Psi_A, 1, 2, \cdots, 16$ can be decomposed into a $Cl(4, C) \otimes Cl(2, C) \otimes Cl(2, C)$ form as follows $\Psi_{\alpha,a,b}$ with $\alpha = 1, 2, 3, 4$; $a = 1, 2$; $b = 1, 2$. The spinorial index $\alpha = 1, 2, 3, 4$ is associated to the complex Dirac algebra $Cl(4, C)$. The index $a = 1, 2$ belonging to the first $Cl(2, C)$ algebra corresponds to the weak isospin $SU(2)$; and the index $b = 1, 2$ belonging to the second $Cl(2, C)$ algebra labels a family-doublet. There are 3 pairings of family-doublets among the 3 generations given by the electron/muon $\Psi^{(e,\mu)}_{\alpha,a}$, electron/tau $\Psi^{(e,\tau)}_{\alpha,a}$, and muon/tau $\Psi^{(\mu,\tau)}_{\alpha,a}$ family doublets, respectively. Under the *triality* automorphism of the $Cl(8, C)$ algebra, these 3 family doublets are rotated into each other in a cyclical way. For further details we refer to [6].

The authors [12] have shown that the Exceptional Jordan algebra $J_3[\mathbf{O}]$ of Hermitian 3×3 octonionic matrices can describe the *internal* space of the fundamental fermions of the Standard Model with 3 generations. An *additional* conjugate Jordan algebra $\bar{J}_3[\mathbf{O}]$ must be introduced in order to describe their *antiparticles*. The pair of Jordan algebras, $J_3[\mathbf{O}]$ and its conjugate $\bar{J}_3[\mathbf{O}]$, globally behave like the $\mathbf{3}, \bar{\mathbf{3}}$ dimensional representations of the complex $su(3)$ algebra [**?**].

The Jordan algebra $J_2[\mathbf{O}]$ of Hermitian 2×2 octonionic matrices is relevant for the description of the internal space of the fundamental fermions of one generation. Once again, *triality* was instrumental to incorporate the internal space of the 3 generations which avoids the introduction of new fundamental fermions and where there is no problem with respect to the electroweak symmetry [12].

The 3 subalgebras $\mathcal{J}_i, i = 1, 2, 3$ of $J_3[\mathbf{O}]$ isomorphic to $J_2[\mathbf{O}]$ were associated to the 3 complete generations of fundamental fermions. $\mathcal{J}_1$ consists of the matrices of $J_3[\mathbf{O}]$ having vanishing elements in the first row and the first column. $\mathcal{J}_2$ consists of the matrices having vanishing elements in the second row and the second column while $\mathcal{J}_3$ consists of the matrices having vanishing elements in the third row and the third column.

$\mathcal{J}_1$ is associated to the first generation containing the leptons e and ν_e. $\mathcal{J}_2$ is associated to the second generation containing the leptons μ and ν_μ, and $\mathcal{J}_3$ is associated to the third generation containing the leptons τ and ν_τ.

The automorphism groups of $J_3[\mathbf{O}]$ and $\mathcal{J}_2[\mathbf{O}]$ are F_4 and $Spin(9)$ respectively. The intersection in F_4 of $Spin(9)$ with $SU(3) \times SU(3)/Z_3$ is precisely the Standard Model group $G_{SM} = SU(3) \times SU(2) \times U(1)/Z_6$ [12]. The first $SU(3)$ factor is common for all $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ and is the color group $SU(3)_c$. The second $SU(3)$ factor projects for each of the 3 generations to its electroweak symmetry $U(2)$. It is why its natural interpretation is that of the extended electroweak symmetry of the Standard Model with 3 generations and was denoted by $SU(3)_{ew}$ in [12].

Thus the gauge theory giving the interactions is based on $SU(3)_c \times SU(3)_{ew}/Z_3$ as structure group with the gauge part $SU(3)_{ew}$ broken via the Higgs mechanism. It is expected that the corresponding Higgs fields are parts of the connection in the quantum directions [12]. With this brief introduction we turn next to our main construction.

2 The $J_3[\mathbf{C} \otimes \mathbf{O}] \otimes \mathbf{Cl(4, C)}$ Algebra and Three Generations

We show next that the model described in this letter can account for all the degrees of freedom of three generations of fermions in addition to the Standard Model gauge symmetries. Our novel approach in this work is based on the algebra $J_3[\mathbf{C} \otimes \mathbf{O}] \otimes \mathbf{Cl(4, C)}$, where we include explicitly the Dirac spinorial degrees of freedom of the fermions, and their antiparticles in $4D$. Let us denote the 3×3 **block** matrix by $\mathbf{J}_{3,\alpha\beta}$

$$\mathbf{J}_{3,\alpha\beta} \equiv \begin{pmatrix} \lambda_1^A \ \Gamma_{\alpha\beta}^A & \Phi_1^A \ \Gamma_{\alpha\beta}^A & \bar{\Phi}_2^A \ \Gamma_{\alpha\beta}^A \\ \bar{\Phi}_1^A \ \Gamma_{\alpha\beta}^A & \lambda_2^A \ \Gamma_{\alpha\beta}^A & \Phi_3^A \ \Gamma_{\alpha\beta}^A \\ \Phi_2^A \ \Gamma_{\alpha\beta}^A & \bar{\Phi}_3^A \ \Gamma_{\alpha\beta}^A & \lambda_3^A \ \Gamma_{\alpha\beta}^A \end{pmatrix}. \ A = 1, 2, \cdots, 16; \ \ \alpha, \beta = 1, 2, 3, 4 \tag{1}$$

where each block is comprised of 4×4 matrices belonging to the $Cl(4, C)$ algebra. The components $\Phi_1^A, \Phi_2^A, \Phi_3^A$ are complex-octonionic-valued, and $\lambda_1^A, \lambda_2^A, \lambda_3^A$ are complex-valued. A summation over the A index from 1 to 16 is implied in (1).

The bar operation in the entries of the matrix in eq-(1) denotes *octonionic* conjugation $e_a \rightarrow -e_a, a = 1, 2, \cdots, 7$ associated with the 7 imaginary units of the octonions. We should emphasize that there is **no** complex conjugation appearing in the entries of (1). For example, there is no complex conjugation of the 16 Gamma matrices $\Gamma_{\alpha\beta}^A$ associated with the 16-dim Clifford algebra $Cl(4, C)$, nor in the diagonal coefficients $\lambda_1^A, \lambda_2^A, \lambda_3^A$.

We shall show that the three generations of fermions can be assigned to the complex-octonionic entries of $\mathbf{J}_{3,\alpha\beta}$. In order to incorporate their *antiparticles* one requires to include the conjugate $\bar{\mathbf{J}}_{3,\alpha\beta}$ matrix which is obtained by taking the ordinary complex conjugate of the entries of $\mathbf{J}_{3,\alpha\beta}$ in eq-(1).

In order to establish this correspondence one needs to start with the following spinors (one for each generation) with complex-valued entries such that

$$(\xi \ \xi^T)_{\alpha\beta} = \begin{pmatrix} \xi^{(1)} \\ \xi^{(2)} \\ \xi^{(3)} \\ \xi^{(4)} \end{pmatrix} \begin{pmatrix} \xi^{(1)} & \xi^{(2)} & \xi^{(3)} & \xi^{(4)} \end{pmatrix} = \sum_{A=1}^{16} \lambda^A \ \Gamma_{\alpha\beta}^A \tag{2}$$

and the spinors (one for each generation) with complex-octonionic entries such that

$$(\Psi \ \Psi^T)_{\alpha\beta} = \begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \\ \Psi^{(3)} \\ \Psi^{(4)} \end{pmatrix} \begin{pmatrix} \Psi^{(1)} & \Psi^{(2)} & \Psi^{(3)} & \Psi^{(4)} \end{pmatrix} = \sum_{A=1}^{16} \Phi^A \ \Gamma_{\alpha\beta}^A \tag{3}$$

Setting aside the trivial zero elements of the complex octonions, these equations establish a one-to-one correspondence among the entries of $\mathbf{J}_{3,\alpha\beta}$ in eq-(1) with the entries of the spinors in eqs-(2-3). These relations are just a manifestation of the fact that spinors have a mixed Clifford grade : spinors are made of a Clifford scalar, vectors, bivectors, trivectors, $\cdots$.

Each spinor $\Psi_{j,\alpha}$, where $j = 1, 2, 3$ labels the three generations, has $8\times2\times4 = 64$ real components that match the number of spinorial degrees of freedom of each fermion generation in $4D$. Their antiparticles have also 64 spinorial degrees of freedom bringing the total to 128. Namely, the 16 fermions of the first generation are $\nu_e, e, u^r, u^b, u^g, d^r, d^b, d^g$, plus their *antiparticles*, given by the electron neutrino, electron, up red/blue/green quark, down red/blue/green quark, and their antiparticles. The 4 complex entries of the Dirac spinors $\Psi^{(f)}_\alpha$ in $4D$ corresponding to each fermion $f = 1, 2, \cdots, 8$ leads to $8 \times 4 \times 2 = 64$ real degrees of freedom. By including their *antiparticles* yields a total of 128 real spinorial degrees of freedom for each generation in $4D$.

Given the algebra $\mathbf{C} \otimes \mathbf{O} \cong \mathbf{O} \oplus \mathbf{O}$ one then has two copies of $\mathbf{O}$. The Hilbert space of the states of the leptons and quarks with three colors red, blue and green is $\mathbf{C} \oplus \mathbf{C^3}$ [4]. From the correspondence described in [12] $\mathbf{C} \oplus \mathbf{C^3} \leftrightarrow \mathbf{O}$ we then have that one copy of $\mathbf{O}$ corresponds to the electron neutrino ν_e and the up quarks with three colors u^r, u^b, u^g. And the second copy of $\mathbf{O}$ corresponds to the electron e and the down quarks with three colors d^r, d^b, d^g. This will allow us to find the strict correspondence among the complex-octonionic-valued spinors $\mathbf{\Psi}$ and the Standard Model fermions. In the Clifford algebra based model by [24] octonions are also used to describe the fermion generations.

Denoting the right/left handed components of the Dirac spinors by R, L, and spin up/down by $1, 2 =\uparrow, \downarrow$, the one-to-one correspondence among the 8 fermions (particles) of the first generation with the complex-octonionic entries $\mathbf{\Psi}$ of (3) is given by

$$\Psi^{(1)} \leftrightarrow \begin{pmatrix} \nu_{e,R1} & u^r_{R1} & u^b_{R1} & u^g_{R1} \\ e_{R1} & d^r_{R1} & d^b_{R1} & d^g_{R1} \end{pmatrix}; \ \Psi^{(2)} \leftrightarrow \begin{pmatrix} \nu_{e,R2} & u^r_{R2} & u^b_{R2} & u^g_{R2} \\ e_{R2} & d^r_{R2} & d^b_{R2} & d^g_{R2} \end{pmatrix} \tag{4}$$

$$\Psi^{(3)} \leftrightarrow \begin{pmatrix} \nu_{e,L1} & u^r_{L1} & u^b_{L1} & u^g_{L1} \\ e_{L1} & d^r_{L1} & d^b_{L1} & d^g_{L1} \end{pmatrix}; \ \Psi^{(4)} \leftrightarrow \begin{pmatrix} \nu_{e,L2} & u^r_{L2} & u^b_{L2} & u^g_{L2} \\ e_{L2} & d^r_{L2} & d^b_{L2} & d^g_{L2} \end{pmatrix} \tag{5}$$

The correspondence with the *antiparticles* of the first generation requires *swapping* the up $\leftrightarrow$ down entries of the $SU(2)$ doublets, the Left $\leftrightarrow$ Right chirality of the spinors [4], and taking the complex conjugation $\Psi \leftrightarrow \Psi^*$:

$$\Psi^{*(1)} \leftrightarrow \begin{pmatrix} e^+_{L1} & \bar{d}^r{}_{L1} & \bar{d}^b{}_{L1} & \bar{d}^g{}_{L1} \\ \bar{\nu}_{e,L1} & \bar{u}^r{}_{L1} & \bar{u}^b{}_{L1} & \bar{u}^g{}_{L1} \end{pmatrix}; \ \Psi^{*(2)} \leftrightarrow \begin{pmatrix} e^+_{L2} & \bar{d}^r{}_{L2} & \bar{d}^b{}_{L2} & \bar{d}^g{}_{L2} \\ \bar{\nu}_{e,L2} & \bar{u}^r{}_{L2} & \bar{u}^b{}_{L2} & \bar{u}^g{}_{L2} \end{pmatrix} \tag{6}$$

$$\Psi^{*(3)} \leftrightarrow \begin{pmatrix} e^+_{R1} & \bar{d^r}_{R1} & \bar{d^b}_{R1} & \bar{d^g}_{R1} \\ \bar{\nu}_{e, R1} & \bar{u^r}_{R1} & \bar{u^b}_{R1} & \bar{u^g}_{R1} \end{pmatrix}; \ \Psi^{*(4)} \leftrightarrow \begin{pmatrix} e^+_{R2} & \bar{d^r}_{R2} & \bar{d^b}_{R2} & \bar{d^g}_{R2} \\ \bar{\nu}_{e, R2} & \bar{u^r}_{R2} & \bar{u^b}_{R2} & \bar{u^g}_{R2} \end{pmatrix} \quad (7)$$

Since $\Psi^{(1)}, \cdots$, and $\Psi^{*(1)}, \cdots$ are complex-octonionic valued, they have 16 real components each, and which match the number of $8 \times 2 = 16$ real components associated with each single one of the 8 sets of 8 complex-valued entries appearing in the right-hand side of eqs-(4-7).

Repeating this assignment with the other complex-octonionic entries of $\mathbf{J}_{3,\alpha\beta}$, and $\bar{\mathbf{J}}_{3,\alpha\beta}$ leads to the correspondence with the 8 fermions, and their antiparticles, of the second and third generation. Therefore, the basis of quantum states for the fermions and their antiparticles of the Standard Model, including *all* their spinorial degrees of freedom, can be described in terms of the off-diagonal complex-octonionic entries of the Jordan *pair* of $(\mathbf{J}_{3,\alpha\beta}, \bar{\mathbf{J}}_{3,\alpha\beta})$ block matrices. This construction extends the work of [12] by including the antiparticles and the explicit spinorial degrees of freedom.

The $Cl(6, C)$ algebraic assignment of the fermions of one generation described by [7], [8], [11] was succintly summarized by [6] via the left and right action of $Cl(6, C)$ on the primitive idempotent $\mathbf{p} \equiv \gamma_1\gamma_2\gamma_3\gamma_3^\dagger\gamma_2^\dagger\gamma_1^\dagger$, $\mathbf{p}^2 = \mathbf{p}$:

$$Cl(6, C) \ \rightarrow \ \mathbf{p} \ \leftarrow \ Cl(6, C) \quad (8)$$

The left action of $Cl(6, C)$ on $\mathbf{p}$ generates the 8 states of the first ideal $\mathbf{P}_1$ and which can be assembled into the 8 entries of the first column. While the right action on the first column generates the remaining 7 ideals $\mathbf{P}_2, \mathbf{P}_3, \cdots \mathbf{P}_8$. The full process will lead to $8 \times 8 = 64$ entries associated with the 64 states represented by the 64 complex entries in eqs-(4-7). The latter 64 complex valued entries amount to the 128 real spinorial degrees of freedom of one fermion generation, and which match the $dim_R \ Cl(6, C) = 2 \times 2^6 = 128$. More recently, the authors [6] extended this procedure to the $Cl(8, C)$ case to account for the three fermion generations via triality.

We continue this discussion by focusing on the Jordan products. The commutative but non-associative Jordan product $X \circ Y$ of two Jordan matrices is given by the following anticommutator

$$X \circ Y \ \equiv \ \frac{1}{2} \ (X \ Y \ + \ Y \ X) \quad (9)$$

and obeying the Jordan identity $(X \circ Y) \circ X^2 = X \circ (Y \circ X^2)$.

If one were to define the Jordan product of matrices given by the tensor products $X^A \otimes \Gamma_A$ and $Y^B \otimes \Gamma_B$ as

$$\mathbf{X} \circ \mathbf{Y} \ = \ \frac{1}{2} \ \{\mathbf{X}, \ \mathbf{Y}\} \ = \ \frac{1}{2} \ \{ \ X^A \otimes \Gamma_A, \ Y^B \otimes \Gamma_B \ \} =$$

$$\frac{1}{4} \ \{X^A, \ Y^B\} \otimes \{\Gamma_A, \ \Gamma_B\} \ + \ \frac{1}{4} \ [X^A, \ Y^B] \otimes [\Gamma_A, \ \Gamma_B] \ =$$

$$\mathbf{Z} \ = \ Z^C \otimes \Gamma_C \tag{10}$$

this would lead to a 3×3 matrix

$$Z^C \ = \ \frac{1}{4} \ d_{AB}^{\ \ C} \ \{X^A, \ Y^B\} \ + \ \frac{1}{4} \ f_{AB}^{\ \ C} \ [X^A, \ Y^B] \tag{12}$$

which is *no* longer Hermitian because when $d_{AB}^{\ \ C}, f_{AB}^{\ \ C}$ are the real-valued structure constants of the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} \ = \ d_{AB}^{\ \ C} \ \Gamma_C, \ \ [\Gamma_A, \Gamma_B] \ = \ f_{AB}^{\ \ C} \ \Gamma_C \tag{13}$$

the 3×3 matrix Z^C in eq-(12) will be comprised of a Hermitian *plus* an anti-Hermitian matrix, respectively, when X^A, Y^B (for $A, B = 1, 2, \cdots, 16$) are 3×3 Hermitian Jordan matrices. The anti-commutator of two Hermitian matrices is Hermitian, while the commutator is anti-Hermitian. Therefore, the product in eq-(10) does *not* lead to a matrix of the form $J_3^C \otimes \Gamma_C$ because the 3×3 matrix Z^C is *not* Hermitian. Furthermore, a careful inspection reveals also that the product (11) is *not* consistent with the Jordan identity $(\mathbf{X} \circ \mathbf{Y}) \circ \mathbf{X}^2 = \mathbf{X} \circ (\mathbf{Y} \circ \mathbf{X}^2)$.

For this reason one must *modify* the product (10) so that the 3×3 matrix Z^C is Hermitian. Given $\mathbf{X} = X^A \otimes \Gamma_A$, and $\mathbf{Y} = Y^A \otimes \Gamma_A$ described by eq-(1), the modified Jordan product $\bullet$ is now defined as

$$\mathbf{X} \bullet \mathbf{Y} \ \equiv \ \frac{1}{4} \ d_{AB}^{\ \ C} \ \{X^A, \ Y^B\} \otimes \Gamma_C \ = \ \mathbf{Z} \ = \ Z^C \otimes \Gamma_C \tag{14}$$

where X^A, Y^B and Z^C (for $A, B, C = 1, 2, \cdots, 16$) are 3×3 Hermitian Jordan matrices. Despite that one has attained *closure* in the product $\mathbf{X} \bullet \mathbf{Y} = \mathbf{Z}$, with $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in J_3^A \otimes \Gamma_A$, one should point out that this modified product $\bullet$ does *not* obey the Jordan identity

$$(\mathbf{X} \bullet \mathbf{Y}) \bullet \mathbf{X}^2 \ = \ \mathbf{X} \bullet (\mathbf{Y} \bullet \mathbf{X}^2) \tag{15}$$

It remains to explore further the physical role of the algebra $J_3[\mathbf{C} \otimes \mathbf{O}] \otimes \mathbf{Cl(4, C)}$ beyond the mere counting of degrees of freedom associated to a vector space. Exceptional Jordan F_4, E_6 (Chern-Simons-like) matrix models involving similar tensor products of Jordan matrices $J_3[\mathbf{O}], J_3[\mathbf{C} \otimes \mathbf{O}]$ with $u(N)$-valued matrix generators were constructed by [15], [16]. F_4 and E_6 are the automorphism groups of the Jordan algebras $J_3[\mathbf{O}], J_3[\mathbf{C} \otimes \mathbf{O}]$, respectively. The large N limit of these Exceptional Jordan matrix models and its relation to a non-perturbative bosonic formulation of M-theory in $D = 27$ was analysed by [17].

Following these findings we shall repeat their procedure and construct an action by replacing the $u(N)$ algebra with the Clifford $Cl(4, C)$ one. Given the structure constants d_{ABC}, f_{ABC} of the Clifford algebra, one candidate action is given by the Jordan trace of the cubic form

$$S_1 \ = \ (X^A, \ X^B, \ X^C) \ d_{ABC} \ = \ tr_3(X^A \circ (X^B \times_F X^C)) \ d_{ABC} \tag{16}$$

involving the Jordan product $\circ$, and the symmetric Freudenthal product

$$Y \times_F Z \;=\; Y \circ Z \;-\frac{1}{2}\; tr_3(Y) Z \;-\frac{1}{2} tr_3(Z) Y \;+\frac{1}{2}\; tr_3(Y)\; tr_3(Z) \;-\frac{1}{2} tr_3(Y\circ Z)\; \mathbf{1} \tag{17}$$

Another candidate action is

$$S_2 \;=\; (\rho^2(X^{[A}),\; \rho(X^B),\; X^{C]})\; f_{ABC} \tag{18}$$

where $\rho, \rho^3 = 1$ is the cycle mapping based on the triality symmetry of $SO(8)$ that takes the index $I \to I+1$ modulo 3. This cycle mapping is essential otherwise the action (18) would be identically zero due to the fact that the cubic form is symmetric in its entries while f_{ABC} is antisymmetric. A more general action is given by the sum $S_1 + S_2$ of eqs-(16,18).

Ohwashi [16] has shown that the cubic action (18) (when the structure constants f_{ABC} correspond to the $u(N)$ matrix algebra) is invariant under global rigid E_6 transformations which are encoded as automorphisms of the $J_3[\mathbf{C} \otimes \mathbf{O}]$ algebra under the transformations $\mathbf{J} \to \alpha \mathbf{J}$, where α is a 3×3 matrix whose entries are numerical constants. The action also posseses the $u(N)$ gauge symmetry resulting from $f_{ABC} = 2Tr_N(\mathbf{T_A}[\mathbf{T_B}, \mathbf{T_C}])$; the cycle symmetry with respect to the fields, and the matrix translation symmetry with respect to the diagonal part of the fields. The explicit components of the complex-valued action (18) can be found in [16]. The real-valued Smolin action was based on the trilinear form $tr_3(X \circ (Y \circ Z)$ instead of the cubic form associated with the Freudenthal product $tr_3(X \circ (Y \times Z)$.

In this letter the $u(N)$ symmetry in [15], [16], [17] is replaced by the Clifford algebra $Cl(4, C) \simeq M(4, C)$ where $M(4, C)$ is the 4×4 matrix algebra over the complex numbers. Given that the algebra $M(4, C) \simeq gl(4, C)$ is also the complexification of $u(4)$ ($sl(4, C)$ is the complexification of $su(4)$), the $Cl(4, C)$ algebra can be decomposed into two copies of $u(4)$: $Cl(4, C) \simeq u(4) \oplus u(4)$. And the latter $u(4) \oplus u(4)$ algebra is large enough to accomodate the Standard Model and Pati-Salam algebras, respectively, $su(3) \oplus su(2) \oplus u(1)$ and $su(4) \oplus su(2) \oplus su(2)$.

Concluding, the model described in this letter can account for all the degrees of freedom of three generations of fermions and the Standard Model gauge symmetries. It remains to find a physical interpretation to the 3 diagonal entries of the matrix in eq-(1), and which can be expressed in terms of the spinors ξ (3 spinors in total), as shown in eq-(2). Since these 3 spinors ξ_1, ξ_2, ξ_3 lie outside the Standard Model, it is tempting to speculate that they could be dark matter candidates. Since we do not have supersymmetry in our model these spinors cannot be Higgsinos.

3 Vinberg algebras and Exceptional Periodicity

There is yet another approach to incorporate spinorial degrees of freedom within the context of Jordan algebras. The authors [20], [21] have shown that the exceptional Jordan algebras are part of an infinite family of finite-dim matrix algebras corresponding to a particular class of cubic (ternary) Vinberg's **T**-algebras [23]. In particular, $\mathbf{e}_8$ is part of an infinite family of non-Lie algebras coined "Magic Star" algebras $\mathbf{e}_8^{(n)}$ for $n = 2, 3, \cdots$ and that resemble lattice vertex algebras [20], [21]. Only at level $n = 1$ one recovers the Lie-algebra $\mathbf{e}_8^{(1)} \equiv \mathbf{e}_8$. At higher levels the magic star (MS) algebras of exceptional periodicity (EP) are non-Lie because the Jacobi identity does not hold on their non-reductive components [20].

A generic element of the rank-3 algebra $\mathbf{T}_3^{8,n}$, at level n, can be written in terms of the 3×3 Hermitian matrix as follows

$$\begin{pmatrix} \lambda_1 & \lambda_v & \bar{\lambda}_{s^+} \\ \bar{\lambda}_v & \lambda_2 & \lambda_{s^-} \\ \lambda_{s^+} & \bar{\lambda}_{s^-} & \lambda_3 \end{pmatrix} \tag{19}$$

whose entries are 3 scalar diagonal elements $\lambda_1, \lambda_2, \lambda_3$; a vector λ_v in $8n$-dimensions, and the pair of conjugate Majorana-Weyl (MW) spinors $\lambda_{s^+}, \lambda_{s^-}$ with 2^{4n-1} real components. $\bar{\lambda}_v, \bar{\lambda}_{s^\pm}$ denote the vectors and spinors in the *dual* space with respect to the appropriate bilinear forms [20].

Given two elements $\mathbf{X}, \mathbf{Y}$ of $\mathbf{T}$ the Vinberg ternary algebraic operation is defined by $\mathbf{X} \circ \mathbf{Y} = \frac{1}{2}[\ [\mathbf{X},\ \mathbf{I}^-],\ \mathbf{Y}\ ]$ where I^- is a rank-3 element of the algebra. See [20] for the rigorous mathematical details of the construction of Exceptional Periodicity based on the Vinberg **T** algebra.

For our purposes, the relevant algebra is $\mathbf{T}_3^{8,2}$ corresponding to the level $n = 2$ and admits the following $SO(17,1)$-covariant Peirce decomposition [20]. [21]

$$\mathbf{T}_3^{8,2} \ = \ (\mathbf{17} + \mathbf{1}) \oplus (\mathbf{128}_+ \oplus \mathbf{128}_-) \oplus \mathbf{1} \tag{20}$$

The above Peirce decomposition is associated with a $\mathbf{128}_+$ Majorana-Weyl (MW) spinor and its conjugate $\mathbf{128}_-$ in a $17+1$-dim spacetime. The extra $\mathbf{1}$ element is a rank-1 idempotent of the $\mathbf{T}_3^{8,2}$ algebra. Therefore, at level $n = 2$, the 3 diagonal entries of the matrix in eq-(19) correspond to the *two* light-cone directions, and *one* spatial direction of a $18+1$-dim spacetime. Furthermore, those 3 directions can be assigned to the $2+1$-dim world-volume of a membrane embedded in the $18+1$-dim spacetime background [21].

This construction can be generalized to higher levels $n > 2$

$$\mathbf{T}_3^{8,n} \ = \ (\mathbf{8n} + \mathbf{2}) \oplus (\mathbf{2^{4n-1}}_+ \oplus \mathbf{2^{4n-1}}_-) \oplus \mathbf{1} \tag{21}$$

and where the 3 diagonal entries are now associated with the $2+1$-dim world-volume of a membrane embedded in a $(8n+2)+1$-dim spacetime background [21].

Following similar arguments as described above for Jordan matrices, the fermions will be described by $\mathbf{T}_3^{8,2}$ and their *antiparticles* by $(\bar{\mathbf{T}}_3^{8,2})$. A careful inspection of the pair of conjugate Majorana-Weyl (MW) spinors $\lambda_{s^+}, \lambda_{s^-}$ with 128 real components *each*, reveals that they match the number of real spinorial degrees of freedom corresponding to **4** generations of fermions in $4D$. The breakdown is $\mathbf{128}_\pm = \sum_{j=1}^4 \mathbf{32}_\pm^{(j)}$, where j labels the four generations $j = 1, 2, 3, 4$, and $\pm$ label the right/left handed components of each fermion. A similar countdown follows for their antiparticles.

Therefore, there is a one-to-one correspondence among the real spinorial degrees of freedom of **4** generations in $4D$ with the off-diagonal $\lambda_{s+}, \lambda_{s-}$ entries, plus their complex conjugates $\lambda^*_{s+}, \lambda^*_{s-}$, of the spinorial elements of the *pair* $\mathbf{T}_3^{8,n}, (\bar{\mathbf{T}}_3^{8,n})$ of Vinberg matrices at level $n = 2$. These results can be generalized to higher levels $n > 2$ leading to a *higher* number of generations beyond **4**. Three *pairs* of $\mathbf{T}$ algebras and their conjugates $\bar{\mathbf{T}}$ were essential in the Magic Star description/construction of Exceptional Periodicity [20] that extends $\mathbf{e}_8$ to $\mathbf{e}_8^{(n)}$ with n integer.

This counting of degrees of freedom is consistent with supersymmetry. Novel higher dimensional super Yang-Mills (SYM) theories were constructed by [21]. These SYM's were coined *exceptional* SYM's and based on the notion of Exceptional Periodicity (EP) [20]. One example of these exceptional SYM is the chiral $\mathcal{N} = (1, 0)$ Poincare superalgebra in a $17 + 1$-dim spacetime

$$\{Q_\alpha,\ Q_\beta\} \ = \ (\gamma^\mu)_{\alpha\beta}\ P_\mu \ + \ \gamma^{\mu_1\mu_2\cdots\mu_5} C^{-1}\ Z_{\mu_1\mu_2\cdots\mu_5} \ + \ \gamma^{\mu_1\mu_2\cdots\mu_9} C^{-1}\ Z_{\mu_1\mu_2\cdots\mu_9} \tag{22}$$

with C denoting the charge conjugation matrix. The rank-5 and rank-9 central charges (p-forms) correspond to electric 5-branes and their dual magnetic 9-branes in $D = 17 + 1 = 5 + 9 + 4$.

There are 16 transverse degrees of freedom of a bosonic membrane moving in $18 + 1$ spacetime dimensions ($19 - 3 = 16$). A massless YM field A_μ in $D = 17+1$ spacetime dimensions has also 16 transverse degrees of freedom, and its massless fermionic gluino superpartner Ψ_α is a MW spinor with $2^{9-2} = 128$ complex components (256 real components), and which match the number of $128_+ + 128_-$ real components of a MW spinor, and its conjugate, in the 16 transverse dimensions. This MW spinor and its conjugate, are precisely the spinors $\lambda_{s+}, \lambda_{s_-}$ discussed above and which is consistent with supersymmetry.

To conclude our brief discussion on $\mathbf{T}_3^{8,2}$ algebras, the spinorial degrees of freedom of **4** fermion generations in $4D$ match the spinorial degrees of freedom of the pair $\mathbf{T}_3^{8,2}, (\mathbf{T}_3^{8,2})$ of Vinberg matrices. The reason one does not have 3 generations is because *triality* is *lost*. A lot of work remains to be done. In particular, to generalize the construction of the magic star of Exceptional Periodicity to the case of Superalgebras [19], the study of novel dark matter proposals, the Leech lattice, the Monster and the Moonshine structure of M, F-theory and beyond [25].

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