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Title :THE PROBLEM P=NP (version 4)

ABSTRACT:

In this article, we are going to solve the problem P=NP for a particular kind of problems called *basic problems of numerical determination*. We are going to propose 3 fundamental Axioms permitting to solve the problem P=NP for basic problems of numerical determination, those Axioms can also be considered as pure logical assertions, intuitively evident and never contradicted, permitting to understand the solution of the problem P=NP for basic problems of numerical determination. We will see that those Axioms imply that the problem P=NP in undecidable for basic problems of numerical determination. Nonetheless we will see that it is possible to give a theoretical justification (which is not a classical proof) of the proposition "P≠NP". We will then study a 2nd problem, named "P_N=DP_N problem" analogous to the problem P=NP but which is fundamental in mathematics.

I)INTRODUCTION

In this article, we are going to give a solution to the problem P=NP. To begin with, we are going to define a class of problems called *basic problems of numerical determination*. This definition is important because it contains a very general kind of problems that are potentially of class P or of class NP, and consequently because it constitutes a very concrete basis permitting to justify intuitively the Axioms that we are going to introduce, and also to test their validity. We remind that the conjecture P=NP (Any problem belonging to class P belongs to class NP and conversely) has never been proved nor its negation P≠NP. In this article we are going to propose 3 assertions of pure logic, intuitively evident and never contradicted, called *Axioms* for this reason, permitting to give a solution to the problem P=NP for basic problems of numerical determination. It is legitimate in a mathematical theory to introduce Axioms, propositions that are evident or own an intuitively evident justification and that have never been contradicted. Moreover, the fact that none fundamental results linked to the problem P=NP have ever been obtained using classical mathematical theories suggests that obtaining the solution of the problem P=NP needs to introduce and to use new Axioms, and cannot be obtained using only classical mathematical theories. It seems to be evident that the Axioms that we are going to admit cannot be proved using classical mathematical theories.

The theory presented on this article is not a purely mathematical theory but a logical mathematical theory. It is very likely that the solution of the problem P=NP obtained by this theory could not be obtained by a purely mathematical theory (Not containing logical concepts). We can consider the theory exposed in this article as a logical theory. Nonetheless, it is apparently in agreement with all (established) results relative to the problem P=NP.

In this article, the approach of the problem P=NP is completely new and does not use any anterior published article concerning this problem. But we will see that its conclusion is in agreement with all articles previously published about this problem. We are going to justify that it is not possible to prove P=NP nor P≠NP for basic problems of numerical determination. Nonetheless, we will see that we can give a theoretical justification, which is not a classical proof but is based on the laws of random in number theory, of the proposition "P≠NP". We will then study a 2^{nd} problem, named "P_N=DP_N problem" analogous to the problem P=NP but which is fundamental in mathematics.

II)SOLUTION OF THE PROBLEM P=NP

A)BASIC PROBLEMS OF NUMERICAL DETERMINATION (DEFINITION).

By definition, *a basic problem of numerical determination* contains the following data:

For any natural n different from 0:

-A finite set A(n) defined as a function of n.

-An increasing function k(n) belonging to $F(\mathbf{N},\mathbf{N})$ (That is possibly constant), with a given polynomial P_k of $F(\mathbf{N},\mathbf{N})$ such that for any natural n $k(n)<P_k(n)$.

-In some cases r (r being a given number) finite sets $B_1(n),..,B_r(n)$ with for i belonging to $\{1,..,r\}$ $B_i(n)$ defined by a proposition of the kind $P_{Bi}(B_i(n),A(n),n)$. (Some B_i can also be defined simultaneously by a proposition of the kind $P_{Bi1,..,Bis}(B_{i1}(n),..,B_{is}(n),A(n),n)$. It can exist several possible $B_i(n)$ for a natural n, but the number of possible $B_i(n)$ must be finite).

-A proposition $P(a_1,..,a_{k(n)},A(n),n, k(n), B_1(n),..,B_r(n))$ defined for any sequence $(a_1,..,a_{k(n)})$ of elements of A(n).

We define an *explicit numerical function* as a function belonging to $F(A^p,B)$, with p natural >0, A,B=**N** or **Z**, (We will admit that we can replace A^p by the set S_{FA} whose the elements are the elements of A and the finite sequences whose the terms belong to A) defined explicitly with the classical operations in **N** or **Z** (addition, multiplication, power….) or using Int(f(n)) f being an increasing reel function tending towards infinity with n, and explicitly defined using classical reel functions (Log, exp..) and the classical operations in **R** (and symbols representing reels ex. "1/3", "3/7"…) r_{DE}, r_{DE} function defined by, p, q being 2 integers with q≠0 $r_{DE}(p,q)$ remainder of the Euclidean division of p by q, and the sum of the product of the terms of a finite sequence of integers (Represented by "Π" and "Σ"), or "absolute value"(Of an integer), or the symbols representing integers "0", "1", "2"….and the terms $a_1,\dots,a_p$ of an element of the starting set.

For instance with the starting set S_{FA} :

$f(a_1,\dots,a_p)=\Pi(a_1,\dots,a_p)$, represented par $f(a_1,\dots,a_p)=\Pi_{i=1,..,p}a_i$.

Starting set **N** :

$f(n)=\Pi(1,..,n)$ represented par $f(n)=\Pi_{i=1,..,n}i$ for n>0 and f(0)=0.

$f(n)=n^3+n^2+1$.

Starting set $\mathbf{N}^2$:

$f(a_1,p)=a_1^p$ except if $(a_1,p)=(0,0)$ then f(0,0)=0.

So it will always be possible to identify an explicit numerical function with an infinite subset of $A^p\times B$. In the case $A^p=\mathbf{N}$, f being an explicit numerical function element of *F*(**N**,**N**), then f will be identified with the infinite sequence (That is also a set) $((i,f(i)))_{\mathbf{N}}$.

We will admit that it will be always possible to find an equivalent problem in which the elements of A(n) and $B_i(n)$ are integers or finite families or finite sequences defined using integers and finite sequences or families. We will call a numerical basic problem of numerical determination such a problem. We will also assume that in this numerical problem, the propositions defining A(n), $B_i(n)$, k(n) and $P(a_1,..,a_{k(n)},A(n),n, k(n), B_1(n),..,B_r(n))$ use and define only sets or sequences whose the elements or the terms integers or are defined from integers and finite families or sequences (And consequently are themselves integers or finite family or finite sequence) and all the infinite sets that it use are either of the kind A^p or S_{FA} (A=**N** or **Z**) or are explicit numerical functions. We will say that the elements of the sets or the terms of the sequences used by a numerical proposition are *numerical elements* that can be defined by the 2 following rules:

a) Any integer is a numerical element.

b) Any family or finite set whose the elements are numerical elements is a numerical element.

There will be an exception: The previous propositions could use sets of the kind *F*($\mathbf{N}^p$,**N**) but only in order to define an explicit numerical function. Nonetheless we can avoid this exception replacing "f is element of *F*($\mathbf{N}^p$,**N**)" by "f is a mapping between $\mathbf{N}^p$ and **N**". (Those propositions are analogous to propositions defined in Section E and named *numerical propositions*. It will be possible to add conditions in order that they be numerical propositions.)

A basic problem of numerical determination cannot use mathematical algorithms, formal system or any equivalent mathematical logical concept. We take this condition in order to avoid logical singularities and also because all classical interesting problems linked to the problem P=NP verify this condition.

P_B being a basic problem of numerical determination, we will define $P_B(n)$ as the part of P_B relative to n (with $A(n)$,$B_i(n)$,$k(n)$…). We will say that $P_B(n)$ is the *order* n of P_B.

By definition solving this problem signifies that for each natural n, we determine (if it exists) a sequence $(a_1,..,a_{k(n)})$ of elements of A(n) verifying $P(a_1,..,a_{k(n)},A(n),n, k(n), B_1(n),..,B_r(n))$. (We will then say that $(a_1,…,a_{k(n)})$ is a solution of $P_B(n)$.).

(We will extend the previous definition of basic problems of numerical determination to similar problems with the difference that k(n) must also be determined with the condition $k(n)<P_k(n)$, P_k given polynomial of $F(\mathbf{N},\mathbf{N})$. What follows remains valid if we consider a much greater class of problems.

According to the definition of a problem of class P, P_B being a basic problem of numerical determination, we will say that P_B is *of class* P it exists a sequence of algorithms $(Alg_{DPB}(n))_{\mathbf{N}}$ (In what follows we will omit the sign "**N**" to represent that a sequence indexed on **N**), *polynomial* (Which means that it exists a polynomial P_{DPB} such that for any natural n, the running time of $Alg_{DPB}(n)$ is inferior to $P_{DPB}(n)$) and *determinist* which means that $Alg_{DPB}(n)$ determines a sequence $(a_1,..,a_{k(n)})$ solution of $P_B(n)$ (If it exists). $Alg_{DPB}(n)$ can naturally use the data of P_B (A(n), $B_i(n)$…We will say that $(Alg_{DPB}(n))$ is a *polynomial determinist sequence of algorithms solving* P_B.

According to the definition of class NP, P_B being a basic problem of numerical determination, we will say that P_B belongs to class NP if it belongs to class P or if its exists a polynomial sequence of algorithms $(Alg_{NDPB}(n))$ *non-determinist* which means that for any sequence $(a_1,..,a_{k(n)})$ of elements of A(n), $Alg_{NDPB}(n)$ permits to verify if $(a_1,..,a_{k(n)})$ is a solution of $P_B(n)$. We will then say that $(Alg_{NDPB}(n))$ is a *polynomial non-determinist sequence of algorithms verifying* P_B.

What follows remains valid if in previous definitions we do not take P is included in NP or if $Alg_{DPB}(n)$ must permit to obtain all the solutions of $P_B(n)$).

It exists some problems that are of class P or of class NP that are not basic problems of numerical determination but this latter kind of problems constitute most of the interesting and classical of problems of class P or class NP.

For instance we can consider the classical basic problem of numerical determination consisting in finding if it exists 2 distinct divisors a_1 and a_2 of a natural n.

Then we have for this basic problem of numerical determination $A(n)=\{1,..,n\}$,$k(n)=2$ and $P(a_1,a_2,n)$: « a_1 et a_2 are 2 distinct naturals different from n and $a_1\times a_2=n$ ».

Concerning the example of the Clay mathematics institute [(1)] we consider 400 students, we have 100 rooms and a list L_P containing 100 pairs of students, and we want to find a list L_{RN} of 100 students with the condition that this list does not contain any pair of students belonging to L_P. This problem is equivalent to the following basic problem of numerical determination with n=400:

-A(n) is a set containing the names of n students (Represented by $e_1,..,e_n$). In the example, n=400.

-k(n)=Int(n/4) (k(400)=100 in the example.

-$B_1(n)$ is defined by $P_{B1}(B_1(n),A(n),n)$: "Card(B_1(n)=Int(n/4) and the elements of $B_1(n)$ are pairs of elements of A(n)". ($B_1(400)=L_P$ in the example).

-P($a_1,\dots,a_{k(n)}$,k(n),B_1(n)): "For any i,j distinct of {1,..,k(n)}, {a_i,a_j} does not belong to B_1 and $a_i \neq a_j$}. ($a_1,\dots.,a_{k(400)}$ are the elements of L_{RN} in the example).

(We can replace e_i by (1,i) to obtain an equivalent numerical problem.)

P=NP, for the basic problems of numerical determination, signifies that any basic problem of numerical determination belonging to class P belongs to class NP and conversely. We are going to prove, using assertions of pure logic intuitively evident and never contradicted that we called "Axioms", that this problem is *undecidable* which means:

-If P=NP, it is impossible to prove it.

-If P≠NP, it is impossible to prove it.

In what follows we will consider only basic problems of numerical determination.

B)IMPOSSIBILITY TO PROVE P≠NP FOR BASIC PROBLEMS OF NUMERICAL DETERMINATION.

In this section we will consider only basic problems of numerical determination.

In order to prove P≠NP, we must prove either than P is not included in NP either that NP is not included in P. According to the definition that we gave P is included in NP. Consequently we must prove that NP is not included in P, which means finding a problem of class NP that is not of class P.

But we admit the following Axiom:

AXIOM 1:

It is impossible to prove that a basic problem of numerical determination does not admit a polynomial determinist sequence of algorithms solving it (Meaning that it is not of class P).

This Axiom 1, that is not demonstrable formally as any axiom, admits the following intuitive justification:

By hypothesis, P_B being a basic problem of numerical determination, Card(A(n) and Card(B_i(n) will be defined by explicit numerical functions, $P(a_1,..,a_{k(n)},A(n),n, k(n), B_1(n),..,B_r(n))$ will also possibly use some explicit numerical functions. So $P(a_1,..,a_{k(n)},A(n),n, k(n), B_1(n),..,B_r(n))$ will express properties of naturals (Including properties of Card(A(n)), A(n) defined for any natural n or properties of explicit numerical functions). Because of this a sequence (Alg(n)) solution of P_B will possibly be defined using very complex explicit numerical functions. Intuitively we understand that it is very likely that the basic Axioms (Axioms of set theory, Peano's Axioms (Defining **N**) are insufficient for permitting to prove that it does not exist any solution for any given basic problem of numerical determination. Moreover, this Axiom 1 has never been contradicted, it has never been proved that a basic problem of numerical determination does not admit any solution, which would be necessary in order to prove the invalidity of this Axiom 1. We remind that any Axiom, even if is true does not admit formal proof.

So because of this Axiom 1, it is impossible to prove that a given basic problem of numerical determination is not of class P, and consequently to prove that NP is not included in P, and consequently to prove P≠NP for basic problems of numerical determination.

C)IMPOSSIBILITY TO PROVE P=NP FOR BASIC PROBLEMS OF NUMERICAL DETERMINATION.

In this section we will consider only basic problems of numerical determination.

In order to prove P=NP, we must prove that any problem of class NP is of class P.

But we admit the following Axiom:

AXIOM 2:

In order to prove that a basic problem of numerical determination is of class P, we must necessarily give a polynomial determinist sequence of algorithms solving it.

This Axiom 2 owns the following intuitive justification: It does not exist general Axioms permitting to justify that it exists a polynomial determinist sequence of algorithms solving a given basic problem of numerical determination without explicitly determining such a sequence or a way to obtain it. This is confirmed also by the fact that it has never been proved that a given basic problem of numerical determination was of class P by another way than giving explicitly a polynomial determinist sequence of algorithms solving it. As the Axiom 1 it has never been contradicted. In order to prove the invalidity of this Axiom 2, it should be necessary to prove that a basic problem of numerical determination belongs to class P without giving explicitly the polynomial determinist sequence of algorithms solving it (or a way to obtain this polynomial determinist sequence of algorithms).

A consequence of this Axiom 2 is the following logical assertion, that can also be considered as its 2nd part:

ASSERTION 1:

In order to prove that NP is included in P (For the basic problems of numerical determination), it will be necessary to provide a (general) polynomial algorithm permitting to determine, for any basic problem of numerical determination P_B of class NP, a polynomial sequence of algorithms solving P_B.

But it is evident that it will be impossible to find such a polynomial algorithm which we admit in the 3rd following Axiom:

AXIOM 3:

It does not exist a general algorithm permitting to determine, for any basic problem of numerical determination P_B, a polynomial determinist sequence of algorithms solving P_B.

Proving the invalidity of this Axiom would imply to find such a general algorithm, which seems to be completely impossible. Indeed it is evident that, we do not have enough information permitting to build a general algorithm obtaining a polynomial determinist sequence solving P_B for any P_B, even if we have P=NP. We have not a single line of such a general algorithm.

So we justified using the preceding Axioms 2,3 that it is impossible to prove P=NP, for basic problems of numerical determination because in order to prove P=NP we must prove that NP is included in P.

So we have as consequence of the preceding Axioms 1,2,3 the following theorem:

THEOREM 1:

Considering the class of the basic problems of numerical determination, the proposition P=NP is undecidable.

So we justified, using the previous Axioms 1,2,3 (That can also be considered as evident logical assertions) that even whatever P=NP be true or wrong for the basic problems of numerical determination, it is in both case impossible to prove it. Using the Axiom 1 and generalizing immediately Axioms 2 and 3, with the same intuitive justifications, we obtain that the problem P=NP is also undecidable for all basic problems of numerical determination with the condition upon k(n):$k(n)<n^3+100$. (We can replace n^3+100 by any non-constant polynomial of $F(\mathbf{N}_{,N})$ or take k(n) as a determinate function of $F(\mathbf{N},\mathbf{N})$ tending towards infinity when n tends towards infinity, for instance k(n)=Int(n/4) as in our 2nd example).

So as a consequence of the Axiom 1 and of the generalizations of Axioms 2 and 3 the following Corollary:

COROLLARY 1:

If we consider the class of basic problems of numerical determination P_B with the condition on $k_{PB}(n)$, $k_{PB}(n)<Pol(n)$, Pol non constant polynomial of $F(\mathbf{N},\mathbf{N})$, or $k_{PB}(n)=f(n)$, f

increasing function of n belonging to $F(\mathbf{N},\mathbf{N})$ tending towards infinity when n tends towards infinity, then also for this class of problems P=NP is undecidable.

So we justified using the introduced Axioms that if P=NP is true, we cannot prove it and if P≠NP, we also cannot prove it.

We remind that we can also consider the Axioms 1,2,3 as assertions of pure logic, never contradicted, that permit to solve the problem P=NP and understand why it is undecidable for basic problems of numerical determination. The fact that we considered only one kind of problems, basic problems of numerical determination, makes more easily to understand and test Axioms (Or logical assertions) 1,2,3. As usual Axioms, even if they are true, it is quasi-certain that Axioms (Or logical assertions) 1,2,3 cannot be demonstrated formally.

D)THEORETICAL JUSTIFICATION OF "P≠NP" BY THE RANDOM NUMBER THEORY.

We have proved, using the introduced Axioms, that for the basic problems of numerical determination, it was impossible to prove classically P=NP nor P≈NP. Nonetheless, it exists a theory, Random Number Theory (2), that can be used in order to give a theoretical justification (which is not a classical proof) of the proposition P≠NP. We have established (2) that this theory gave a theoretical justification to weak and strong GOLDBACH Conjectures and also to weak and strong twin primes Conjectures, and that it would be very possible that there are the only ones theoretical justifications of those Conjectures, classical proofs on those Conjectures or of their negations not existing. The Random Number Theory shows that the random laws in number theory can lead to justify the validity of some propositions for which it does not exist any classical proof nor for their negations. The Random Number Theory permits to obtain theoretical justifications of some propositions, using laws of random in numbers, permitting to understand the origin of the validity of those propositions, the same way as for classical proofs.

Using some elements of the Random Number Theory (That we are going to remind), we are going to expose a theoretical justification based on random in numbers, of the proposition P≠P.

The Random Number Theory is based on some propositions, named *random pseudo-Axioms*, which are analogous to classical Axioms but express the laws of random in number theory. Those random pseudo-Axioms bring to probabilistic models that can be valid with a good approximation but also completely wrong. Despite of this problem, random pseudo-Axioms appear to be the only one solution in order to use random laws in numbers, those ones permitting to obtain theoretical justifications for propositions in number theory for which it does not exist classical proofs nor for their negations. We will use the following random pseudo-Axiom:

Pseudo-Axiom 1:

If A is a finite set , B a subset of A, x an element of A, the proposition "x is element of B" is modeled by the event of the equiprobable probability space (A,P(A),peqA) "x is element of B".

Inside Random Number Theory, according to the definition of "is modeled by", that can exist between a proposition and an event of a probability space, if P is modeled by an event Ev, then Non(P) is modeled by the event Non(Ev).

If a proposition P is modeled by an event Ev whose the probability is very close to 1 (We will say that P owns a *random pseudo-proof*, which is the obtainment of "P is modeled by Ev" using classical Axioms and at least one random pseudo-Axiom), we assume in the Random Number Theory the fundamental and intuitively evident deduction process which leads to admit that P could be true as a consequence of its modelization by an event with a probability close to 1. So we will consider that a the random pseudo-proof of P is a theoretical justification based on the laws of random in number theory of the validity of P. More generally if a proposition Q is obtained using classical Axioms and Propositions owning a random pseudo-proof, we will say that the obtainment of Q is also a *random pseudo-proof*. Nonetheless, contrary to the case in which P has a classical proof, P will not be compulsory true if it owns a random pseudo-proof. It is only in the case in which it does not exist a classical proof of P nor of Non(P) (And therefore that none have ever be found) that a random pseudo-proof of P will be interesting. And it is the case for the proposition P=NP. Mod(P,Ev): "P is modeled by the event Ev", with p(Ev)≈1 will mean intuitively: "If the random laws used in order to obtain Mod(P,Ev) are valid with a sufficiently good approximation, then P behaves as it had a probability very close to 1 to be true".

We illustrate the concepts of random pseudo-proof and of a theoretical justification based on random the following way:

We consider a box containing 1000 balls, 999 red balls and one white ball. John draws a ball from the box without looking at it.

The random laws in ball's boxes are expressed by the following random pseudo-Axiom 2:

Pseudo-AXIOM 2:

If A is a finite set (or "box") containing some objects (ex.ball, cards..) and B is a subset of A, then if N. (ex. John) draws an object x of A without looking at it, then we will have: "The proposition "x is an element of B" is modeled by the event of the equiprobable probability space (A,*P*(A),p_{eq}) "x is an element of B"(Therefore with a probability equal to Card(B)/Card(A))" (The proposition "x is an element of B" will be equivalent to the proposition "N. draws a ball belonging to B")

According to the random laws of balls' boxes, if P is the proposition "John draws a red ball", P is modeled by an event Ev, with p(Ev)=0,999. This means that P behaves as it had a probability equal to 0,999 to be true. The obtainment of Mod(P,Ev), using the random laws of balls' boxes (Expressed by the pseudo-Axiom 2) will be called the *random pseudo-proof* of P. If P is true, it will be considered as a *theoretical justification based on random* of P.

Nonetheless in the preceding example, we are sure of the probabilistic model permitting to obtain Mod(P,Ev). It is generally not the case if P is a proposition in number theory, because then the probabilistic model could be valid with a good approximation or completely wrong.

Let H be the set of the basic problems of numerical determination of class NP written with a number of symbols (letters, figures, punctuation..) inferior to 10^5 (Therefore H is finite but Card(H) is high. We will assume Card(H)>10000). Let A=P(H). Let x be the subset of H containing all the basic problems of numerical determination of class P. So we have "x is element of A". Let B={H}. So B is included in A. According to the random pseudo-Axiom 1 we have:

The proposition "x is element of B" is modeled by the event Ev of $(A,\mathcal{P}(A),p_{eqA})$ "x is element of B" with $p_{eqA}(B)=1/Card(A)=1/2^{Card(H)}\approx 0$ because Card(B)=1 and Card(H)>10000.

But the proposition "x is element of B" is equivalent to the proposition "P=NP for the basic problems of numerical determination written with less than 105 symbols".

Therefore, Non("P=NP for the considered problems") is modeled by Non(Ev) with a probability very close to 1 and consequently the proposition "P≠NP for the considered problems" owns a random pseudo-proof. Using that "P≈NP for the considered problems" involves "P≠NP in the general case", "P≠NP in the general case" also owns a random pseudo-proof. So we proved that the laws of random in number theory could be the origin of the validity of the proposition P≠NP.

It is very likely that the preceding example of Clay mathematics Institute be a basic problem of numerical determination belonging to class NP and not belonging to class P, in agreement with the theoretical justification based on random that P≠NP. But it will be impossible to prove this classically if the Axiom 1 is valid.

We could also replace H by H/X, X being the set of the basic problems of numerical determination for which we proved they are of class P.

E)THE PROBLEM $P_N=DP_N$.

We are now going to study a 2^{nd} problem, named problem "$P_N=DP_N$", completely analogous to the problem P=NP in its resolution, but which is fundamental in mathematics.

We define the problem $P_N=DP_N$ the following way: We will say that a proposition P belongs to the class P_N (We will then say that P is a *numerical proposition*) if P is a mathematical proposition using exclusively, in addition to the symbol that it defines:

(i)Primitive relational concepts between propositions ("Non", "or", "is equivalent to"…)

(ii)Typical concepts of number theory: "is prime", "is prime with", "is pair", "divides", "0","1", "2"..,$\mathbf{N}^p$,$\mathbf{Z}^p$, "finite", " infinite", "is congruent modulo p" (Defined by, a, b being any integers: "a is congruent modulo p with b is equivalent to "It exists an integer k such that a-b=kp), explicit numerical functions, "Card", "=" (Between sets or integers or sequences), ">", "<" (Between integers), or "N_S" giving the number of terms of a finite sequence, or d_S(i,) i natural non nil, giving the ith term of a finite sequence with at least i terms.

(iii)Explicit numerical functions and expressions defining explicit numerical functions.

We remind that an *explicit numerical function* is a function belonging to $F(A^p,B)$, with p natural >0, A,B=**N** or **Z**, (We admit that we can replace A^p by the set S_{FA} whose the elements are the elements of A and the finite sequences whose the terms belong to A) defined explicitly with the classical operations in **N** or **Z** (addition, multiplication, power….).or using some expressions Int(f(n)), f increasing reel function tending towards the infinity with n, explicitly defined using classical reel functions (exp, Log..) and classical operations in **R**, (And symbols representing reels, ex. "1/3", "3/7"…) or the function r_{DE} or the product or the summation of the terms of a given finite sequence of integers (Represented by "Π" and "Σ"), or the absolute value (Of an integer) or the symbols representing integers "0", "1", "2"….So we can identify an explicit numerical function with an infinite set, and with a sequence indexed on **N** if A^p=**N**.

(iv)The classical concepts in set theory "is element of", "is included in", "is a (finite, infinite) set", "is a (finite) sequence (on an infinite subset of **N**)".

(v)The classical quantificators: "Whatever be"; "It exists", and "(is) defined by".

(vi)Infinite or finite sets B, subsets of A^p, A=**N** or **Z**, (p natural) defined by an expression "B is defined by $P(B,A^p,B_1,..,B_k,a_1,..,a_t)$", $P(B,A^p,..,B_k,a_1,..,a_t)$ mathematical proposition using points (i) to (v), the infinite sets B_j defined as B or equal to A^q (A=**N** or **Z**), numerical elements (meaning defined from integers and finite sets or sequences) defined by B, pre-defined numerical elements $a_1,..,a_t$, A^p, $B_1,..,B_k$ and as infinite sets only among B, A^p, $B_1,..,B_k$ and explicit numerical functions. The definition of B can also use none B_j or none a_j or not use A^p. We could suppress the condition that B must be a subset of A^p but then we will have always the condition expressed by point (ix).

(vii)Sets C(n) defined for n belonging to B, B subset of **N** defined as in (vi), with C(n) defined by an expression such as ""C(n) is defined by $P(C(n),n, C_1(n),..,C_l(n), B_1,..,B_k,a_1,..,a_t)$", $P(C(n), n, C_1(n),..,B_k,a_1,..,a_t)$ mathematical proposition using elements defined in points (i) to (v), numerical elements defined by C(n), $C_i(n)$, pre-defined numerical elements a1,..,at and the B_j, the sets B_j being defined as in (vi), the finite sets $C_i(n)$ defined as C(n) and as infinite sets only the B_j and some explicit numerical functions. The proposition could use none B_j or none a_j or none $C_i(n)$ or not n if it uses at least one $C_i(n)$.

If C(n) is defined using none $C_i(n)$, the numerical proposition could then use the sequences $(C(n))_B$ or $(Card(C(n))_B$.

(viii)Expressions of the kind "$\lim_{n\to\infty}$(Card(C(n))/f(n))=1", with $(C(n))_B$ sequence of finite sets defined as in (vii) and f explicit numerical function.

(ix) Any set or sequence used in a numerical proposition will have elements or terms that are defined using integers and finite families or sequences. We will say that the elements of the sets or the terms of the sequences used by a numerical proposition are *numerical elements* that can be defined by the 2 following rules:

a)Any integer is a numerical element.

b)Any family or finite set whose the elements are numerical elements is a numerical element.

(There is an exception, a numerical proposition could use sets of the kind $F(\mathbf{N}^p,\mathbf{N})$ in order to define an explicit numerical function (We can avoid this exception replacing "f is element of $F(\mathbf{N}^p,\mathbf{N})$" by "f is a mapping from $\mathbf{N}^p$ to $\mathbf{N}$".)

We admit that a numerical proposition can use numerical elements defined from sets A^p, A=**N** or **Z**, sets B as defined in point (vi), and sets C(n) and $(C(n))_B$ as defined in point (vii).

(x)A numerical proposition can only use (except in point (viii) as infinite sets (Named *numerical sets*) only A^p, S_{FA} (A=**N** or **Z**), sets B defined as in point (vi) or sequences defined as in point (vii). We can generalize point (viii) to numerical sets that are sequences of integers indexed on an infinite subset of **N**. We could obtain S_{FA} as a set B of the point (vi) without the condition "B is included in A^p".

Therefore, a numerical proposition also cannot use logical concepts such as "a proposition", "a proof"…or equivalent concepts, because such a proposition do not use concepts defined from points (i) to points (x).

For instance the BEZOUT Theorem, the FERMAT's Conjecture, the weak and strong GOLDBACH Conjectures are numerical propositions. But it is not the case for the CAILEY-HAMILTON Theorem, the CANTOR-BERSTEIN Theorem, the Continuum Hypothesis, the problems "P=NP" or "$P_N=D_{PN}$".

Nonetheless, if we consider matrix whose coefficients are integers, we can prove that the CAILEY-HAMILTON Theorem is equivalent (with the meaning "own the same signification, is interchangeable in its use with") to a numerical proposition. We remind that the CAILEY-HAMILTON Theorem is expressed by the proposition: "For any non nil natural n, for any reel matrix n×n A, $X_A(A)=0_{n\times n}$, with $X_A(x)$ is the polynomial $\det(A-xId_{n\times n})$."

We then define the following numerical proposition:

n being a non nil natural, we define classically the matrix n×n with integers coefficients:

$A(n)=\{((1,1),a_{11}),\ldots, ((n,n),a_{nn})\}$, $a_{11},..,a_{nn}$ being integers.

We then define the explicit numerical function $f_{XA(A)}$ from S_{FZ} to **Z** by:

For any sequence $(n,i,j,a_{11},\ldots,a_{nn})$, with n non nil natural, i,j non nil naturals inferior or equal to n, $a_{11},\ldots,a_{nn}$ being integers, then $f_{XA(A)}((n,i,j, a_{11},\ldots,a_{nn}))$ defines the term (i,j) of the matrix $X_A(A)$, as a function of $a_{11},\ldots,a_{nn}$. (We will admit as evident that $f_{XA(A)}$ exists, because $x_{i,j}$ is defined using $a_{11},..,a_{nn}$ and the operations in **Z** "+", "×", "-"…).

In order to define $f_{XA(A)}$, first we define the explicit numerical function g_{XA} such that for any natural i inferior or equal to n, $g_{XA}(n,i,a_{11},\ldots,a_{nn})$ is the coefficient of x^i in $X_A(x)$.

Then we define the explicit numerical function h_{Ap} such that if k is a non nil natural and i,j non nil naturals inferior or equal to n, $h_{Ap}(k,n,i,j,a_{11},\ldots,a_{nn})$ is the term (i,j) of $A(n)^k$. (h_{Ap} could be defined by induction on k).

Then using g_{XA} and h_{Ap}, we can define $f_{XA(A)}$.

For an element s of S_{FZ} that is not of the previous kind, we will take $f_{XA(A)})(s)=1$.

Then the CAILEY-HAMILTON Theorem for matrix with integers coefficients is equivalent to :

For any non nil natural n, for any i,j non nil naturals inferior or equal to n, for any integers $a_{11},\ldots,a_{nn}$, $f_{XA(A)}((n,i,j, a_{11},\ldots, a_{nn})=0$.

It is clear that we could enlarge the definition of a numerical proposition given by points (i) to (viii), for instance permitting to a numerical proposition to use sequences $(d(i,r))_{\mathbf{N}^*}$, r reel and d(i,r) ith decimal of r (numerical set). We can easily prove the fundamental result that if a sequence on an infinite subset B of **N** $(u(n))_B$ is a numerical set and for any n element of B u(n) belongs to the starting set of an explicit numerical function f, then $(f(u(n))_B$ is a sequence on B of integers that is also a numerical set. The point (ix) is necessary in order to obtain that any set used in a numerical proposition is numerable. Concerning numerical sets defined in point (viii) we saw that we had an exception if we use point (viii) because in that case we use sets that are not numerical sets (For instance division in **R** or in **Q**).

Nonetheless, point (viii) is not necessary. Indeed we name strictly numerical proposition a proposition defined as a numerical proposition, but suppressing point (viii). Then we can show that the proposition of point (viii) is equivalent (with the meaning, "owns the same signification, is interchangeable in its use with") to a strictly numerical proposition, which means that point (viii) is the consequence of the other points defining a strictly numerical proposition.

In order to show this, we define the sequence on **N*** $(\varepsilon_p)_{\mathbf{N}^*}$, with $\varepsilon_p=1/10^p$. We remind that the proposition of point (viii) signifies:

P1:"For any ε element of $\mathbf{R}^{*+}$, it exists N element of **N** such that for any n superior to N and element of B, $|Card(C(n))/f(n)-1|<\varepsilon$".

In order to obtain a strictly numerical proposition equivalent to P1, we replace "For any ε element of $\mathbf{R}^{*+}$" by "For any p element of $\mathbf{N}^*$" and "<ε" by "<$\varepsilon_p=1/10^p$". We then obtain immediately a strictly mathematical proposition equivalent to P1, multiplying the 2 terms of the strict inequality by $f(n)10^p$.

We can generalize what precedes, replacing $(Card(C(n))_B$ by a sequence $(u(n))_B$ that is a numerical set. Indeed, then u(n) is a numerical element that can be used in a strictly numerical proposition.

It is very likely that we can admit **Q** among numerical sets, and that an explicit numerical function from set $\mathbf{Q}^p$ or S_{FQ} to set **Q**. We will name such a function an *explicit numerical function of rationals* and explicit numerical functions for which the starting set is equal to A^p or S_{FA} (A=**N** or **Z**) *explicit numerical functions of integers*. The definition of an explicit numerical function of rationals will be completely analogous to the definition of an explicit numerical function of integers except for the function 'power" because we will admit that an explicit numerical function of rationals can use an expression a^p only if p is an integer. This condition is necessary in order that an equality (or an inequality) between 2 rationals defined using rationals and explicit numerical function of rationals be equivalent to an equality between 2 integers defined using integers and explicit numerical function of integers.

Nonetheless, it seems that any numerical proposition using **Q** as a numerical set or explicit numerical functions of rationals is equivalent to a strictly numerical proposition, the same way as for the proposition of point (viii)..

P being a numerical proposition, we will say that it belongs to class D_{PN} if P or Non(P) can be proved by a classical proof, this proof possibly containing any propositions that have proved and that are not numerical propositions. For instance FERMAT Theorem is a numerical proposition belonging to class D_{PN} despite that its proof uses many propositions that are not numerical propositions. The class D_{PN} is therefore included in class P_N, and the problem P_N=DP_N consists in searching to determine if the class DP_N is equal to class P_N, in complete analogy with the problem P=NP that consisted to research if the class P was equal to class NP.

The problem P_N=DP_N can be solved exactly the same way as the problem P=NP: We begin to prove that it is impossible to prove classically P_N=DP_N or $P_N \neq D_{PN}$:

We set the Axiom 1B, analogous to the Axiom 1:

AXIOM 1B:

P being a numerical proposition, it is impossible to prove that neither P nor Non(P) have a classical proof. (Meaning that P does not belong to the class DP_N).

This Axiom 1B can be justified intuitively the same way as Axiom 1: P being a numerical proposition, it expresses properties of naturals (including properties of Card(A(n)), (A(n)) sequence of finite sets or properties of explicit numerical functions). Because of this it is very possible that P admit very complex classical proofs using many complex explicit numerical functions. And we understand intuitively that basic Axioms (Axioms of the classical set theory, Peano's Axioms (permitting to define **N**), Axiom of choice) seem to be insufficient to be able to prove that neither P nor Non(P) admit any classical proof, even among the more complex classical proofs. Moreover it has never been proved, for any numerical proposition P such that we defined, that neither P nor its negation admitted a classical proof. We remind that this Axiom 1B, as any axiom cannot be formally proved even if it is the case.

We remark that this Axiom 1B cannot be generalized to any mathematical proposition and not even to any *strictly mathematical proposition*. (We define a *strictly mathematical proposition* as a proposition not using, directly or indirectly paramathematical logical concepts as for instance "a proposition", " a demonstration", "a problem"..).

Indeed that Axiom 1B cannot be applied to the Continuum Hypothesis :H_C: "It does not exist a set whose the cardinality is strictly between **N** and **R**". Indeed, this proposition is clearly not a numerical proposition, it uses the set *F*(**N**,**N**) or **R**, that does not correspond to points (vi) and (vii) of the definition of a numerical proposition and moreover whose the elements are defined using infinite sequences of naturals which is not possible for sets used in a numerical proposition (*F*(**N**,**N**)) is not used to define an explicit numerical function). We can understand intuitively why we cannot apply the Axiom 1B to H_C because H_C uses concepts directly obtained from the basic Axioms, replacing **R** by *F*(**N**,**N**) (Contrary to numerical propositions that use explicit numerical functions) and consequently we understand intuitively that it should be easier to prove the undecidability of such a proposition relative to numerical propositions. Moreover it is intuitively evident that the proof of H_C or of Non(H_C) do not require the use of explicit numerical functions (Which is not the case for numerical propositions) and consequently in order to prove the undecidability of H_C we do not need to consider proofs using explicit numerical functions, which is clearly not the case for numerical propositions. We remind that the undecidability of the continuum hypothesis has been proved using basic Axioms (3).

As a consequence of the Axiom 1 we have the impossibility to prove classically $P_N \neq DP_N$.

In order to prove the impossibility to prove classically $P_N=DP_N$, we admit the Axiom 2B, analogous to the Axiom 2:

AXIOM 2B:

In order to prove that a numerical proposition P can be proved classically, it is necessary to give explicitly a classical proof of P, or an algorithm permitting to obtain it.

This Axiom 2B can be intuitively justified exactly the same way as the Axiom 2.

Using this Axiom 2B, exactly the same way as for the problem P=NP, we obtain that in order to prove classically $P_N=DP_N$, then it would be necessary to provide an algorithm permitting to give a classical proof to any numerical proposition or its negation, which is clearly impossible. So we proved the impossibility to prove classically P_N=DP_N.

So using the preceding Axioms we proved the impossibility to prove classically P_{SM}=DP_{SM} or its negation.

Nonetheless proceeding exactly the same way as for the problem P=NP, we can prove that random laws bring to justify theoretically $P_N \neq DP_N$.

We remark that contrary to the case P=NP, it exists strictly mathematical propositions for which we can think intuitively that they do not belong to the class DP_N, meaning that they cannot be proved classically not their negation. It is the case for instance for the weak and strong GOLDBACH Conjectures or Twin Prime Conjectures. In agreement with Axiom 1B it should be impossible to prove this classically but it is remarkable that those Conjectures own theoretical justifications based on random that can be obtained in the Random Number Theory (2). It is precisely those theoretical justifications based on random, and also the fact that a classical proof for those Conjectures nor for their negation have never be obtained, that bring to believe that probably those Conjectures do not own classical proof nor there negations.

We remark that the problem P_N=DP_N is equivalent to the fundamental problem: "Is number theory (Restricted to numerical propositions) is complete?"

The numerical propositions are interesting because they are simple (They are defined very easily with basic concepts of number theory) and we can find easily an infinity of numerical propositions, defined using complex explicit numerical functions (For instance E(Log(n)), that seem to be undecidable. The existence of such propositions is in agreement with the justification based on random of the incompleteness of number theory. It is also this kind of proposition for which the Random Theory of Numbers (2) can be applied, giving simple theoretical justification based on random for propositions that seem to be true but for which a classical proof has never be proved nor for their negation. The use of the concept of numerical proposition is also useful in order to give a kind of strictly mathematical propositions to which it is possible to apply the Axiom 1B. We remind that it is not valid for all strictly mathematical propositions. They also permit to understand the solution of the problem P=NP.

If a basic problem of numerical determination P_B belongs to class NP but not to class P, then by definition, it will be not possible to find a sequence of algorithms solving it. Moreover because of the Axiom 1, it will be not possible to prove classically (Meaning using basic Axioms) that it does not admit a solution.

Of a proposition P belongs to class P_N but not to class DP_N, by definition it will be not possible to prove classically P or Non(P). Moreover because of the Axiom 1B it will be not possible to prove classically that neither P nor Non(P) admit a classical proof.

All brings to believe that previous basic problem of numerical determination or propositions exist, considering random pseudo-proof of the Axioms 1 and 1B or also numerical propositions that admit a theoretical justification based on random and for which a classical proof has never been found.

We could name the analogous Axioms 1 and 1B "Axioms of indeterminism in Number Theory". We remind that those Axioms, as any Axiom, cannot be proved classically, they admit an intuitive justification, they have a very simple expression, they have never been contradicted despite that they concern an infinity of propositions and they are fundamental.

It is possible that the definition of a concept equivalent to a classical formal system could be obtained by a numerical proposition. (Even if this needs to be proved).

Let Sys_F be such a formal system, defined by a numerical proposition.

Let us now consider the classical proposition belonging to Sys_F: "P:P is not demonstrable in Sys_F".

We assume that the proposition Q:"P is true in Sys_F" is also equivalent (meaning with the same signification) to a numerical proposition. (This should be true if Sys_F is itself defined by a numerical proposition").

We know that Sys_F being a classical formal system (as formal systems considered by GODEL), it is possible to prove classically that Q is true. Therefore we have not the proposition "Q and Non(Q) do not admit classical proof" and we have not a contradiction of the Axiom 1B.

Now we consider the proposition R: "P is demonstrable in Sys_F".

We also assume that R is equivalent to a numerical proposition. We know that we can prove classically that R is not true, and therefore, as for Q there is none contradiction of the Axiom 1B.

We remind that classical proofs of Q and of Non(R) constitute the classical proof of the GODEL incompleteness Theorem. We have showed that even if a classical formal system could be defined by a numerical proposition, the GODEL theorem would not invalidate the Axiom 1B. We remark that this Axiom 1B concerns also all proofs obtained by classical formal systems, because classical formal systems are defined in order that their proofs can be identify with classical proofs.

Thus, if a numerical proposition P is such that neither P nor Non(P) admit a classical proof (As it seems to be the case for GOLDBACH weak and strong Conjectures and for an infinity of numerical propositions admitting a theoretical justification based on random, that seem to be true but for which a classical proof has never been found), it will be never possible for a classical formal system, meaning a formal system whose the rules model basic mathematical Axioms, to give a demonstration of P or of Non(P). Moreover if the Axiom AB is valid, it will be never possible for the preceding formal system to give a demonstration that neither P nor Non(P) admit a classical proof and if the Axiom 1 is valid, to give a demonstration that a basic problem of numerical determination does not admit any solution.

III)CONCLUSION

So we did not prove that neither P=NP nor P≠NP was true for basic problems of numerical determination, but we justified that in both cases it will be impossible to prove it. We remark that the fact that for basic problems of numerical determination it is impossible to prove P=NP implies that it is also the case in the general case.

The theory that we presented is not a purely mathematical theory but is a logical mathematical theory. It is very likely that it is not possible to obtain the solution that we gave of the problem P=NP with a purely mathematical theory. We remind that we can consider the Axioms 1,2,3 as assertions of pure logic admitted because they have an intuitive evident justification and have never been contradicted permitting to solve the problem P+NP for basic problems of numerical determination and to understand why it is undecidable. Those Axioms can be applied to basic problems of numerical determination but also to much more general classes of problems. In fact the conclusion of Axioms 1,2 (It is impossible to prove P=NP) can be applied to all mathematical problems concerned by the problem P=NP, and it should be very difficult to find a single mathematical problem concerned by the problem P=NP and to which the Axiom 1 (Permitting to justify the impossibility to prove P≠NP) is contradicted. Moreover, the fact that we have never obtained fundamental result concerning the problem P=NP suggests that the solution of this problem needs compulsory to introduce new Axioms, and cannot be obtained using only classical mathematical theories. It is very possible that any theory solving the problem P=NP must admit Axioms analogous to Axioms we introduced in this article. We remind that as usual Axioms, even if they are true, it is quasi-certain that Axioms (Or logical assertions) 1,2,3 cannot be demonstrated formally. Nonetheless, it is in agreement with all the (established) equations relative to the problem P=NP. We remind that our theory can be considered either as a mathematical logical proof (using intuitive Axioms) either as a logical justification (Considering our Axioms as logical assertions with intuitive justification). In both cases the conclusion is a fundamental mathematical result that a priori cannot be obtained without using logical assertions analogous to those introduced in this article.

We remind that the definition of a *basic problem of numerical determination* is important because it contains a very general kind of problems that are potentially of class P or of class NP, and consequently because it constitutes a very concrete basis permitting to justify intuitively the Axioms that we introduced, and also to test their validity.

So we did not prove P=NP nor P≠NP for basic problems of numerical determination but we solved the problem P=NP the same way the proof that it did not exist any algorithm permitting to obtain the trisection of the angle or the quadrature of the circle with a compass solved those problems. The conclusion of this article is therefore in agreement with all articles previously published about the problem P=NP. In order to prove P=NP or its contradiction, it should be necessary at first to prove that the theory exposed in this article and its Axioms are wrong, which should be clearly much easier than proving P=NP or its negation.

Nonetheless we showed that the random laws in number theory could be at the origin of the validity of P≠NP, the theoretical justification of P≠NP being fundamental because neither P=NP neither its negation have already been classically proved (And moreover, if our Axioms are true, it is impossible to prove any of them.).

To end we proposed a solution analogous to the solution of the problem P=NP to the problem $P_N=DP_N$, this last problem being fundamental in mathematics because it is equivalent to the problem "Is Number Theory complete?".

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