Quantum Entanglement and Application

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***Abstract*-Quantum entanglement, a phenomenon occurring when states of pairs or groups of particles cannot be described independently, is one of the most 'spooky action' but truly exists. In this paper, entangled states and few of its application is introduced.**

***Keywords*-quantum entanglement; Bell states; quantum teleportation; superdense coding;**

I. Introduction

Quantum entanglement is a physical phenomenon which occurs when pairs or groups of particles are generated or interact in ways such that the quantum state of each particle cannot be described independently of the state of the others, even when the particles are separated by a large distance—instead, a quantum state must be described for the system as a whole. Such phenomena were the subject of a 1935 paper by Albert Einstein, Boris Podolsky, and Nathan Rosen, describing what came to be known as the Einstein-Podolsky-Rosen paradox, or EPR paradox. At the very beginning, Einstein and others considered such behavior to be impossible. Later, however, such counterintuitive predictions of quantum mechanics were verified experimentally by producing violations of Bell's inequality.

Entanglement has many applications in quantum information theory. Among the best-known applications of entanglement are superdense coding and quantum teleportation.

II. EPR paradox, and Bell's inequality

EPR paradox of 1935 is a thought experiment claimed by Albert Einstein, Boris Podolsky, and Nathan Rosen, to demonstrate the 'incompleteness' of quantum mechanics. One version of the thought experiment is as following: Prepare an EPR pair consisted by particle A and particle B with zero total spin. Then separate them in space. Set the separation of two particles to be a, then if position of particle A is measured to be x, particle B would be at position $(x-a)$. Also if momentum of particle B is measured to be p, by conservation of momentum, another particle A would have momentum $-p$. Then position and momentum of each individual particle can be know, but this violate principle of uncertainty. In addition, if one is particle is measured to be spin up, then another one must be spin down. [1]

Einstein believed that to explain this paradox, either one of the following has to be true: (i) there exists action at a distance, so that measuring position of one particle can affect momentum of another particle instantaneously. (ii) quantum theory is not complete, and need to extended with hidden variables. [1]

Bohr on the other hand, come up with the resolution as follows: for two "entangled" particles, measurable properties have well-defined meaning only for the ensemble system. Properties of constituent subsystems, considered individually, remain undefined. Which eliminates the need for hidden variables, action at a distance, or other schemes introduced over time, in order to explain the phenomenon. [2]

A preference for the latter resolution is supported by experiments suggested by Bell's theorem of 1964, which exclude some classes of hidden variable theory. Notate a and a' to be detector settings on side A, b and b' on side B. Classic theory suggest that the correlation of measured result of a particle pair should follow the inequality (detailed derivation in Appendix I): [3][4]

$$E(a,b) + E(a,b') + E(a',b) - E(a',b') \leq 2 \qquad (2.1)$$

where E refers to correlations of particle pair.

Lager number of experiments has been done since Bell proposed his theory. Experimental evidence of the violation of Bell's inequality suggests that predictions of quantum mechanics are correct.

III. Entanglement and Bell states [4]

Consider a two-particle (2-qubit) system and each particle (qubit) can in either one of the two states:

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \text{ or } |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{3.1}$$

There are 4 ways they form a pair:

$$\begin{aligned}
|00\rangle &= |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \\
|01\rangle &= |0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
|10\rangle &= |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \\
|11\rangle &= |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}
\end{aligned} \tag{3.2}$$

Entangled states are states that cannot be represent by tensor product of two qubits. Because of the entanglement, measurement of one qubit will assign a value to the other qubit immediately. There are four ways one qubit assigned the value to the other qubit depend on how these two particle are entangled. The state of their entanglement can be represented by Bell states. Bell state of a pair of qubit can be expressed as:

$$\begin{aligned}
|\beta_{00}\rangle &= \frac{(|00\rangle + |11\rangle)}{\sqrt{2}} \\
|\beta_{01}\rangle &= \frac{(|01\rangle + |10\rangle)}{\sqrt{2}} \\
|\beta_{10}\rangle &= \frac{(|00\rangle - |11\rangle)}{\sqrt{2}} \\
|\beta_{11}\rangle &= \frac{(|01\rangle - |10\rangle)}{\sqrt{2}}
\end{aligned} \tag{3.3}$$

($|00\rangle$ with $|10\rangle$, or $|11\rangle$ with $|01\rangle$ can be represent by tensor product of 2 qubits, so they are not entangled states)

Or

$$\begin{aligned}
|\Psi^{\pm}\rangle &= \frac{(01\rangle \pm |10\rangle)}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle \pm |1\rangle \otimes |0\rangle) \\
|\Phi^{\pm}\rangle &= \frac{(|00\rangle \pm |11\rangle)}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle)
\end{aligned} \tag{3.4}$$

IV. Quantum teleportation [5]

Using the fact that measurement of one qubit is related to its entangled partner, a complete new method of sending information is proposed. The seminal paper first expounding the idea of quantum teleportation was published by C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters in 1993. Quantum teleportation is not a form of transportation, but of communication: it provides a way of transporting a qubit from one location to another without having to move a physical particle along with it. The protocol of quantum teleportation is as follows: [6]

1. An EPR pair is generated, one qubit sent to location A, the other to B.

2. At location A, a measurement of system of the qubit of the EPR pair and anther qubit contained information is performed, yielding one of four measurement outcomes, which can be encoded in classical information.

3. The classical information are sent from A to B.

4. As a result of the measurement performed at location A, the EPR pair qubit at location B is in one of four possible states. Which of these four possibilities actually obtained, is encoded in the classical information. With the information of the classical bits and state of qubit at location B, the information contained in qubit at location A can be reproduced.

Mathematical representation for the process listed above is as follows:

Using an EPR pair (particle 2 and 3) prepared in bell state $|\Psi^{-}_{23}\rangle$ as an example:

$$\begin{aligned}
|\Psi^{-}_{23}\rangle &= \frac{(|01\rangle - |10\rangle)}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}(|0_2\rangle \otimes |1_3\rangle - |1_2\rangle \otimes |0_3\rangle)
\end{aligned} \tag{3.4}$$

Particle 2 is send to A and particle 3 is send to B. Information need to be sent from A to B can be stored into a third particle 1. If the information is (a, b), the third particle can be made as:

$$|\varphi_1\rangle = \mathrm{a}|0_1\rangle + \mathrm{b}|1_1\rangle \tag{4.1}$$

State of the whole system (particle 1, 2, and 3) can be written as:

$$
\begin{aligned}
|\Psi_{123}\rangle &= |\varphi_1\rangle \otimes |\Psi_{23}^-\rangle \\
&= \frac{a}{\sqrt{2}}(|0_1\rangle \otimes |0_2\rangle \otimes |1_3\rangle - |0_1\rangle \otimes |1_2\rangle \otimes |0_3\rangle) \\
&+ \frac{b}{\sqrt{2}}(|0_0\rangle \otimes |0_2\rangle \otimes |1_3\rangle - |0_0\rangle \otimes |1_2\rangle \otimes |0_3\rangle)
\end{aligned} \tag{4.2}
$$

Since particle 1 and 2 are at A, people at 1 and 2 can only measure the state of system consisted by particle 1 and 2. When measurement is performed on the system of particle 1 and 2, process called entanglement swapping occur, and particle 1 and 2 are now an EPR pair.

Rewritten Eq.4.2 as a linear combination of bell states of particle 1 and 2, since measurement for system of particle 1 and 2 only gives information of EPR pair of particle 1 and 2.

$$
\begin{aligned}
|\Psi_{123}\rangle = |\mathrm{I}_3\rangle \otimes |\Psi_{12}^-\rangle + |\mathrm{II}_3\rangle \otimes |\Psi_{12}^+\rangle \\
+ |\mathrm{III}_3\rangle \otimes |\Phi_{12}^-\rangle + |\mathrm{IV}_3\rangle \otimes |\Phi_{12}^+\rangle
\end{aligned} \tag{4.3}
$$

where $|\mathrm{I}_3\rangle$, $|\mathrm{II}_3\rangle$, $|\mathrm{III}_3\rangle$, and $|\mathrm{IV}_3\rangle$ are states of particle 3.

Since Bell states are orthonormal, we can find $|\mathrm{I}_3\rangle$, $|\mathrm{II}_3\rangle$, $|\mathrm{III}_3\rangle$, and $|\mathrm{IV}_3\rangle$ using inner product:

$$
\begin{aligned}
\langle \Psi_{12}^- | \Psi_{123}\rangle = |\mathrm{I}_3\rangle \langle \Psi_{12}^- | \Psi_{12}^-\rangle \\
+ |\mathrm{II}_3\rangle \langle \Psi_{12}^- | \Psi_{12}^+\rangle \\
+ |\mathrm{III}_3\rangle \langle \Psi_{12}^- | \Phi_{12}^-\rangle \\
+ |\mathrm{IV}_3\rangle \langle \Psi_{12}^- | \Phi_{12}^+\rangle
\end{aligned} \tag{4.4}
$$

therefore:

$$
\begin{aligned}
|\mathrm{I}_3\rangle &= \langle \Psi_{12}^- | \Psi_{123}\rangle = \cdots \\
&= -\frac{1}{2}(\mathrm{a}|0_3\rangle + \mathrm{b}|1_3\rangle)
\end{aligned} \tag{4.5}
$$

Use same method, we get:

$$
\begin{cases}
|\mathrm{I}_3\rangle = \langle \Psi_{12}^- | \Psi_{123}\rangle = -\frac{1}{2}(\mathrm{a}|0_3\rangle + \mathrm{b}|1_3\rangle) \\
|\mathrm{II}_3\rangle = \langle \Psi_{12}^+ | \Psi_{123}\rangle = -\frac{1}{2}(\mathrm{a}|0_3\rangle - \mathrm{b}|1_3\rangle) \\
|\mathrm{III}_3\rangle = \langle \Phi_{12}^- | \Psi_{123}\rangle = \frac{1}{2}(\mathrm{a}|0_3\rangle + \mathrm{b}|1_3\rangle) \\
|\mathrm{IV}_3\rangle = \langle \Phi_{12}^+ | \Psi_{123}\rangle = \frac{1}{2}(\mathrm{a}|0_3\rangle - \mathrm{b}|1_3\rangle)
\end{cases} \tag{4.6}
$$

express the above four state in matrix representation:

$$
\begin{cases}
|\mathrm{I}_3\rangle = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\
|\mathrm{II}_3\rangle = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\
|\mathrm{III}_3\rangle = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\
|\mathrm{IV}_3\rangle = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix}
\end{cases} \tag{4.7}
$$

It is clear that $|\mathrm{I}_3\rangle$, $|\mathrm{II}_3\rangle$, $|\mathrm{III}_3\rangle$, and $|\mathrm{IV}_3\rangle$ are all the original state of particle 1 (a, b) under change of a unitary matrix. Therefore once measurement at A is done, people at A know which bell states is particle 1 and 2 are in. The measurement result is one of $|\Psi^-_{12}\rangle$, $|\Psi^-_{12}\rangle$, $|\Psi^-_{12}\rangle$, and $|\Psi^-_{12}\rangle$, and those state indicate particle 3 is in state $|\mathrm{I}_3\rangle$, $|\mathrm{II}_3\rangle$, $|\mathrm{III}_3\rangle$, or $|\mathrm{IV}_3\rangle$ respectively. Then next thing people at A have to do is just sending the corresponded unitary matrix to people at B, so people at B can use the matrix and particle 3 to recover the original state of particle 1 and get the information (a, b).

Quantum teleportation is almost a perfect way to send information securely. Since tapping on classical trennel can only get the unitary matrix, but to get the information, both the unitary matrix and qubit sent to B is required. In addition, because the no-cloning theorem (details in Appendix II), any attempt to measure or copy the qubit sent to B change the state of the qubit, and the information then is 'destroyed'.

V. Superdense coding [7]

Superdense coding is a technique that allows increasing the classical information content that can be encoded within a number of qubits. In its simplest form, the protocol is used to encode two classical bits of information in the state of a single qubit. It can be thought of as the inverse of quantum teleportation, in which one transfers a quantum state from one party to the other via communication of a number of classical bits.

	Quantum teleportation	Superdense coding
Channel used	Classical channel	Quantum channel
Information	1 qubit	2 classical bits
Information sent via channel	2 classical bits	1 qubit

Table 1. Comparison between quantum teleportation and superdense coding.

Suppose people at A want to send 2 bits of classical information to people at B using only 1 qubit. First, people at A and B need to have an EPR pair, one at qubit at A and the other at B, for example:

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_b + |1\rangle_A \otimes |1\rangle_B) \quad (5.1)$$

where subscribes mark the location of qubits.

Table 2 lists messages people at A can send, and in order to send a specific message, what operator needed to performed on qubit at A.

info	operator	result of operation
00	no operator	$\|\beta_{00}\rangle$
01	X-Gate	$X\|\beta_{00}\rangle$ $= \frac{1}{\sqrt{2}}((X\|0\rangle_A)\otimes\|0\rangle_b + (X\|1\rangle_A)\otimes\|1\rangle_B)$ $= \frac{1}{\sqrt{2}}(\|1\rangle_A\otimes\|0\rangle_b + \|0\rangle_A\otimes\|1\rangle_B)$ $= \|\beta_{01}\rangle$
10	Z-Gate	$Z\|\beta_{00}\rangle$ $= \frac{1}{\sqrt{2}}((Z\|0\rangle_A)\otimes\|0\rangle_b + (Z\|1\rangle_A)\otimes\|1\rangle_B)$ $= \frac{1}{\sqrt{2}}(\|0\rangle_A\otimes\|0\rangle_b - \|1\rangle_A\otimes\|1\rangle_B)$ $= \|\beta_{10}\rangle$
11	X-Gate then Z-gate	$ZX\|\beta_{00}\rangle$ $= \frac{1}{\sqrt{2}}((ZX\|0\rangle_A)\otimes\|0\rangle_b + (ZX\|1\rangle_A)\otimes\|1\rangle_B)$ $= \frac{1}{\sqrt{2}}(-\|1\rangle_A\otimes\|0\rangle_b + \|0\rangle_A\otimes\|1\rangle_B)$ $= \|\beta_{00}\rangle$

Table 2. Information want to send and corresponded operator need to be performed.

X-Gate and Z-Gate are quantum logic gates used in quantum computing. For a 2-qubit system, X-Gate and Z-Gate's matrix representation as following: [8]

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (5.2)$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (5.3)$$

Then people at A send the qubit been operated to B, and people at B measure the state of the pair of qubits. Each Bell state is corresponded to a 2-bit classical message.

Again, superdense coding is a very security way to communicate. Even if the qubit sent from A is somehow 'stolen' by a third party, without the qubit at B, no useful information can be extract.

VI. Conclusion

Property of quantum entanglement makes entangled pair to be perfect media for transmitting information. Applications such as quantum teleportation and superdense coding guarantee secure communication.

The main challenge faced by the field of quantum communication nowadays is the effect so-called quantum decoherence, which refers to when a quantum system is not perfectly isolated, but in contact with surroundings, coherence decays with time and eventually system can be no longer considered to be entangled. Once the problem brought by quantum decoherece is resolved, quantum communication over long distance can be possible.

VII. Reference

[1] Einstein, A; B Podolsky; N Rosen (1935). "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" Physical Review. 47 (10): 777–780.

[2] Bohr, N. (1935). "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" Physical Review. 48 (8): 696–702.

[3] Bell, John (1964). "On the Einstein Podolsky Rosen Paradox". Physics. 1 (3): 195–200.

[4] J. S. Bell, in Foundations of Quantum Mechanics, Proceedings of the International School of Physics "Enrico Fermi", Course XLIX, B. d'Espagnat (Ed.) (Academic, New York, 1971), p. 171 and Appendix B. Pages 171-81 are reproduced as Ch. 4 of J. S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press 1987)

[5] Nielsen M, Chuang I. 2.6 EPR and the Bell inequality. In: "Quantum Computation and Quantum Information". Cambridge. UK: Cambridge University Press, 2000, p. 111–118.

[6] Nielsen M, Chuang I. 1.3.7 Example: quantum tleportation. In: "Quantum Computation and Quantum Information". Cambridge. UK: Cambridge University Press, 2000, p. 26-28.

[7] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W. K. Wootters. (1993). "Teleporting an Unknown Quantum State via Dual Classical and Einstein–Podolsky–Rosen Channels". Phys. Rev. Lett. 70, p. 1895–1899.

[8] Nielsen M, Chuang I. 2.3 Application: superdense coding. In: "Quantum Computation and Quantum Information". Cambridge. UK: Cambridge University Press, 2000, p. 97-98.

[9] Nielsen M, Chuang I. 1.3 Quantum computation and 1.4 Quantum algorithms. In: "Quantum Computation and Quantum Information". Cambridge. UK: Cambridge University Press, 2000, p. 97-98.

Appendix I

Bell inequality[i]

Assume two sides A and B are independent for any selected value of the 'hidden variable' λ, so joint probabilities of pairs of outcomes can be obtained by multiplying the separate probabilities. λ is assumed to be drawn from a fixed distribution of possible states of the source, the probability of the source being in the state λ for any particular trial being given by the density function ρ(λ), the integral of which over the complete hidden variable space is 1. Thus:

$$E(a,b) = \int \bar{A}(a,\lambda)\bar{B}(b,\lambda)\rho(\lambda)d\lambda \qquad \text{(I.1)}$$

where $\underline{A}$ and $\underline{B}$ are the average values of the outcomes. Since the possible values of A and B are −1, 0 and +1, it follows that:

$$|\bar{A}| \le 1, |\bar{B}| \le 1 \qquad \text{(I.2)}$$

Then, if *a*, *a'*, *b* and *b'* are alternative settings for the detectors:

$$\begin{aligned} &E(a,b) - E(a',b') \\ &= \int \left(\bar{A}(a,\lambda)\bar{B}(b,\lambda) - \bar{A}(a',\lambda)\bar{B}(b',\lambda)\right)\rho(\lambda)d\lambda \\ &= \int \bar{A}(a,\lambda)\bar{B}(b,\lambda)[1 \pm \bar{A}(a',\lambda)\bar{B}(b',\lambda)]\rho(\lambda)d\lambda \\ &- \int \bar{A}(a,\lambda)\bar{B}(b',\lambda)[1 \pm \bar{A}(a',\lambda)\bar{B}(b,\lambda)]\rho(\lambda)d\lambda \end{aligned} \qquad \text{(I.3)}$$

Take absolute value to both side and by the triangle inequality:

$$\begin{aligned} &|E(a,b) - E(a',b')| \\ &\le \left| \int \bar{A}(a,\lambda)\bar{B}(b,\lambda)[1 \pm \bar{A}(a',\lambda)\bar{B}(b',\lambda)]\rho(\lambda)d\lambda \right. \\ &\left. - \int \bar{A}(a,\lambda)\bar{B}(b',\lambda)[1 \pm \bar{A}(a',\lambda)\bar{B}(b,\lambda)]\rho(\lambda)d\lambda \right| \end{aligned} \qquad \text{(I.4)}$$

Since $[1\pm \bar{A}(a',\lambda)\bar{B}(b',\lambda)]$ and $[1\pm \bar{A}(a',\lambda)\bar{B}(b,\lambda)]$ are both negative, right-hand side of Eq.I.4 can be written as:

$$\begin{aligned} &\left| \int \bar{A}(a,\lambda)\bar{B}(b,\lambda)[1 \pm \bar{A}(a',\lambda)\bar{B}(b',\lambda)]\rho(\lambda)d\lambda \right| \\ &+ \left| \int \bar{A}(a,\lambda)\bar{B}(b',\lambda)[1 \pm \bar{A}(a',\lambda)\bar{B}(b,\lambda)]\rho(\lambda)d\lambda \right| \end{aligned} \qquad \text{(I.5)}$$

By Eq.I.2 and assumption that the integral of ρ(λ) is 1, Eq.I.5 turns into:

$$\begin{aligned} &2 \pm \Big[\int \bar{A}(a',\lambda)\bar{B}(b',\lambda)\rho(\lambda)d\lambda \\ &+ \int \bar{A}(a',\lambda)\bar{B}(b,\lambda)\rho(\lambda)d\lambda\Big] \end{aligned} \qquad \text{(I.6)}$$

which is equal to 2±[*E(a',b')+E(a'b)*].
Put it back into the inequality and apply the triangle inequality again, we can obtain:

$$\begin{aligned} 2 &\ge |E(a,b) - E(a,b')| + |E(a',b') + E(a'b)| \\ &\ge |E(a,b) - E(a,b') + E(a',b') + E(a',b)| \end{aligned} \qquad \text{(I.7)}$$

Which is exactly Eq.2.1.

Appendix II

No-cloning theorem[ii]

The no-cloning theorem can be explained based on the linearity of quantum mechanics.

Let A and B be two quantum system, and they are in $|\varphi_A\rangle$, and $|0_B\rangle$. $|0_B\rangle$ is 'blank' state of B. Suppose there are some way to copy any state from quantum system A to another B:

$$|\phi_A\rangle \otimes |0_B\rangle \xrightarrow{copy} |\phi_A\rangle \otimes |\phi_B\rangle \qquad \text{(II.1)}$$

Same operation should copy another state from A to B as well:

$$|\psi_A\rangle \otimes |0_B\rangle \xrightarrow{copy} |\psi_A\rangle \otimes |\psi_B\rangle \qquad \text{(II.2)}$$

Superposition of states of A:

$$|\Psi_{A_{total}}\rangle = |\phi_A\rangle + |\psi_A\rangle \qquad \text{(II.3)}$$

By assumption, the total state can be copied to system B as well:

$$\begin{aligned} |\Psi_{A_{total}}\rangle \otimes |0_B\rangle &\xrightarrow{copy} |\Psi_{A_{total}}\rangle \otimes |\Psi_{B_{total}}\rangle \\ &= (|\phi_A\rangle + |\psi_A\rangle) \otimes (|\phi_B\rangle + |\psi_B\rangle) \\ &= |\phi_A\rangle \otimes |\phi_B\rangle + |\phi_A\rangle \otimes |\psi_B\rangle \\ &\quad + |\psi_A\rangle \otimes |\phi_B\rangle + |\psi_A\rangle \otimes |\psi_B\rangle \end{aligned} \qquad \text{(II.4)}$$

However because of linearity:

$$\begin{aligned} |\Psi_{A_{total}}\rangle \otimes |0_B\rangle &= (|\phi_A\rangle + |\psi_A\rangle) \otimes |0_B\rangle \\ &= |\phi_A\rangle \otimes |0_B\rangle + |\psi_A\rangle \otimes |0_B\rangle \\ &\xrightarrow{copy} |\phi_A\rangle \otimes |\phi_B\rangle + |\psi_A\rangle \otimes |\psi_B\rangle \end{aligned} \qquad \text{(II.5)}$$

Eq.II.4 and II.5 are not equal, so this mean the assumption 'there are some way to copy any state from one quantum system to another' cannot be true.

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[i] J. S. Bell, in Foundations of Quantum Mechanics, Proceedings of the International School of Physics "Enrico Fermi", Course XLIX, B. d'Espagnat (Ed.) (Academic, New York, 1971), p. 171 and Appendix B. Pages 171-81 are reproduced as Ch. 4 of J. S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press 1987)

[ii] Nielsen M, Chuang I. Box 12.1: The no-cloning theorem. In: "Quantum Computation and Quantum Information". Cambridge. UK: Cambridge University Press, 2000, p. 532.