Divergence Series and Integrals From the Viewpoint of the Division by Zero Calculus

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Abstract: In this short note, we would like to refer to the fundamental new interpretations that for the fundamental expansion $1/(1-z) = \sum_{j=0}^{\infty} z^j$ it is valid in the sense $0 = 0$ for $z = 1$, for the integral $\int_1^{\infty} 1/x dx$ it is zero and in the formula $\int_0^{\infty} J_0(\lambda t)dt = 1/\lambda$, it is valid with $0 = 0$ for $\lambda = 0$ in the sense of the division by zero.

Key Words: Division by zero, divergence series, divergence integral, division by zero calculus, infinity, zero, point at infinity, Riemann sphere, horn torus model.

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1 Introduction

In this short note, we would like to refer to the fundamental new interpretations that for the fundamental expansion on $|z| < 1$

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j, \tag{1.1}$$

it is valid in the sense $0 = 0$ for $z = 1$, for the integral

$$\int_1^\infty \frac{1}{x} dx, \tag{1.2}$$

it is zero and in the formula

$$\int_0^\infty J_0(\lambda t) dt = \frac{1}{\lambda}, \tag{1.3}$$

it is valid with $0 = 0$ for $\lambda = 0$ in the sense of the division by zero.

These formulas are not curious more, but we are already familiar with the new concept of the division by zero calculus. Indeed, we discussed the relationship of infinity and zero ([8]) and their relation is realized clearly by the horn torus model ([9]). However, the results will be very surprising for many people and so, we were not able to state them clearly. Now, we would like to state the results clearly. We will be able to see some new idea on the universe. We will need some more time to accept the new concept on our division by zero calculus; so we are presenting our results in the viXra series for some general discussions asking for suggestions and valuable comments.

For the general situation on our division by zero calculus, see [10].

2 Division by zero calculus

In order to state the results in a self-contained manner, we will recall the division by zero calculus.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z-a)^n,$$

we **define** the division by zero calculus by the identity

$$f(a) = C_0.$$

For many basic properties and applications of the division by zero calculus, see [10] and the references.

In particular, note that for the base of our division by zero, we need only the assumption of the definition of the division by zero calculus, as in an axiom.

As a simple and direct result, for the mapping

$$W = f(z) = \frac{1}{z}, \tag{2.1}$$

the image of $z = 0$ is $W = 0$. This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere. As the representation of the point at infinity on the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value at the origin by limiting like $+\infty$ and $-\infty$ or by the point at infinity as ∞. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences.

In addition, for the fundamental function (2.1), note that the function is odd

$$f(z) = -f(-z)$$

and if the function may be extended as an odd function at the origin $z = 0$, then the identity $f(0) = 1/0 = 0$ has to be satisfied. Further, if the equation

$$\frac{1}{z} = 0$$

has a solution, then the solution has to be $z = 0$.

The strong discontinuity of the division by zero around the point at infinity will appear as the destruction of various figures. These phenomena may be looked in many situations as the universal one. However, the simplest cases are the disc and sphere (ball) with their radius R. When $R \to +\infty$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $R = \infty$ (the point at infinity), they are zero, because they become the half-plane and half-space, respectively. These facts may be also looked by analytic geometry. However, the results are clear already from the definition of the division by zero.

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$. We thus see that

$$\lim_{z \to \infty} z = \infty, \tag{2.2}$$

however,

$$[z]_{z=\infty} = 0, \tag{2.3}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (2.2) and (2.3) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x \to +\infty} x = +\infty, \quad \lim_{x \to -\infty} x = -\infty,$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0.$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [5], we gave many examples for this property.

In particular, in $z \to \infty$ in (2.2), ∞ represents the topological point on the Riemann sphere, meanwhile ∞ in the left hand side in (2.2) represents the limit by means of the ϵ - δ logic. That is, for any large number M, when we take for some large number N, we have, for $|z| > N$, $|z| > M$.

In conclusion, the point at infinity is represented by zero and its geometrical interpretation is given clearly by the Puha-Däumler's horn torus model. See [8, 9].

3 Interpretations

For the most elementary formula (1.1), we will state the result. Of course, we know its meaning of the expansion (1.1) that is valid in the open unit disc $|z| < 1$. Now, how will be the expansion for the point at $z = 1$? Usually, we will consider that

$$\lim_{z \to 1} \frac{1}{1-z}$$

diverges to infinity, meanwhile,

$$\lim_{N\to\infty}\sum_{j=0}^{N} 1^j = +\infty.$$

Of course, these are right. However, now we can consider that by the division by zero calculus

$$\frac{1}{1-z} = 0$$

at the point $z = 1$ and

$$\lim_{N\to\infty}\sum_{j=0}^{N} 1^j = 0.$$

Then the expansion (1.1) is still valid for the point $z = 1$.

Next, for the integral (1.2), we will consider that

$$\int_1^\infty \frac{1}{x}dx = \lim_{R\to\infty}\int_1^R \frac{1}{x}dx = \lim_{R\to\infty}\log R = +\infty. \tag{3.1}$$

However, by the new idea of the point at infinity, this is zero ([4]), since $\log\infty = 0$.

We have, by the division by zero calculus idea,

$$\int_0^\infty J_0(0t)dt = \int_0^\infty 1dt = \frac{1}{\lambda}|_{\lambda=0} = 0. \tag{3.2}$$

4 Further examples

We can find many and many formulas in the divergence series and integrals that may be applied the idea of this paper. We will show examples.

In the formula ([2], page 153), for $0 \le x, t \le \pi$

$$\sum_{n=1}^{\infty}\frac{\sin ns \sin nt}{n} = \frac{1}{2}\log|\sin((s+t)/2)/\sin((s-t)/2)|,$$

for $s = t = 0, \pi$, we can obtain that

$$0 = \frac{1}{2}\log\frac{0}{0} = \frac{1}{2}\log 0.$$

In general, for $s=t$, we may consider that

$$\sum_{n=1}^{\infty}\frac{\sin^2 ns}{n}=\frac{1}{2}\log|\sin((s+s)/2)/0)|$$

$$=\frac{1}{2}\log|\sin ns/0|=\frac{1}{2}\log 0=0.$$

By Poisson's formula, we have

$$\sum_{n=-\infty}^{\infty}e^{-\alpha|n|}=\sum_{n=-\infty}^{\infty}\frac{2\alpha}{\alpha^2+(2\pi n)^2}.$$

For $\alpha=0$, the both sides are zero.

For the integrals, for non-negative integer n and $a>0$,

$$\int_0^{\infty}t^{2n}e^{-at^2}dt=\frac{\Gamma(n+(1/2))}{2a^{n+1}}$$

and

$$\int_0^{\infty}t^{2n+1}e^{-at^2}dt=\frac{n!}{2a^{n+1}},$$

for $a=0$, we see that they are zero.

Meanwhile, from the well-known expansion ([1], page 807) of the Riemann zeta function

$$\zeta(s)=\frac{1}{s-1}+\gamma-\gamma_1(s-1)+\gamma_2(s-1)^2+...,$$

we see that the value at $s=1$ is the Euler constant γ; that is,

$$\zeta(1)=\gamma. \tag{4.1}$$

Meanwhile, from the expansion

$$\zeta(z)=\frac{1}{z}-\sum_{k=2}^{\infty}C_k\frac{z^{2k-1}}{2k-1}$$

([1], 635 page 18.5.5), we have

$$\zeta(0)=0.$$

From our idea that the point at infinity is represented by zero, the result (4.1) is contrary curious. However, we should consider that

$$\sum_{n=1}^{\infty} \frac{1}{n} = 0, \tag{4.2}$$

because the limit tends to infinity or N terms sums are not bounded. We should consider some problem for its representation $\zeta(1)$ by the Riemann zeta function by its analytic extension. For this point, recall that

$$\zeta(z) = -\frac{1}{12} + (x+1)\left(\frac{1}{12} - \log C\right) + ...$$

for $x = -1$; that is $\zeta(-1) = -1/12$ by the division by zero calculus, of course, it does not represent the series

$$1 + 2 + 3 + ... + n + ...$$

5 Conclusion

In many places in mathematical science, infinity notation may be replaced by zero, by the concept of the division by zero calculus. How to represent the formulas? We will need some time in order to see some global situation. These facts anyhow will show a new meaning of ZERO.

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