Inertial motion of the quantum self-interacting electron

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Abstract

Attempts represent the self-interacting quantum electron as the cyclic motion on the stable attractor has been discussed. This motion subjects quantum inertia principle expressed by the parallel transported energy-momentum generator along a closed geodesic in the space of the unlocated quantum states $CP(3)$. The affine gauge potential in the complex projective state space (similar to the Higgs potential) seriously deforms the Jacobi fields in the vicinity of the "north pole". It was assumed that the divergency of the Jacobi field may be compensated by the fields of the Poincaré generators representing EM-like "field shell" of the electron in the dynamical spacetime. Thereby, the spacetime looks as ultimately deprecated in the role of the "container of matter" and it appears as the accompanied to the quantum electron functional space (dynamical spacetime). Meanwhile, the dynamics of the self-interacting electron is essentially non-linear and deterministic.

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1 Introduction. Motivation of the return to reality

Take the famous lectures of R. Feynman for "a nontechnical audience" in QED [1]. It is clear that there is some fundamental symmetry demonstrated along all the lectures but this fact was did not mentioned due to elementary character of the discussions. Namely, the arbitrariness of the choice of the "initial" arrow length and the clock orientation tell us that there is hidden projective symmetry of the quantum theory. This fact was know initially as from the first publications of Schrödinger. Since 1926 there where two ways before physicists: the linear approach in the framework of the probabilistic paradigm, or highly non-linear complex projective geometry with a misty perspective. The mathematical beauty of the works of Dirac and Fock provides the shocking efficiency of the quantum statistic in linear Hilbert state spaces (Fock space). However, one has in fact more difficult problem: the consistent theory of a single self-interacting quantum particles that connected with the problems of the localization, divergency, inertial mass, stability, etc.

I would like to discuss here the formulation of non-linear QED of the single quantum extended electron based on the Quantum Relativity [4, 5, 6, 7]. The sectional curvature of the coset state space is the main principle ingredient in my approach.

2 How the Quantum Relativity intended to solve the localization and divergency problems?

Let me start with simple Dirac's example of the divergences problem [2]. Dirac took the fermionic model Hamiltonian $\hat{H} = \frac{1}{2}(a_{mn}\eta_m\eta_n - \bar{a}_{mn}\bar{\eta_m}\bar{\eta_n})$. It is assumed that $\bar{\eta_n}|S >= 0$ for the all $1 \leq n \leq \infty$, where $|S>$ is a "standard" vector. The matrix a_{mn} is defined by Dirac as follows: $a_{mn} = \delta_{m+1,n} - \delta_{m,n+1}$. It is easy to see $\hat{H}^2|S >= -\frac{1}{2}Tr(\bar{a}a)|S >= \infty|S >$. It should be noted that the 'deformation' of the standard state vector $|S>$ in this artificial example has only 'longitudinal' character, i.e. only phase of the state vector $|S>$ is changed, not its direction. Formally one can find Schrödinger solution with this Hamiltonian $|\Psi(t) >= cos(\frac{t}{\hbar}Tr(\bar{a}a))|S >$, corresponding to the infinitely fast phase oscillation. Definitely physicist has some aversion to such behavior, but from the point of view of the projective geometry (and postulate of the ordinal quantum mechanics) 'deformed' vector $|\Psi(t)>$ belongs to the same ray as $|S>$, i.e. this is the same quantum state. This gives us the main idea to avoid the divergences problem. Namely, the orthogonal projection along of the vacuum state is the subtraction the 'longitudinal' component of the variation velocity of the action state $|\Psi(t)>$. Let put η_n to be creation operator and $|\xi_n >= \eta_n|S >= \alpha_n|S > +|n)$ is deformed standard vector, so that $(n|S >= 0$, therefore, $< S|\xi_n >=< S|\eta_n|S >= \alpha_n < S|S >$ and, hence, $\alpha = \frac{<S|\eta_n|S>}{<S|S>}$. Now we can express the 'transversal' part of the standard vector deformation

$$|n) = |\xi_n > - \frac{< S|\xi_n >}{< S|S >}|S >, \tag{1}$$

so that it is orthogonal to $< S|$. Let me calculate only the 'transversal' components of the $|\Psi(t) >= \exp(\frac{i}{\hbar}\hat{H}t)|S >$ which I will define as follows:

$$|\Psi(t)) = |\Psi(t) > - \frac{< S|\Psi(t) >}{< S|S >}|S >= \exp(\frac{i}{\hbar}\hat{H}t)|S > - \frac{|S >< S|}{< S|S >}\exp(\frac{i}{\hbar}\hat{H}t)|S > \tag{2}$$

Now I apply this definition to calculation of all orders of $|\Psi(t)>$. All detail may be found in the [3]. One sees that

$$|\Psi(t)_3) = \frac{t^3}{3!}(\hat{H}^3|S > - \frac{< S|\hat{H}^3|S >}{< S|S >}|S >)$$

$$= \frac{t^3}{3!}\frac{1}{8}(a_{mn}a_{pq}a_{rs}\eta_m\eta_n\eta_p\eta_q\eta_p\eta_s - 2Tr(\bar{a}a)a_{mn}\eta_m\eta_n)|S> \tag{3}$$

and that the divergences alive in the third order. Since the indefinite trace $Tr(\bar{a}a) = -\infty$ is a coefficient before the transversal to the vacuum two-fermionic term, the compensation projective term does not help. Nevertheless, we can extract the useful hint: the vacuum vector (the standard vector in Dirac's example) should be smoothly changed, and, furthermore, the transversal component should be reduced during the "smooth" evolution. One may image some a smooth surface with a normal vector, taking the place of the vacuum vector. Then the orthogonal projection acting continuously is in fact the covariant differentiation of the tangent Hamiltonian vector field. One has in fact the modification of the creation-annihilation operators of quantum particles. Let me recall that the main technical result of Dirac approach [2] is the calculations of the coefficients Y_{in} and Z_{in} modifying the initial creation-annihilation operators.

I assumed that the projective symmetry hidden in quantum mechanics and QFT may be applied not only to the elementary particles and more complex quantum systems but even for pure quantum degrees of freedom like electric charge, spin, etc. Then the smooth manifold of the vacuum will be represented by the complex projective Hilbert space $CP(N-1)$ or $CP(\infty)$. The "elementary particles" will be represented by stable or unstable motions of points in $CP(N-1)$. The present work deals only with $CP(3)$ of the quantum self-interacting electron.

3 Quantum Relativity

Two simple observations may serve as the basis of the intrinsic unification of relativity and quantum principles. The first observation concerns interference of quantum amplitudes in a fixed quantum setup.

A. The linear interference of quantum amplitudes shows the symmetries relative spacetime transformations of whole setup. This interference has been studied in the "standard" quantum theory. Such symmetries reflects, say, the *first order of relativity*: the physics is same if any *complete setup* subject (kinematical, not dynamical!) shifts, rotations, boosts as whole in the single Minkowski space-time. According to our notes given some time ago [9, 12] one should add to this list a freely falling quantum setup (super-relativity).

The second observation concerns a dynamical "deformation" of some quantum setup as the quantum analog of the Newton's force.

B. If one dynamically changes the setup configuration or its "environment", then the amplitude of an event will be generally changed. Nevertheless there is a different type of tacitly assumed symmetry that may be formulated on the intuitive level as the invariance of physical properties of "quantum particles", i.e. the invariance of their quantum numbers like mass, spin, charge, etc., relative variation of quantum amplitudes. This means that properties of, say, physical electrons in two different setups S_1 and S_2 are the same.

One may postulate that the invariant content of this quantum numbers may be kept if one makes the infinitesimal variation of some "flexible quantum setup" reached by a small variation of some fields by adjustment of tuning devices.

The invariant content of these properties will be discussed here under the infinitesimal variation of the "flexible quantum setup" [11] described by the amplitudes $|\Psi(\pi, P) >$ due to a small variation of the boson electromagnetic-like field $P^{\alpha}(\pi)$ treated as *the set of the scalar functions* relative π^i coordinates in $CP(N-1)$. The DST dependence of $P^{\alpha}(\pi)$ will be established after the separation of the shifts, boosts and rotations in the manifold of the $SU(4)$ generators.

The mathematical formulation of the QR principle is based on the *similarity* of any physical systems which are built on the "elementary" particles. This similarity is obvious only on the level of the pure quantum degrees of freedom of quantum particles. Therefore, all "external" details of the "setup" should be discarded as non-essential and only the relations of components of the "unitary spin" like

$$\pi^i_{(j)} = \begin{cases} \frac{\Psi^i}{\Psi^j}, & \text{if } 1 \leq i < j \\ \frac{\Psi^{i+1}}{\Psi^j} & \text{if } j \leq i < N-1 \end{cases}, \tag{4}$$

i.e. only inhomogeneous local coordinates of the state rays should be taken into account. These relations will be taken as the local projective coordinates in the complex projective Hilbert space $CP(N-1)$. *They are independent on ordinary macroscopic time and coordinates. I think just the opposite conjecture takes the place: the intrinsic dynamics of the quantum degrees of freedom is accompanied by the quantum dynamical spacetime (DST).*

The points of the complex projective Hilbert space CP^{∞} or its finite dimension subspace $CP(N-1)$ represent generalized coherent states (GCS) that will be used thereafter as fundamental physical concept instead of "material point". This space will be treated as the space of *"unlocated quantum states"* in the analog of the *"space of unlocated shapes"* of [20]. The problem we will dealing with is the lift the quantum dynamics from $CP(N-1)$ into the *space of located quantum states*. That is, the difference between the Shapere and Wilczek construction and our scheme is that not a self-deformation of 3D-shapes should be represented by the motions of a spatial reference frame but the dynamics of the unlocated quantum states should be represented by the motions of the localizable 10D "field-shell" in DST.

4 Electron as flexible reference frame

The plane wave $\exp i(Et - \vec{P}\vec{x})$ was used by Einstein and de Broglie in two opposite directions. For quantization of the EM wave, i.e. for energy of photons $E = \hbar\omega$ (Einstein (1905)), for momentum of the photons $\vec{P} = \hbar\vec{k}$ (Einstein (1917)), and for the electron waves energy $E = mc^2 = \hbar\omega$ and the momentum $\vec{P} = \hbar\vec{k}$ (de Broglie (1923)). Furthermore, the plane wave solutions being substitute in the Dirac equation gives the correct on-shell dispersion law for point-wise free electrons. The linear superposition of the plane waves connecting two electrons gives in the first approximation of the second quantization the Coulomb potential [21, 22]. Nevertheless, the plane wave solutions of the Dirac equations cannot be associated with a single quantum electron [23].

The mass-shell restriction dictates the Clifford algebra for the matrices of Dirac belonging to the $AlgSU(4)$ and the plane wave solution of the Dirac equation for free electron leads to the well known eigenvalue problem [24]. In such a formulation the "on-shell" condition takes the form of a solvability of the homogeneous linear system $D = (E^2 - m^2c^4 - c^2|\vec{p}|^2)^2 = 0$. These four solutions

$$
\begin{aligned}
&u_1 = 1, u_2 = 0, u_3 = \frac{cp_z}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}}, u_4 = \frac{c(p_x + ip_y)}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}},\\
&u_1 = 0, u_2 = 1, u_3 = \frac{c(p_x - ip_y)}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}}, u_4 = \frac{-cp_z}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}},\\
&u_1 = \frac{-cp_z}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}}, u_2 = \frac{-c(p_x + ip_y)}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}}, u_3 = 1, u_4 = 0,\\
&u_1 = \frac{-c(p_x - ip_y)}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}}, u_2 = \frac{cp_z}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}}, u_3 = 0, u_4 = 1. \qquad (5)
\end{aligned}
$$

may be rewritten in inhomogeneous coordinates $\pi^i_{(j)} = \frac{u_i}{u_j}$ in the four maps $U_1 : \{u_1 \neq 0\}, U_2 : \{u_2 \neq 0\}, U_3 : \{u_3 \neq 0\}, U_4 : \{u_4 \neq 0\}$. If one decides to use the local inhomogeneous coordinates π^i of the state vector initially, then the single-value solutions of the tree linear inhomogeneous equation may be obtained in Cramer's rule under the condition $D = (E^2 - m^2c^4 - c^2|\vec{p}|^2)^2 \neq 0$ [10, 8]. Thereby, the "off-shell" zone will be accessible for the internal field dynamics of the electron. Say, solution in the map $U_4 : \{u_4 \neq 0\}$ may be written in the form

$$
\begin{aligned}
\pi^1 &= \frac{-c(p_x - ip_y)}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}}\frac{c|\vec{p}|}{c|\vec{p}|};\\
\pi^2 &= \frac{cp_z}{mc^2 + \sqrt{m^2c^4 + c^2|\vec{p}|^2}}\frac{c|\vec{p}|}{c|\vec{p}|};\\
\pi^3 &= 0. \qquad (6)
\end{aligned}
$$

Then introducing $\tan(\theta) = \frac{c|\vec{p}|}{mc^2+\sqrt{m^2c^4+c^2|\vec{p}|^2}}$ and $f^1 = -(p_x - ip_y), f^2 = p_z, f^3 = 0$ with $g^2 = |f^1|^2 + |f^2|^2 + |f^3|^2 = |\vec{p}|^2 = p_x^2 + p_y^2 + p_z^2$, one may rewrite the local coordinates of the UQS of the geodesic in $CP(3)$ as follows

$$
\begin{aligned}
\pi^1(\theta) &= \frac{f^1}{g}\tan(\theta);\\
\pi^2(\theta) &= \frac{f^2}{g}\tan(\theta);\\
\pi^3 &= 0. \qquad (7)
\end{aligned}
$$

One may note that the mass m as a parameter may be deleted from the $\tan(\theta)$.

It is clear that whole geodesic (7) contains the plane waves with the full spectrum of the wave lengths $0 \leq |\vec{p}| < \infty$. Probably, it is difficult to show that all geodesics rotated by the $H = U(1) \times U(3)$ contains all directions of the $\vec{p}$ but the "bundle" of nearby geodesics in the vicinity of the basic geodesic (7) will contains the plane waves with small deviations around $\vec{p}$.

Therefore, the main idea is to replace the wave packet of the plane waves that is unstable due to the dispersion by the stable "bundle" of the close geodesics in $CP(3)$. But this "bundle" is governing by the "quantum Newton equation", i.e. Jacobi equation that includes the holomorphic sectional curvature κ and the affine gauge potential Γ^i_{km}. Commonly used solutions of the Jacobi equations in the parallel transported along a geodesic reference frame [25, 26] eliminates the action of the affine potential. This is an analogous of the local "freely falling down frame" where gravitation effects does not exists. I think, however, that the affine Higgs-like potential plays the essential role in the quantum intrinsic dynamics. The second derivative of the Jacobi field defined by the curvature of $CP(3)$, serves as an field quantum analog of the classical point-wise acceleration. In order to suppress the divergency of the geodesic "bundle", I introduced the compensation fields of the $SU(4)$ generators corresponding to the shifts, rotations and boosts in $CP(3)$ (instead of the creation-annihilation operators) arising from the matrices of Dirac [5]. Thereby, the dynamical deformation of the spin/charge UQS opens the real way to the interpretation of the boosts as internal electric field, rotations as the internal magnetic field of the spin, and the shifts as the quantum inertial and potential terms.

In view of the future discussion of infinitesimal unitary transformations, it is useful to compare *velocity* of variation of the Berry's phase

$$\dot{\gamma}_n(t) = -\mathbf{A}_n(\mathbf{R})\dot{\mathbf{R}}, \tag{8}$$

where $\mathbf{A}_n(\mathbf{R}) = \Im < n(\mathbf{R})|\nabla_{\mathbf{R}} n(\mathbf{R}) >$ [18] with the affine parallel transport of the vector field $\xi^k(\pi^1, ..., \pi^{N-1})$ given by the equations

$$\frac{d\xi^i}{d\tau} = -\Gamma^i_{kl}\xi^k \frac{d\pi^l}{d\tau}. \tag{9}$$

The affine parallel transport is the fundamental because this agrees with Fubini-Study "quantum metric tensor"

$$G_{ik^*} = \kappa^{-1}[(1 + \sum |\pi^s|^2)\delta_{ik} - \pi^{i^*}\pi^k](1 + \sum |\pi^s|^2)^{-2}, \tag{10}$$

in the base manifold $CP(N-1)$ [19]. The affine gauge field given by the connection

$$\Gamma^i_{mn} = \frac{1}{2}G^{ip^*}(\frac{\partial G_{mp^*}}{\partial \pi^n} + \frac{\partial G_{p^*n}}{\partial \pi^m}) = -\frac{\delta^i_m \pi^{n^*} + \delta^i_n \pi^{m^*}}{1 + \sum |\pi^s|^2}. \tag{11}$$

whose potential shape for $CP(1)$ is depicted in Fig.1 is similar to the Higgs potential. It is involved in the affine parallel transport of LDV's which agrees with the Fubini-Study metric [11, 13, 10].

It is interesting that Anandan and Aharonov [27] insisting on the reasonable usage of the the local projective coordinates in $CP(N-1)$ instead of the coordinates $\mathbf{R}$ in the parameter space a lá Berry consistently avoided to use the affine

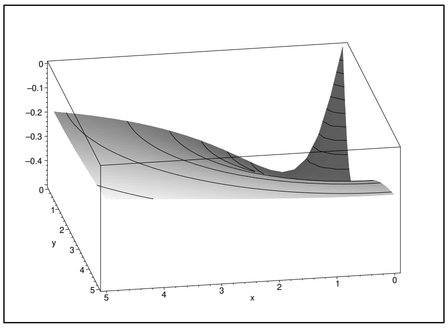

Figure 1: The shape of the gauge potential associated with the affine connection in CP(1): $\Gamma = -2\frac{|\pi|}{1+|\pi|^2}, \pi = x + iy$.

gauge potential replaced this by the "distant parallelism" arisen due to the introduction of the "second particle" in the role of the "quantum environment" with the common Hilbert space. However, this model is in fact non-relativistic and semi-classical. In the framework of the true QED model one has the interaction of two currents with all well known problems like divergences, non-unitary representations, etc.

The affine Cartan's moving reference frame takes here the place of "flexible quantum setup", whose motion refers to itself with infinitesimally close coordinates. Thus we will be rid of necessity in "second particle" as an external reference frame [27, 9, 12, 10]. Such construction perfectly fits for the quantum formulation of the quantum inertia principle [8] since the affine parallel transport of energy-momentum vector field in $CP(N-1)$ expresses the self-conservation and the conditions of stability of, say, electron.

5 Dynamical generators of Poincaré and dynamical spacetime

On the classical level the existence of bodies and forces are simply assumed. Now the existence of "elementary particles" reduced to the notion of "observable". But observation (measurement) could not be even provided. Nevertheless, particles exist! Intuitively it is clear that existence somehow connected with some kind of conservation laws and stability. I will discuss some dynamical conservation laws for quantum extended electron treated as a dynamical process.

In order to formulate the quantum (internal) energy-momentum conservation law in the state space one needs the invariant classification of quantum motions [9, 12]. This invariant classification is the quantum analog of classical conditions of inertial and accelerated motions. They are rooted into the global geometry of the dynamical group manifold. Namely, the geometry of $G = SU(N)$, the isotropy group $H = U(1) \times U(N-1)$ of the pure quantum state, and the coset $G/H = SU(N)/S[U(1) \times U(N-1)]$ geometry, play an essential role in classification of quantum state motion [11]. It will be used in the model of self-interacting quantum electron where spin/charge degrees of freedom in C^4 have been taken into account [5, 6].

There are a lot of attempts to build speculative model of electron as extended compact object in existing space-time [15]. The model of electron proposed here is quite different. Self-interacting quantum electron is *a cyclic motion of quantum degrees of freedom along closed geodesics* in projective Hilbert state space $CP(3)$. Namely, it is assumed that the motion of spin/charge degrees of freedom comprises of stable attractor in the state space, whereas its "field-shell" in dynamical space-time arises as a consequence of the local conservation law of energy-momentum vector field. *This conservation law leads to PDE's whose solution give the distribution of energy-momentum in DST that keeps motion of spin/charge degrees of freedom along geodesic in* $CP(3)$. The cyclic motion of quantum spin/charge degrees of freedom generated by the coset transformations from $G/H = SU(4)/S[U(1) \times U(3)] = CP(3)$ will be associated with inertial "mechanical mass" and the gauge transformations from $H = U(1) \times U(3)$ rotates closed geodesics in $CP(3)$ as whole will be associated with the "field-shell" (mostly electromagnetic) energy.

In order to build the LDV corresponding to the internal energy-momentum of relativistic quantum electron I will use the matrices of Dirac corresponding to the dynamical shift $\hat{\gamma}_t, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3$

$$\Phi^i_\mu = \lim_{\epsilon \to 0} \epsilon^{-1} \left\{ \frac{[\exp(i\epsilon\hat{\gamma}_\mu)]^i_m \psi^m}{[\exp(i\epsilon\hat{\gamma}_\mu)]^j_m \psi^m} - \frac{\psi^i}{\psi^j} \right\} = \lim_{\epsilon \to 0} \epsilon^{-1} \{\pi^i(\epsilon\hat{\gamma}_\mu) - \pi^i\}, \tag{12}$$

[5, 6] that gives the coefficient functions

$$\begin{aligned}
\Phi^1_0(\hat{\gamma}_t) &= i(\pi^3 - \pi^1\pi^2), \quad \Phi^2_0(\hat{\gamma}_t) = i(1-(\pi^2)^2), \quad \Phi^3_0(\hat{\gamma}_t) = i(\pi^1 - \pi^2\pi^3); \\
\Phi^1_1(\hat{\gamma}_1) &= -i(\pi^2 - \pi^1\pi^3), \quad \Phi^2_1(\hat{\gamma}_1) = -i(-\pi^1 - \pi^2\pi^3), \quad \Phi^3_1(\hat{\gamma}_1) = -i(-1-(\pi^3)^2); \\
\Phi^1_2(\hat{\gamma}_2) &= -i(i(\pi^2 + \pi^1\pi^3)), \quad \Phi^2_2(\hat{\gamma}_2) = -i(i(\pi^1 + \pi^2\pi^3)), \quad \Phi^3_2(\hat{\gamma}_2) = -i(i(-1+(\pi^3)^2)); \\
\Phi^1_3(\hat{\gamma}_3) &= -i(-\pi^3 - \pi^1\pi^2), \quad \Phi^2_3(\hat{\gamma}_3) = -i(-1-(\pi^2)^2), \Phi^3_3(\hat{\gamma}_3) = -i(\pi^1 - \pi^2\pi^3).
\end{aligned} \tag{13}$$

for the local "spacetime shift" generator

$$\vec{P}_\mu = \frac{\partial}{\partial x^\mu} = \Phi^i(P_\mu)\frac{\partial}{\partial \pi^i} + c.c. \tag{14}$$

Such choice of the vector fields leads to the "imaginary" basic in local DST which conserves $4D$ Eucledian geometry along geodesic in $CP(3)$ for real four vectors (p^0, p^1, p^2, p^3) and correspondingly $4D$ pseudo-Eucledian geometry for four vectors (ip^0, p^1, p^2, p^3). There are six products of the these matrices generating rotations $\hat{R}_x = (i/2)\hat{\gamma}_y\hat{\gamma}_z, \hat{R}_y = (i/2)\hat{\gamma}_z\hat{\gamma}_x, \hat{R}_z = (i/2)\hat{\gamma}_x\hat{\gamma}_y$ and boosts $\hat{B}_x = (i/2)\hat{\gamma}_t\hat{\gamma}_x, \hat{B}_y = (i/2)\hat{\gamma}_t\hat{\gamma}_y, \hat{B}_y = (i/2)\hat{\gamma}_t\hat{\gamma}_z$ of the Poincaré group. The corresponding coefficient functions of the vector fields of the Lorentz generators is as follows for boosts

$$\Phi^i(\hat{B}_\alpha) = \lim_{\epsilon \to 0} \epsilon^{-1} \left\{ \frac{[\exp(\epsilon\hat{B}_\alpha)]^i_m \psi^m}{[\exp(\epsilon\hat{B}_\alpha)]^j_m \psi^m} - \frac{\psi^i}{\psi^j} \right\} = \lim_{\epsilon \to 0} \epsilon^{-1} \{\pi^i(\epsilon\hat{B}_\alpha) - \pi^i\}, \tag{15}$$

$$\Phi^1(\hat{B}_x) = \frac{1}{2}(1-(\pi^1)^2), \Phi^2(\hat{B}_x) = \frac{-1}{2}(\pi^3+\pi^1\pi^2), \Phi^3(\hat{B}_x) = \frac{-1}{2}(\pi^2+\pi^1\pi^3),$$
$$\Phi^1(\hat{B}_y) = -\frac{i}{2}(1+(\pi^1)^2), \Phi^2(\hat{B}_y) = -\frac{i}{2}(\pi^3+\pi^1\pi^2), \Phi^3(\hat{B}_y) = \frac{i}{2}(\pi^2-\pi^1\pi^3),$$
$$\Phi^1(\hat{B}_z) = -\pi^1, \Phi^2(\hat{B}_z) = -\pi^2, \Phi^3(\hat{B}_z) = 0, \quad (16)$$

and for rotations

$$\Phi^i(\hat{R}_\alpha) = \lim_{\epsilon\to 0} \epsilon^{-1} \left\{ \frac{[\exp(\epsilon\hat{R}_\alpha)]^i_m \psi^m}{[\exp(\epsilon\hat{R}_\alpha)]^j_m \psi^m} - \frac{\psi^i}{\psi^j} \right\} = \lim_{\epsilon\to 0} \epsilon^{-1}\{\pi^i(\epsilon\hat{R}_\alpha) - \pi^i\}, \quad (17)$$

$$\Phi^1(\hat{R}_x) = \frac{i}{2}(1-(\pi^1)^2), \Phi^2(\hat{R}_x) = \frac{i}{2}(\pi^3-\pi^1\pi^2), \Phi^3(\hat{R}_x) = \frac{i}{2}(\pi^2-\pi^1\pi^3),$$
$$\Phi^1(\hat{R}_y) = \frac{1}{2}(1+(\pi^1)^2), \Phi^2(\hat{R}_y) = -\frac{1}{2}(\pi^3-\pi^1\pi^2), \Phi^3(\hat{R}_y) = \frac{1}{2}(\pi^2+\pi^1\pi^3),$$
$$\Phi^1(\hat{R}_z) = -i\pi^1, \Phi^2(\hat{R}_z) = 0, \Phi^3(\hat{R}_z) = -i\pi^3, \quad (18)$$

Then the three generators

$$\vec{B}_\alpha = \Phi^i(\hat{B}_\alpha)\frac{\partial}{\partial \pi^i} + c.c. \quad (19)$$

define the boosts and three generators

$$\vec{R}_\alpha = \Phi^i(\hat{R}_\alpha)\frac{\partial}{\partial \pi^i} + c.c. \quad (20)$$

define the rotations. The commutators of these vector fields may be found in [5].

6 Quantum formulation of the inertia principle

Two aspects of a classical force action: acceleration relative inertial reference frame and deformation of the body are very important already on the classical level as it has been shown by Newton's bucket rotation. The second aspect is especially important for quantum "particles" since the classical acceleration requires the point-like localization in space-time; such localization is, however, very problematic in quantum theory. Nevertheless, almost all discussions in foundations of quantum theory presume that spacetime structure is close with an acceptable accuracy to the Minkowski geometry and may be used without changes in quantum theory up to Planck's scale or up to topologically different space-time geometry of string theories. Under such approach, one loses the fact that spacetime relationships and geometry for quantum objects should be reformulated totally *at any space-time distance* since from the quantum point of view such fundamental dynamical variables as "time-of-arrival" of Aharonov

[17]and the position operator of Newton and Wigner [16] representations are state-dependent [9, 12, 10]. Therefore the spacetime itself should be built in the frameworks of a new "quantum geometry".

In such a situation, one should make accent on the second aspect of the force action – the body deformation. In fact, microscopically, it is already a *different body* with different temperature, etc., since the state of body is changed [12]. In the case of inertial motion one has the opposite situation – the internal state of the body does not change, i.e. body is self-identical during inertial spacetime motion. In fact this is the basis of all classical physics. Generally, spacetime localization being treated as ability of the coordinate description of an object in classical relativity closely connected with operational identification of "events". It is tacitly assumed that all classical objects (frequently represented by material points) are self-identical and they cannot disappear during inertial motion because of the energy-momentum conservation law.

The inertia law of Galileo-Newton ascertains this self-conservation "externally". Einstein, however, clearly understood the logical inconsistence of the classical formulation of the inertia principle: "The weakness of the principle of inertia lies in this, that it involves an argument in a circle: a mass moves without acceleration if it is sufficiently far from other bodies; we know that it is sufficiently far from other bodies only by the fact that it moves without acceleration" [14]. This argument may be repeated with striking force being applied to non-localizable quantum objects since for such objects the "sufficiently far" distance is simply not defined. One should find more reliable footing for the consistent theory.

In order to tear off the logical circle one should use in quantum area more primitive primordial elements than bodies. Even "elementary particles" are not sufficiently primitive. I will use the pure quantum degrees of freedom like spin, charge, etc. Then the distance between two unlocated quantum states has been used as a basic concept instead of the distance between two bodies. Such eigen-dynamics of the unlocated quantum states based on the invariant geometric classification of quantum motions [11] The *existence* of electron and other quantum particles may be physically provided by the self-interaction that should lead to stable periodic process a lá de Broglie. Closed geodesics in complex projective Hilbert space $CP(N-1)$ is the simplest and natural possibility to describe such internal gauge invariant motions. The coset manifold $G/H_{|\psi>} = SU(N)/S[U(1) \times U(N-1)] = CP(N-1)$ contains locally unitary transformations *deforming* "initial" quantum state $|\psi>$. This means that $CP(N-1)$ contains physically distinguishable, "deformed" quantum states. Thereby the unitary transformations from $G = SU(N)$ of the basis in the Hilbert space may be identified with the unitary state-dependent gauge field $U(|\psi>)$ that may be represented by the N^2-1 unitary generators as functions of the local projective coordinates $(\pi^1, ..., \pi^{N-1})$.

I formulate the following requirement: the projection of the trajectory of a single quantum particle onto $CP(N-1)$ should be a geodesic since all geodesics in $CP(N-1)$ are closed that provide the periodicity by the natural manner.

Then the speed of the UQS components

$$T^i = \frac{d\pi^i}{d\tau} = \frac{c}{\hbar}[P^\mu \Phi^i_\mu + K^\alpha \Phi^i(B_\alpha) + M^\alpha \Phi^i(R_\alpha) + J^i] \quad (21)$$

should obey the nullification of the covariant derivative in the sense of the Fubini-Study metric

$$T^i_{;k} = (P^\sigma \Phi^i_\sigma)_{;k} + J^i_{;k} = \frac{\partial P^\sigma}{\partial \pi^k}\Phi^i_\sigma + P^\sigma(\frac{\partial \Phi^i_\sigma}{\partial \pi^k} + \Gamma^i_{kl}\Phi^l_\sigma) + J^i_{;k} = 0. \quad (22)$$

The Jacobi fields to be taken in the fixed basis [6]. One may assume that the "Schrödinger equation"

$$i\hbar \frac{d\Psi(\pi, x_\mu, \vec{u}, \vec{\omega})}{d\tau}$$
$$= [cP^\mu \Phi^i_\mu + K^\alpha \Phi^i(B_\alpha) + M^\alpha \Phi^i(R_\alpha) + J^i]\frac{\partial \Psi(\pi, x_\mu, \vec{u}, \vec{\omega})}{\partial \pi^i} + c.c. = 0, \quad (23)$$

where the coordinates $(\pi^i, x_\mu, \vec{u}, \vec{\omega})$ correspond to the shifts, rotations, boosts and gauge parameters of the local DST and τ is the *quantum elapsed time counted from the start of the internal motion.* This equation expresses the conservation of the action for the electron. The calculation of the self-energy of the electron postponed for future work. It contains the non-Abelian field current interacting with EM-like "field shell" of the electron contains as some part of the internal energy of electron compensating the "divergency" of the Jacobi field. The relativistic Hamiltonian vector field

$$\vec{H} = c[P^\mu \Phi^i_\mu + K^\alpha \Phi^i(B_\alpha) + M^\alpha \Phi^i(R_\alpha) + J^i]\frac{\partial}{\partial \pi^i} + c.c. \quad (24)$$

may be used for the eigen-value problem in terms of the PDE for the total wave function $\Psi(\pi, x_\mu, \vec{u}, \vec{\omega})$.

In order to find physically acceptable solutions of this equation one needs to put the gauge and the "border" restrictions on meanwhile undefined functions P^α. Our requirement tells that the projection of the trajectory of a single quantum particle onto $CP(N-1)$ should be a geodesic. Hence, the covariant derivative in the sense of the Fubini-Study metric of the velocity of UQS $\frac{d\pi^i}{d\tau}$ should be zero

$$(P^\alpha \Phi^i_\alpha)_{;k} = \frac{\partial P^\alpha}{\partial \pi^k}\Phi^i_\alpha + P^\alpha(\frac{\partial \Phi^i_\alpha}{\partial \pi^k} + \Gamma^i_{kl}\Phi^l_\alpha) = 0. \quad (25)$$

One sees that the dynamical system for non-linear field momentum is self-consistent since the speed of the traversing the geodesic in $CP(N-1)$ is not a constant but a variable value "modulated" by the field coefficients P^α.

Let me take initially only the shifts in DST without rotations and boosts. Then in the equation (25) one will have the summation only of four terms

$$(P^\mu \Phi^i_\mu)_{;k} = \frac{\partial P^\mu}{\partial \pi^k}\Phi^i_\mu + P^\mu(\frac{\partial \Phi^i_\mu}{\partial \pi^k} + \Gamma^i_{kl}\Phi^l_\mu) = 0. \quad (26)$$

In order to get the field equations in DST, I will use the definition of the DST derivatives. Thus one may rewrite this equations for $k = i$ as follows

$$\frac{\partial P^\mu}{\partial x^\mu} + P^\mu(\frac{\partial \Phi^i_\mu}{\partial \pi^i} + \Gamma^i_{il}\Phi^l_\mu) = -J^i_{;i}. \tag{27}$$

Thus one has the gauge restriction in the form of the field equation. For the parallel transported Φ^i_μ this gauge restriction coincides with the ordinary Lorentz gauge. This linear PDE has the traveling wave solutions (TWS), say, in the form

$$P^\mu = K^\mu + A^\mu F(\Phi^i_\mu)\tanh(C_0 + C_1 x + C_2 y + C_3 z + C_4 t)$$
$$+B^\mu G(\Phi^i_\mu)\tanh(C_0 + C_1 x + C_2 y + C_3 z + C_4 t)^2 + H^\mu(\Phi^i_\mu). \tag{28}$$

Such solutions realize the state-dependent gauge conditions on the energy-momentum (potentials) and show that in the given definition of the DST coordinates x^μ the complicated highly nonlinear field equations (25) transform into the linear PDE's (27) with soliton-like solution (28) or within more wide class of TWS's.

In general case of the full Poincaré motions in $10D$ DST one has correspondingly

$$\frac{\partial P^\mu}{\partial x^\mu} + P^\mu(\frac{\partial \Phi^i_\mu}{\partial \pi^i} + \Gamma^i_{il}\Phi^l_\mu)$$
$$+\frac{\partial K^\alpha}{\partial u^\alpha} + K^\alpha(\frac{\partial \Phi(B^i_\alpha)}{\partial \pi^i} + \Gamma^i_{il}\Phi(B^l_\alpha)$$
$$+\frac{\partial M^\alpha}{\partial \omega^\alpha} + M^\alpha(\frac{\partial \Phi(R^i_\alpha)}{\partial \pi^i} + \Gamma^i_{il}\Phi(R^l_\alpha)) = -J^i_{;i}. \tag{29}$$

with more complicated but similar TWS solutions. Nevertheless since each such solution contains the π^i coordinates only in the rational manner the PDE's for the parallel transport condition (21) will be pure algebraic. Therefore, one has the field of the energy-momentum in the local $10D$ DST as the functions of π^i on the each physical gauge "sheet" defined by the "border" choice of the integration constants. It is important that DST argument of the TWS function $\xi = \frac{1}{\hbar} q_a C^a, (1 \leq a \leq 10)$ will be equal to the action invariant of the single classical material point

$$S = -a_\mu P^\mu + \frac{1}{2}\Omega_{\mu\nu}M^{\mu\nu} = const \tag{30}$$

under the appropriate choice of these constants. One needs the boundary conditions for the "field shell".

The non-Abelian field equations for the internal current of the electron may be written directly from these commutation relations and may be compared with the Maxwell equations if the dynamical shift will be treated as the differentiation in corresponding direction. The following equations

$$[P_3[P_2, P_1]] = [P_0[P_2, P_1]] = [P_2[P_3, P_1]]$$

$$== [P_0[P_3,P_1]] = [P_1[P_2,P_3]] = [P_0[P_2,P_3]] = 0; \tag{31}$$

are more strong than the Jacobi identity.

The equations

$$[P_1[P_0,P_1]] + [P_2[[P_0,P_2]] + [P_3[P_0,P_3]]$$
$$= (\xi^1 = 12i(\pi^1\pi^2 - \pi^3), \xi^2 = 12i(-1+(\pi^2)^2), \xi^3 = 12i(-\pi^1 + \pi^2\pi^3) \tag{32}$$

are the analog of the Maxwell equation with the vector charge

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho. \tag{33}$$

The following equations

$$[P_1,R_2] - [P_2,R_1] = (\xi^1 = 2i(\pi^1\pi^2 + \pi^3), \xi^2 = 2i(1+(\pi^2)^2), \xi^3 = 2i(-\pi^1 + \pi^2\pi^3)),$$
$$[P_3,R_1] - [P_1,R_3] = (\xi^1 = 2(\pi^2 + \pi^1\pi^3), \xi^2 = 2(\pi^1 + \pi^2\pi^3), \xi^3 = -2(1-(\pi^3)^2),$$
$$[P_2,R_3] - [P_3,R_2] = (\xi^1 = 2i(\pi^1\pi^3 - \pi^2), \xi^2 = 2i(\pi^1 + \pi^2\pi^3), \xi^3 = 2i((1+(\pi^3)^2)) \tag{34}$$

are similar to the equations

$$\nabla \times \vec{B} = \vec{J}. \tag{35}$$

Thereby, old attempts to identify boosts and rotations with electric and magnetic field have to be reformulated as the intrinsic non-Abelian field current of the quantum electron. The generators of the internal fields (14),(19),(20) are in involution, hence, according to the theorem of Frobenius, they are quite integrable. These field currents serve as the natural boundary conditions for the "field shell" equations (29) with wide class of the TWS's [6]. This equation resembles the equation for the so-called quantum potential where the divergency of the Jacobi field $Q = -J^i_{;i}$ plays now the role of such potential. This topic will be discussed elsewhere.

7 Dynamics of the Jacobi fields

The local gauge invariance was ordinary connected with the invariance of the Maxwell equations. The Yang-Mills fields serve as local non-Abelian gauge fields in the Standard Model (SM). These fields generalize the Abelian gauge fields by the formal introducing more general kind of "covariant" derivative in the persisting Minkowski space-time. The principle difficulties and technical obstacles like diffeomorphism anomalies on the road to intrinsic unification of relativity and quantum theory leads to necessary to review the "standard" approach. Shortly speaking the Yang-Mills gauge fields serves for the accommodation of quantum state to the persisting Minkowski spacetime position but my approach seeks the opposite aim: the dynamical spacetime structure accommodation to the quantum state, i.e. the space-time will be now state-dependent. This peculiarity will be realized due to the "inverse representation" mentioned above.

The a current conservation is a consequence of the dynamical self-conservation of the extended electron. Since electron is stable during interaction with electromagnetic field (EMF) in very wide range of the field intensity one may assume that state-dependent gauge invariance is needed for dynamical conservation of the quantum structure of the electron. The PDE's for "field-shell" of the quantum electron supporting the geodesic motion of the generalized coherent state (GCS) along geodesic line in $CP(3)$. Thereby, new formulation of the inertia principle in terms of self-conservation of internal quantum energy-momentum opens a way for a discrete spectrum of mass. This formulation is based on essentially different kind of the gauge group realization that used in present well established physical theories. Invariance of the electron structure under the action of gauge field will be now expressed by the Jacobi geodesic variation of the initial geodesic in $CP(3)$.

I will introduce the universal running coupling constant κ in the following way.

$$\kappa = \frac{e^2}{m_1 c^2}\frac{m_2 c^2}{\hbar c} = \alpha\frac{m_2}{m_1} = \alpha\frac{E_2}{E_1} = \alpha\frac{\lambda_1}{\lambda_2} \tag{36}$$

Furthermore, this running constant will be used as the holomorphic sectional curvature for the specialization of the IC's (foliation) of the coset manifold $CP(N-1)$. If $m_1 = m_2$ then the "classical radius" is equal to the Compton wave length for corresponding particle and one has $\kappa = \alpha = \frac{e^2}{\hbar c} \approx 1/137$. It is the zone of the self-interaction but typically $m_2 << m_1$ and $\kappa \approx 0$ when one uses the "scanning" of quantum particles by the EM waves. The opposite case $m_1 << m_2$ corresponds to a "strong" interaction, say, due to collision whose energy much larger than the rest mass of particles. The question is: does the dynamical instability of the Jacobi field generate mass, electric charge and spin?

I will show here that such variable sectional curvature serves as the bifurcation parameter in the reconstruction of the phase portraits of the Jacobi field dynamics. This demonstration requires of course strict analysis which is postponed for future work. In parallel with few phase portraits for different values of the sectional curvature, I will show the corresponding components of the Jacobi field.

Generally, the Jacobi fields in $CP(N-1)$ obey the equation with the solutions expressed in Heun's -functions. But along the geodesic in the parametric form $\pi^i = \frac{f^i}{g}\tan(g\tau)$ they have more simple form

$$\frac{d^2 J^i}{d\tau^2} - 2g\tan g\tau(\delta^i_k + \frac{f^i f^{k*}}{g^2})\frac{dJ^k}{d\tau} + \kappa g^2(\delta^i_k + \frac{f^i f^{k*}}{g^2})J^k = 0. \tag{37}$$

[5]. Taking for simplicity the set $(f^1 = 1, f^2 = f^3 = 0)$ one get the three equations

$$\frac{d^2 J^1}{d\tau^2} - 4\tan\tau\frac{dJ^1}{d\tau} + 2\kappa J^1 = 0,$$
$$\frac{d^2 J^2}{d\tau^2} - 2\tan\tau\frac{dJ^2}{d\tau} + \kappa J^2 = 0,$$

$$\frac{d^2J^3}{d\tau^2} - 2\tan\tau\frac{dJ^3}{d\tau} + \kappa J^3 = 0. \tag{38}$$

The general solutions of these equations are as follow:

$$\begin{aligned} J^1 &= C_1\cos(\tau)^{-3/2}P(\sqrt{2\kappa+4}-1/2,3/2,\sin(\tau)) \\ &+ C_2\cos(\tau)^{-3/2}Q(\sqrt{2\kappa+4}-1/2,3/2,\sin(\tau)), \\ J^2 &= C_3\cos(\tau)^{-1}\sinh(\sqrt{-1-\kappa\tau}) + C_4\cos(\tau)^{-1}\cosh(\sqrt{-1-\kappa\tau}), \\ J^3 &= C_5\cos(\tau)^{-1}\sinh(\sqrt{-1-\kappa\tau}) + C_6\cos(\tau)^{-1}\cosh(\sqrt{-1-\kappa\tau}), \end{aligned} \tag{39}$$

where $P(\sqrt{2\kappa+4}-1/2,3/2,\sin(\tau)), Q(\sqrt{2\kappa+4}-1/2,3/2,\sin(\tau))$ are the associate Legendre functions of the first and the second kinds. It is clear that more complicated choice for the complex velocity traversing of the basic geodesic $(f^1, f^2, f^3, ..., f^{N-1})$ gives more complicated solutions. Such solutions should be included in the equations (29).

The complicated behavior of these solutions under differen values of the sectional curvature looks like the parametric instability and the decay of the "homogeneous mode" - the oscillations of UQS's and the creation of the EM-like fields of the "field shell" in the DST.

Let me rewrite the Jacobi equation (43) for the first component $J^1(\tau)$ as follows system

$$\begin{aligned} \frac{dJ^1(\tau)}{d\tau} &= V^1(\tau), \\ \frac{dV^1(\tau)}{d\tau} &= 4V^1(\tau)\tan(\tau) - 2\kappa J^1(\tau). \end{aligned} \tag{40}$$

Then one may use "Maple 17" in order to build the phase portraits of the Jacobi field component. Arbitrary units will be used in all graphics.

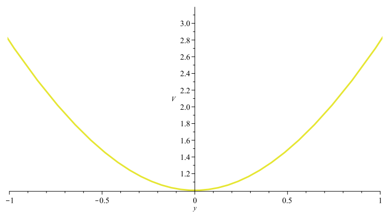

Figure 2: The phase portraits of the Jacobi field at the zero sectional curvature.

One sees that variation of the sectional curvature leads to the reconstruction of the topologically trivial phase portraits to the quite complicated separatrice. The corresponding solutions for the Jacobi fields demonstrate opposite-sine solutions that presumably may be compensated by the opposite charged EM-like "field shell". The question is: could this field configuration may be interpretable as the pairs creation?

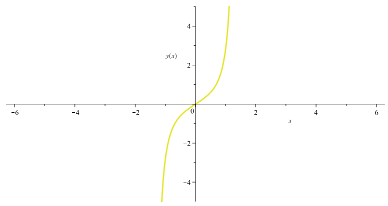

Figure 3: The Jacobi field at zero sectional curvature. Presumably may be compensated by the uncharged and massless gauge field.

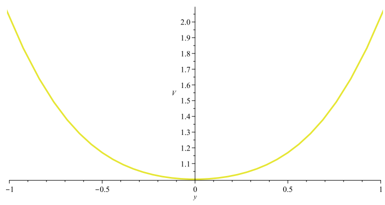

Figure 4: The phase portraits of the Jacobi field at the sectional curvature K=200

8 Conclusion and discussion

Generally speaking the separation of the matter and motion was acceptable in classical physics since the physics and geometry were sharply distinct. Einstein clearly shown that physical measurements taking into account the finite speed of the light lead to the essentially different spacetime geometry. Furthermore, general relativity leads to the blurring border between physics and geometry. But in the quantum physics, due to the quantization of the action and finite uncertainty, the situation has been drastically changed - the quantum geometry is in fact non-distinguishable from the quantum dynamics. First of all because the distance between bodies or particles must be replaced by the distance between unlocated (in spacetime) quantum states corresponding to the pure quantum degrees of freedom. I demonstrated this statement on the example of the self-interacting quantum electron. Localization means the stable motion of the spin/charge UQS along geodesic in $CP(3)$. The stability has been provide by the counterbalance between the Jacobi vector fields and the vector fields of the Poincaré generators built from the matrices of Dirac.

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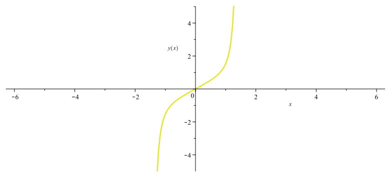

Figure 5: The Jacobi field at the sectional curvature K=200.

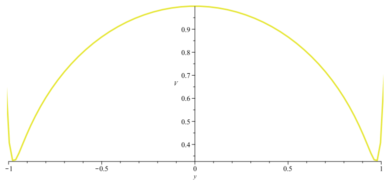

Figure 6: The phase portraits of the Jacobi field at the sectional curvature K=342.3

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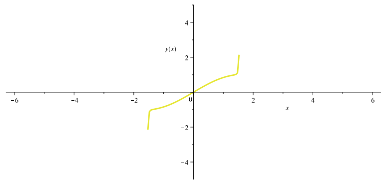

Figure 7: The Jacobi field at the sectional curvature K=342.3

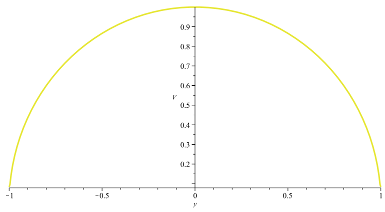

Figure 8: The phase portraits of the Jacobi field at the sectional curvature K=342.5.

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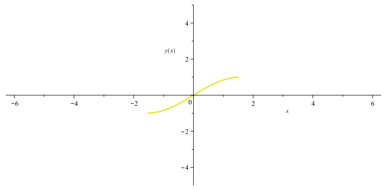

Figure 9: The Jacobi field at the sectional curvature K=342.5.

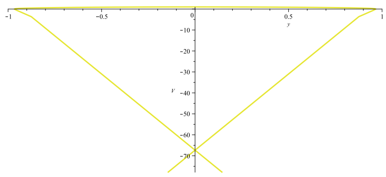

Figure 10: The phase portraits of the Jacobi field at the sectional curvature K=342.7.

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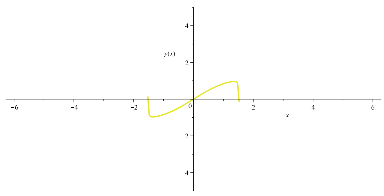

Figure 11: The Jacobi field at the sectional curvature K=342.7.

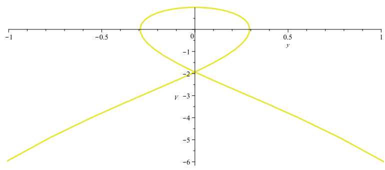

Figure 12: The phase portraits of the Jacobi field at the sectional curvature K=1000.

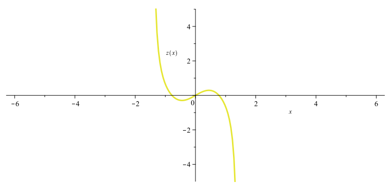

Figure 13: The Jacobi field at the sectional curvature K=1000.

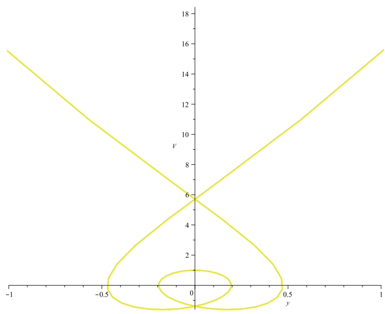

Figure 14: The phase portraits of the Jacobi field at the sectional curvature K=2000.

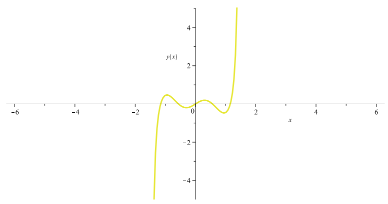

Figure 15: The Jacobi field at the sectional curvature K=2000.

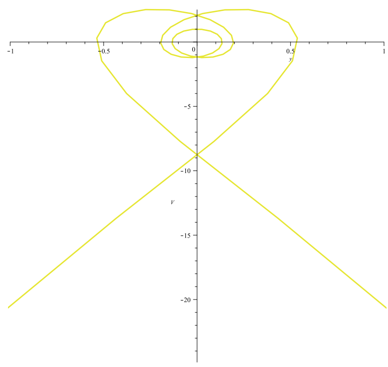

Figure 16: The phase portraits of the Jacobi field at the sectional curvature K=4000.

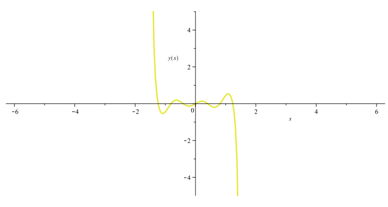

Figure 17: The Jacobi field at the sectional curvature K=4000.