On the Delta Function and its Derivatives

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Abstract

In this writing the conventional law concerning the derivatives of the delta function and those representing the delta function have been considered to bring out certain discrepancies. The errors and their source have been discussed.

Keywords: Delta Function, derivatives of the delta function, theory of Distributions

Introduction

The conventional derivative laws in relation to the delta function and their examples have been analyzed to bring out certain conflicting features.

Inconsistencies with Derivatives of the Delta Function

We consider the fundamental result[1][2] on derivatives of the delta function as given below

$$\int_{-\infty}^{+\infty} f(x)\delta^n(x)\,dx = -\int_{-\infty}^{+\infty} f'(x)\delta^{n-1}(x)\,dx \quad (1)$$

The above holds for any arbitrary function and we have the following result[3]

$$f(x)\delta'(x) = -f'(x)\delta(x) \quad (2)$$

But we have considered the same delta function for all f(x). This notion will be proved erroneous in the article later.

It would be considered more reasonable to consider (1) valid for any subinterval on $(-\infty, +\infty)$ and then conclude(2). If the subinterval does not contain zero each side of (1) would be zero else each side would be the same non zero quantity.

$$\int_{-\epsilon}^{+\epsilon} f(x)\delta^n(x)\,dx = -\int_{-\epsilon}^{+\epsilon} f'(x)\delta^{n-1}(x)\,dx$$

$$\delta'(x) = -\frac{f'(x)}{f(x)}\delta(x) \;\; (3)$$

$$\int_{-\infty}^{+\infty} \delta'(x)dx = -\int_{-\infty}^{+\infty} \frac{f'(x)}{f(x)}\delta(x)$$

$$[\delta(x)]_{-\infty}{}^{+\infty} = -\frac{f'(0)}{f(0)}$$

$$0 = -\frac{f'(0)}{f(0)} \Rightarrow f'(0) = 0 \;\; (4)$$

Since $f(x)$ Is an arbitrary function, well behaved in relation to continuity and differentiability of course,, equation (4) becomes questionable.

Equation (2) is differentiated with respect to x:

$$f'(x)\delta'(x) + f(x)\delta''(x) = -f'(x)\delta'(x) - f''(x)\delta(x)$$

$$2f'(x)\delta'(x) + f(x)\delta''(x) + f''(x)\delta(x) = 0$$

Applying (3) on the last equation we have,

$$-2f'(x)\left[\frac{f'(x)}{f(x)}\delta(x)\right] + f(x)\delta''(x) + f''(x)\delta(x) = 0$$

$$\delta(x)\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right] + f(x)\delta''(x) = 0 \;\; (5)$$

$$\delta''(x) = -\frac{1}{f(x)}\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right]\delta(x) \;\;\; (6)$$

$\delta''(x)$ depends on the nature of the test function $f(x)$ which is not an acceptable idea.

Integrating (4) with respect to x we obtain,

$$\int_{-\infty}^{+\infty} \delta(x)\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right]dx + \int_{-\infty}^{+\infty} f(x)\delta''(x)dx = 0 \; (7)$$

$$\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] + \int_{-\epsilon}^{+\epsilon} f(x)\delta''(x)dx = 0$$

Since for $x \neq 0$,$\delta(x) = 0$ we have $\delta'(x) = 0$ and $\delta''(x) = 0$ [for $x \neq 0$]. Moreover from (5) $\delta''(x)$ is a peaked function like $\delta(x)$: $f(x)$ is expected to vary much slowly than $\delta''(x)$ on an infinitesimally small interval – $\epsilon < x < +\varepsilon$. Therefore

$$\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] + f(0)\int_{-\epsilon}^{+\epsilon} \delta''(x)dx = 0$$

$$\int_{-\epsilon}^{+\epsilon} \delta''(x)dx = -\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] \quad (8)$$

$$[\delta'(x)]_{-\epsilon}{}^{+\epsilon} = -\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right]$$

$$-\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] = 0$$

$$f''(0) = 2\frac{[f'(0)]^2}{f(0)} \quad (9)$$

The above formula[represented by (9)]is not acceptable

We consider the following result[4] :

$$x\delta'(x) = -\delta(x) \quad (10)$$

$$x^n\delta^n(x) = -n!\,(-1)^n\delta(x)$$

$$x^2\delta'^{(x)} = -x\delta(x)$$

$$\int_{-\infty}^{\infty} x^2\delta'\,(x) = -\int_{-\infty}^{+\infty} x\delta(x)\,dx$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^2\delta'\,(x)dx = -\int_{-\epsilon}^{+\epsilon} x\delta(x)\,dx = 0$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^2\delta'\,(x)dx = 0 \; (11)$$

The above is true of any arbitrary interval$(-\epsilon, \epsilon)$. Therefore $x^2\delta'(x)$ should be an odd function. Since $x^2$is an even function $\delta'(x)$ should be odd. That implies $\delta(x)$should be even.

Indeed by integration

$$\int \delta'(x)\,dx = f_{even}(x)$$

$$\delta(x) = f_{even}(x)$$

Since a constant is an even function it may be included in $f_{even}(x)$

[In general any arbitrary function may be expressed as the sum of an even and an odd function. If the even part is not a constant the derivative will be the sum of an even and an odd function.]

Now we consider

$$x^3\delta'(x) = -x^2\delta(x)$$

$$\int_{-\infty}^{\infty} x^3\delta'(x) = -\int_{-\infty}^{+\infty} x^2\delta(x)\,dx$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^3\delta'(x)dx = -\int_{-\epsilon}^{+\epsilon} x^2\delta(x)\,dx = 0$$

$$\int_{-\epsilon}^{\epsilon} x^3\delta'(x)dx = 0 \; (12)$$

The above is true of any $(-\epsilon, \epsilon)$. Therefore δ'(x) should be an even function. With $\delta'(x)$ we have

$$\int \delta'(x)\,dx = f_{odd}(x) + C$$

$$\delta(x) = f_{odd}(x) + C$$

[In general any arbitrary function may be expressed as the sum of an even and an odd function. If the even part is not a constant the derivative will be the sum of an even and an odd function.]

and $\delta(x)$, consequently, an odd function at most with an additive constant as opposed to what we saw earlier: $\delta(x)$= $f_{even}(x)$,

From (10) cannot arrive at (2) by power series technique:$(10) \not\Rightarrow (2)$. Consequently

Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots . + a_2x^2 + a_1x + a_0$$

$$f'(x) = a_n n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + \cdots \ldots . + 2a_2x + a_1$$

From the above expansions it is evident that $f(x)\delta'(x) = -f'(x)\delta(x) \not\Rightarrow x\delta'(x) = -\delta(x)$ and

$x\delta'(x) = -\delta(x) \not\Rightarrow f(x)\delta'(x) = -f'(x)\delta(x)$ though $f'(x) = 1$ if $f(x) = x$

The reason ,as we shall see soon is , that for each function f(x) we require a separate sequence of functions representing the delta function: we have to consider distributions: mapping from functions to real numbers in the form of a linear functional . Even that does not help as we shall see. The delta function, as we know and the idea is a highlighted one in literature , is not a function in the usual sense of being a function. We have ignored this fact while arriving at the contradiction.

Next we consider the standard formula[5]

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dx \quad (13)$$

Differentiating with respect to x, we have,

$$\Rightarrow \delta'(x) = ix\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dk = ik\delta(x)$$

$$\delta'(x) = ix\delta(x) \quad (14)$$

The result given by (14) stands opposed to the standard result given by (10)

$$x\delta'(x) = -\delta(x) \quad (15)$$

The Delta Function in Formal Theory and with Applications

We consider the formal definition of the delta function[6] as a distribution, The Dirac delta function is a linear functional that maps every function to its value at zero. ... In many applications, the Dirac delta is regarded as a kind of limit (a weak limit) of a sequence of functions having a tall spike at the origin (in theory of distributions, this is a true limit).

$$\langle \delta, \varphi \rangle = \varphi(0) \ (16)$$

We have a mapping from a function to a real number[functional]

$\delta: \varphi \rightarrow \varphi(0)$

By way of example the mapping may be achieved as

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(x)dx = \varphi(0) \quad (17)$$

The above example is relevant in applications like physics . As an example[7] we may refer to the derivation of Helmholtz theorem where the following is considered

$$\nabla F = -\nabla^2 U = -\frac{1}{4\pi}\int D(\vec{r}')\nabla^2 \frac{1}{|\vec{r}' - \vec{r}|} dV' = \int D(\vec{r}')\delta(\vec{r}' - \vec{r})dV' = D(\vec{r}) \quad (18)$$

But we have seen the serious errors with

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(x)dx = \varphi(0)$$

when it comes to the derivatives

Further Investigation[for locating the source of error]

A distribution is a mapping from a asset of functions to real numbers. To that end we consider a sequence of functions $G_n(x)$ such that

1. $G_n(x) \neq 0$for $-\epsilon_1(n) < x < \epsilon_2(n)$ else $G_n(x) = 0; \epsilon_1(n) > 0, \epsilon_2(n) > 0$; $G_n(x)$ smooth functions[differentiable and of course continuous]

2. . $lim_{n\to\infty}\epsilon_i(n) = 0; i = 1,2; lim_{n\to\infty}G_n(0) = \infty$;and $lim_{n\to\infty}\int_{-\epsilon_1(n)}^{+\epsilon_2(n)} G_n(x) = 1$

We consider the following identity

$$\int G_n(x)\, G_n'(x)dx = \frac{1}{2}[G_n(x)]^2 + C$$

Since $G_n(-\epsilon_1) = G_n(\epsilon_2) = 0$, then,

$$\int_{-\epsilon_1}^{+\varepsilon_2} G_n(x)G_n'(x)dx = 0$$

If $G_n > 0$on $(-\epsilon_1, \epsilon_2)$ then $G_n'(x)$ will have both positive and negative values on $(-\epsilon, \epsilon)$.,At some point $-\epsilon_1 < c < \epsilon_2$, $G_n'(c) = 0$

$G_n'(x)$ will have at least two turning points

If $G_n(x) > 0$on $(-\epsilon, \epsilon)$ then $G_n'(x) > 0$ on the nearside of $x = -\epsilon$. The peak lying on x=0 or very close to x=0

$$\int_{-\epsilon_1}^{0} f(x)G_n'(x)dx = [f(x)G_n(x)]_{-\epsilon_1}{}^{0} - \int_{-\epsilon_1}^{0} f'(x)G_n(x)dx$$

With $n \to \infty$[for $f(0) \neq 0$]

$$\int_{-\epsilon_1}^{0} f(x)G'(x)dx = \pm\infty - f'(0)$$

With $n \to \infty$[for $f(0) \neq 0$]

$$\Rightarrow \int_{-\epsilon_1}^{0} f(x)G'(x)dx \to \pm\infty$$

Now, $G_n'(-\varepsilon_1) = G_n'(\varepsilon) = G_n'(\varepsilon_2) = 0$; $-\varepsilon_1 < \varepsilon < \varepsilon_2$

$$\int_{-\varepsilon_1}^{\varepsilon} G_n' G_n''(x)dx = \int_{-\varepsilon_1}^{\varepsilon} G_n' dG_n' = \left[\frac{{G_n'}^2}{2}\right]_{-\varepsilon_1}^{\varepsilon} = 0; G_n'(\epsilon) = 0$$

G_n'' will have at least four turning points two on the interval $(-\varepsilon_1, \varepsilon)$ and two on $(\varepsilon, \varepsilon_2)$

Extending our argument we conclude: G_n^s will have at least 2s turning points

Taylor Series Interpretation

We carry out a Taylor expansion of $G_n(x)$ about $x = -\varepsilon_1$

$$G_n(x) = G_n(-\varepsilon_1) + \frac{x+\varepsilon_1}{1!} G_n'(-\varepsilon_1) + \frac{(x+\varepsilon_1)^2}{2!} G_n''(-\varepsilon_1) + \frac{(x+\varepsilon_1)^3}{3!} G_n'''(-\varepsilon_1) + \cdots + \frac{(x+\varepsilon_1)^n}{n!} {G_n}^{(n)}(-\varepsilon_1) + \cdots.$$

The right side is convergent each term being exactly zero if ${G_n}^{(n)}(-\varepsilon_1) = 0$. That implies $G_n(x) = 0$ for any x

We rule out the possibility of derivatives of all orders being zero unless the function is zero everywhere.

Assume derivatives upto order m are zero . We expand about the point $x = -\varepsilon_1$

$$\begin{aligned} G_n(x) &= G_n\big(-\varepsilon_1 + (x+\varepsilon_1)\big) \\ &= \frac{(x+\varepsilon_1)^{m+1}}{(m+1)!} {G_n}^{m+1}(-\varepsilon_1) + \frac{(x+\varepsilon_1)^{m+2}}{(m+2)!} G_n^{m+2}(-\varepsilon_1) \\ &+ \frac{(x+\varepsilon_1)^{m+3}}{(m+3)!} G_n^{m+3}(-\varepsilon_1) + \frac{(x+\varepsilon_1)^{m+4}}{(m+4)!} G_n^{m+4}(-\varepsilon_1) + \cdots + \cdots (A). \end{aligned}$$

For non zero x the series will converge. The derivatives are obviously not of an unfriendly nature

Fo r x=0 for the same set of derivatives at$x = -\varepsilon_1$ the right side will diverge

$$G_n(0) = \frac{(\varepsilon_1)^{m+1}}{(m+1)!} G_n{}^{m+1}(-\varepsilon_1) + \frac{(\varepsilon_1)^{m+2}}{(m+2)!} G_n^{m+2}(-\varepsilon_1) + \frac{(\varepsilon_1)^{m+3}}{(m+3)!} G_n^{m+3}(-\varepsilon_1) + \frac{(\varepsilon_1)^{m+4}}{(m+4)!} G_n^{m+4}(-\varepsilon_1) + \cdots + \cdots . (B)$$

The right side of B will diverge, $G_n{}^{m+k}(-\varepsilon_1)$ remaining the same[for n tending to zero].This is quite unusual .This trouble ensues as an effect of the Taylor series.

Ne3xt we consider the following

$$f(x+2h) = f\big((x+h)+h\big) \;\; (1)$$

Expanding about $(x+h)$

$$f(x+2h) = f(x+h) + \frac{h}{1!} f'(x+h) + \frac{h^2}{2!} f''(x+h) + \frac{h^3}{3!} f'''(x+h) + \cdots \ldots (2)$$

Expanding about $x = x$

$$f(x+2h) = f(x) + \frac{2h}{1!} f'(x+h) + \frac{4h^2}{2!} f''(x+h) + \frac{8h^3}{3!} f'''(x+h) + \cdots \ldots (3)$$

From (2) and (3)

$$f(x+h) + \frac{h}{1!} f'(x+h) + \frac{h^2}{2!} f''(x+h) + \frac{h^3}{3!} f'''(x+h) + \cdots \ldots = f(x) + \frac{2h}{1!} f'(x) + \frac{4h^2}{2!} f''(x) + \frac{8h^3}{3!} f'''(x) + \cdots \ldots$$

$$f(x+h) - f(x) + h[f'(x+h) - 2f'(x)] + \frac{1}{2!} h^2 [f''(x+h) - 4f''(x)] + \frac{1}{3!} h^3 [f'''(x+h) - 8f'''(x)] + \cdots .. = 0 \;(4)$$

$$\frac{f(x+h) - f(x)}{h} \frac{1}{h} + \frac{[f'(x+h) - 2f'(x)]}{h} + \frac{1}{2!} [f''(x+h) - 4f''(x)] + \frac{1}{3!} h [f'''(x+h) - 8f'''(x)] + \cdots .. = 0$$

$$\frac{f(x+h) - f(x)}{h} \frac{1}{h} + \frac{[f'(x+h) - f'(x)]}{h} - \frac{f'(x)}{h} + \frac{1}{2!} [f''(x+h) - 4f''(x)] + \frac{1}{3!} h [f'''(x+h) - 8f'''(x)] + h[\ldots..] = 0$$

With $h \to 0$

$$lim_{h\to 0} f'(x)\frac{1}{h} + f''(x) - lim_{h\to 0} f'(x)\frac{1}{h} - \frac{3}{2!}[f''(x)] - + \cdots .. = 0 \ (5)$$

$$f''(x) - \frac{3}{2!} f''(x) = 0 \Rightarrow f''(x) = 0 \ (6)$$

Applying Integration by Parts

Next we consider for very large 'n'[n tending to infinity] $lim_{n\to\infty} \int_{-\infty}^{+\infty} f(x)G_n(x)\, dx$

$$\int_{-\infty}^{+\infty} f(x)G_n(x)\, dx = \int_{-\infty}^{-\epsilon_1(n))} f(x)G_n(x)dx + \int_{-\epsilon_1(n)}^{+\epsilon_2(n)} f(x)G_n(x)dx + \int_{\epsilon_2(n)}^{+\infty} f(x)G_n(x)dx$$

Since $\int_{-\infty}^{-\epsilon_1(n)} f(x)G_n(x)dx = 0$and $\int_{\epsilon_2(n)}^{+\infty} f(x)G_n(x)dx = 0$ we have

$$\int_{-\infty}^{+\infty} f(x)G_n(x)\, dx = \int_{-\epsilon_1(n)}^{+\epsilon_2(n)} f(x)G_n(x)\, dx \ \ (19)$$

If $f(x)$ changes much slowly with respect to $G_n(x)$ on the interval $-\epsilon_1(n) < x < \epsilon_2(n)$

$$lim_{n\to\infty} \int_{-\infty}^{+\infty} f(x)G_n(x)\, dx = f(0) \ (20)$$

Equation (16) relates to the defining criterion for the delta function

For the nth function

$$\int_{-\infty}^{+\infty} f(x){G_n}'(x)\, dx = \int_{-\epsilon(n)}^{+\epsilon(n)} f(x){G_n}'(x)\, dx = [f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} - \int_{-\epsilon(n)}^{+\epsilon(n)} f'(x)G_n(x)\, dx \ \ (21)$$

$$\int_{-\infty}^{+\infty} f(x){G_n}'(x)\, dx = \int_{-\epsilon(n)}^{+\epsilon(n)} f(x){G_n}'(x)\, dx = [f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} - f'(0) \ \ (22)$$

Case1

We assume

$$lim_{n\to\infty}[f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} = 0$$

together with (17),we have the fundamental laws given by (1)

We have from (18)

$$\int_{-\infty}^{+\infty} f(x){G_n}'(x)\,dx = -\int_{-\epsilon(n)}^{+\epsilon(n)} f'(x)G_n(x)\,dx = -f'(0) = finite$$

for sufficiently large 'n'

$$\Longrightarrow lim_{n\to\infty}\int_{-\infty}^{+x;|x|\epsilon} f(x){G_n}'(x)\,dx = lim_{n\to\infty}a_n(x)bounded, finite$$

Next we consider the following theorem:

If the integral

$$\int_a^c f(x)dx$$

is convergent with $f(x)$ continuous everywhere on (a, c) and then $f(x) > 0[orf(x) < 0\,]$ cannot tend to infinity as x tends to c

Proof[for f(x)>0]:If possible let $f(x) \to \infty$for $x = c; a < c$

We consider the improper integral

$$\int_a^c f(x)dx$$

$$= lim_{\max\{\Delta x_i\}\to 0}\sum_{i=0}^{n-1} f(x_i)\,\Delta x_i; x_0 = a, x_n = c$$

In generalΔx_i are unequal intervals

But we know from our knowledge of integral calculus that the integral expressed by the above limit does not change if we replace the unequal intervals by the equal intervals. That is

$$\int_a^c f(x)dx$$

$$= lim_{h\to 0}\sum_{i=0}^{n-1} f(x_i)\,h; x_0 = a, x_n = c; \Delta x = h \;(23)$$

Now if we look at the infinite sequence expressed by $\sum_{i=0}^{n-1} f(x_i)\,h$ we have successively increasing terms in the vicinity of 'c' and therefore the sum of the corresponding series will be of divergent nature.

Physically or even mathematically , we cannot think of an infinite number of terms. But we can think of an arbitrarily large number of terms with equal intervals $\Delta x_i = h$ so that each successive $f(x_i)h$ is greater than the previous one in the neighborhood of the singularity. In this manner we may think of $\sum_{i=0}^{n-1} f(x_i)\,h$ exceeding any large speculated quantity by consider a suitably large but finite 'n'.

The larger the number of intervals we consider the greater the number of high values of the function that we take into account on the higher side of the interval towards the singularity[but x<c]

For a large speculated[preassigned] quantity A>0 we aspire,

$$\sum_{i=0}^{n-1} f(x_i)\,h > A$$

$$\sum_{i=0}^{n-1} f(x_i)\frac{c-a}{n} > A$$

$$\sum_{i=0}^{n-1} f(x_i)\,(c-a) > nA$$

$$(c-a)\sum_{i=0}^{n-1} f(x_i) > nA \quad (24)$$

We start from a suitable a=a' sufficiently close to c so that $f(x)$ is monotonically increasing on (a', c)

We try for

$$(c-a')f(x_k) > A; a' < x_k < c(25.1)$$

Though c-a' may be small equation (21.1)*is achievable bcause of closeness to the infinite singularity*

At the next step the interval is repartitioned to two equal subintervals with

$$(c-a')\left[f\left(x_{k_1}\right) + f\left(x_{k_2}\right)\right] > 3A \quad (25.2)$$

Equation (25.2) is achievable. We always do have the larger values on the second interval to combat an emergency.

Then we again repartition (a',c) into three equal subintervals such that we have,

$$(c-a')\left[f\left(x_{k_1}\right) + f\left(x_{k_2}\right) + f\left(x_{k_3}\right)\right] > 4A \quad (25.3)$$

Equation (25.3) is achievable. We always do have sufficiently large values on the third interval to combat an emergency.

Finally we have an equation of the type (24)

$$(c-a')\sum_{i=0}^{n-1} f(x_i) > nA \quad (25.4)$$

The last interval is convenient for achieving (25.4)

$\int_{a'}^{c} f(x)dx$ is divergent $\Rightarrow \int_a^c f(x)dx = \int_a^{a'} f(x)dx + \int_{a'}^{c} f(x)dx$ is divergent

Now

$$(c-a')f(x_k)$$

is a finite quantity.

We are not allowed to consider the approximation

$$(c-a')f(x_k) \approx \int_{a'}^{c} f(x)dx$$

since the right side is infinitely large while the left side is finite: $f(x_k)$ is not a representative value of the function on (a',c)

[If $\mathrm{f(x)} < 0$ on $(\mathrm{a,c})$ then we consider $\sum_{\mathrm{i=0}}^{\mathrm{n-1}} \mathrm{f(x_i)\ h} < -A; A > 0$where A is an arbitrarily large but a finite quantity. We are excluding such cases where the

$$\int_a^c f(x)dx$$

covers infinitely large positive and negative areas on (a,c)but is convergent as a whole due to cancelation of infinities resulting in a finite contribution.]

Let us do this in a formal manner . Assume $\int_a^c f(x)dx = I$ is convergent but $f(x)$ blows up at x=c

For any $\epsilon > 0$ we have $N > 0$ such that

$$\left| I - \sum_{i=0}^{n-1} f(x_i)\Delta x_i \right| < \epsilon \quad (25)$$

for$i > N$

On the last interval that is for i=n-1 we can make the value of f(x) arbitrarily large if there is a singulsrity at x=c. The stated inequality will break down

Thus we conclude that for a function continuous on (a, c) the integral

$$\int_a^c f(x)dx$$

can converge only if f(x) does not diverge at x=c We call this our special theorem

Examples:

1.
$$\int_0^{\frac{\pi}{2}} tan^{-1}xdx = \infty[divergent\ integral]$$
2.
$$\int_0^{\pi} tan^{-1}xdx = \int_0^{\pi/2} tan^{-1}\,dx + \int_{\pi/2}^{\pi} tan^{-1} = 0[convergent\ \ integral]$$
The infinities on either side of $x = \pi/2$ cancel out since we have contributions of opposite sign from either side.

By our special theorem we claim that $f(x)G_n{}'(x)$ is finite

That does not match with

$$x\delta'(x) = -\delta(x)$$

For x=0

$$0 \times \delta'(0) = -\infty$$

$\delta'(0)$ should be an infinity. This contradicts $lim_{n\to\infty} G_n'(x)$ =finite

Nevertheless

$$G_n = a_n x + b_n \ (26)$$

As $n \to \infty$, a_n is bounded. Therefore $b_n \to \infty$ as $n \to \infty$,.

G_n becomes discontinuous at $+\epsilon$ with an undefined derivative[right hand and left hand derivatives are unequal: right handed derivative is infinitely larger. The product $f(x)G_n{}'(x)$ has an infinitely large discontinuity and is not integrable. The fundamental law given by (1)will not work.

Case 2

We assume for sufficiently large ‘n’

$$[f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} = a_n = finite\ and\ non\ zero$$

$f(x)$ and $G_n(x)$ are continuous on $(-\epsilon, \epsilon)$

$$\left|\frac{f(\varepsilon_2)G_n(\varepsilon_2) - f(-\varepsilon_1)G_n(-\varepsilon_1)}{\varepsilon_2 - (-\varepsilon_1)}\right| \big(\varepsilon_2 - (-\varepsilon_1)\big) = finite\ non\ zero$$

$$\left|\frac{f(\varepsilon_2(n))G_n\big(\varepsilon_2(n)\big) - f(-\varepsilon_1(n))G_n\big(-\varepsilon_1(n)\big)}{\varepsilon_2 - (-\varepsilon_1)}\right| \Big(\varepsilon_2(n) - \big(-\varepsilon_1(n)\big)\Big) = a_n$$

$$\left|\frac{f(\varepsilon_2)[G_n(\varepsilon_2) - G_n(-\varepsilon_1)] + [f(\varepsilon_2) - f(-\varepsilon_1)]G_n(-\varepsilon_1)}{\varepsilon_2 - (-\varepsilon_1)}\right| (\varepsilon_2 + \varepsilon_1) = a_n$$

$$\left[f(\varepsilon_2)\frac{[G_n(\varepsilon_2) - G_n(-\varepsilon_1)]}{\varepsilon_2 - (-\varepsilon_1)} + G_n(-\varepsilon_1)\frac{[f(\varepsilon_2) - f(-\varepsilon_1)]G_n(-\varepsilon_1)}{\varepsilon_2 - (-\varepsilon_1)}\right] (\varepsilon_2 + \varepsilon_1) = a_n$$

$lim_{n\to\infty}\left[f(\varepsilon_2)\frac{[G_n(\varepsilon_2)-G_n(-\varepsilon_1)]}{\varepsilon_2-(-\varepsilon_1)} + G_n(-\varepsilon_1)\frac{[f(\varepsilon_2)-f(-\varepsilon_1)]G_n(-\varepsilon_1)}{\varepsilon_2-(-\varepsilon_1)}\right] (\varepsilon_2 + \varepsilon_1) = finite\ non\ zero =$
a'[bounded]

$$|f(0)G_n'(0) + G_n(0)f'(0)|(\varepsilon_2 + \varepsilon_1) = a'$$

$$lim_{n\to\infty}\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx + \int_{-\infty}^{+\infty} G_n(x)f'(x)dx\right| = a'$$

$$lim_{n\to\infty}\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx + f'(0)\right| = a'$$

$$lim_{n\to\infty}\int_{-\infty}^{+x:|x|<\epsilon} f(x)G_n'(x)\,dx < a'' = finite\ (27)$$

By our special theorem we claim that $f(x){G_n}'(x)$ is finite

That does not match with

$$x\delta'(x) = -\delta(x)$$

For x=0

$$0 \times \delta'(0) = -\infty$$

$\delta'(0)$ should bean infinity

Nevertheless

$$G_n = a_n x + b_n$$

As $n \to \infty$, a_n is bounded. Therefore $b_n \to \infty$ as $n \to \infty$,.

G_n becomes discontinuous at $+\epsilon$ with an undefined derivative[right hand and left hand derivatives are unequal: right handed derivative is infinitely larger. The product $f(x){G_n}'(x)$ has an infinitely large discontinuity and is not integrable. The fundamental law given by (1)will not work.

Case 3

We assume

$$[f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} = \infty$$

and also that both $f(x)$ and $G_n(x)$ are continuous on $(-\epsilon, \epsilon)$

From (22)

$$\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx$$

becomes divergent

$$\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx + \int_{-\infty}^{+\infty} G_n(x)f'(x)dx\right| = \infty$$

$$\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx + f'(0)\right| = \infty$$

$$\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx\right| = \infty$$

$$\left|\int_{-\epsilon}^{+\epsilon} f(x)G_n'(x)\,dx\right| = \infty$$

$$\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx\right| = \infty$$

The above integral is of divergent nature.

The derivative law in relation to the as given by (10) is at stake in that and we do not have the fundamental theorem for the first order derivative of the delta function. The delta function itself is at stake.

To that end let us consider [for sufficiently large 'n',

$$\int_{-\infty}^{+\infty} G_n(x)dx \approx 1$$

$$\int_{-\infty}^{+x:-\varepsilon x<+\epsilon} G_n(x)dx = a(x) = finite < 1$$

$$lim_{n\to\infty} \int_{-\infty}^{+x} G_n(x)dx = bounded(finite) < 1$$

By our special theorem we claim that $f(x)G_n{}'(x)$ is finite

[We may consider $\int_{-\infty}^{+\infty} G_n(x) = 1$for all n in place of and $lim_{n\to\infty} \int_{-\infty}^{+\infty} G_n(x) = 1$. The conclusions we have arrived at in this article remain unaffected].

Nested Interval Theorem for Open Intervals

First we consider Cantor' s Intersection[6] theorem[a.k.a the nested interval theorem] which applies to a sequence of non empty closed intervals each interval [except the first] being a subset of the previous one. According to the theorem there exists exactly one element at the intersection of the closed nested intervals.

Let this common element be $= c$. From each subset we exclude this common element c and create an open subset against every closed subset. If the subsets remained closed after the exclusion then by applying Cantor's intersection theorem again we will have another element h at their intersection. Therefore the earlier nesting of closed subsets would have two distinct elements at their intersection which is not true according to the theorem: we can have only one element at the intersection for the nesting of closed subsets.

Logically there are two possibilities

1. The null set is not included in the nesting in so far as Cantor's intersection theorem[1] is concerned.
2. We have one or more element at the intersection.

Case 1: The null set standing at the intersection means that there is no common element at the intersection. But any subset[excepting the first one that is the outermost one] that contains an element has a superset. Every non empty subset has a corresponding superset. The nesting of non empty open subsets cannot be without an intersection. So this alternative[case 1] gets ruled out

Case 2. There is one or more element standing at the intersection of the open subsets. If we consider the earlier

More on Taylor Expansion

Next we consider the following

$$f(x+2h) = f\big((x+h)+h\big) \quad (28)$$

Expanding about $(x+h)$

$$f(x+2h) = f(x+h) + \frac{h}{1!}f'(x+h) + \frac{h^2}{2!}f''(x+h) + \frac{h^3}{3!}f'''(x+h) + \cdots \ldots (29)$$

Expanding about $x = x$

$$f(x+2h) = f(x) + \frac{2h}{1!}f'(x+h) + \frac{4h^2}{2!}f''(x+h) + \frac{8h^3}{3!}f'''(x+h) + \cdots \ldots (30)$$

From (29) and (30)

$$f(x+h) + \frac{h}{1!}f'(x+h) + \frac{h^2}{2!}f''(x+h) + \frac{h^3}{3!}f'''(x+h) + \cdots \ldots$$
$$= f(x) + \frac{2h}{1!}f'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \cdots \ldots$$

$$f(x+h) - f(x) + h[f'(x+h) - 2f'(x)] + \frac{1}{2!}h^2[f''(x+h) - 4f''(x)]$$
$$+ \frac{1}{3!}h^3[f'''(x+h) - 8f'''(x)] + \cdots .. = 0 \ (4)$$

$$\frac{f(x+h)-f(x)}{h}\frac{1}{h} + \frac{[f'(x+h) - 2f'(x)]}{h} + \frac{1}{2!}[f''(x+h) - 4f''(x)] + \frac{1}{3!}h[f'''(x+h) - 8f'''(x)]$$
$$+ \cdots .. = 0$$

$$\frac{f(x+h)-f(x)}{h}\frac{1}{h} + \frac{[f'(x+h) - f'(x)]}{h} - \frac{f'(x)}{h} + \frac{1}{2!}[f''(x+h) - 4f''(x)]$$
$$+ \frac{1}{3!}h[f'''(x+h) - 8f'''(x)] + h[\ldots ..] = 0$$

With $h \to 0$

$$lim_{h\to 0}f'(x)\frac{1}{h} + f''(x) - lim_{h\to 0}f'(x)\frac{1}{h} - \frac{3}{2!}[f''(x)] - + \cdots .. = 0 \ (31)$$

$$f''(x) - \frac{3}{2!}f''(x) = 0 \Rightarrow f''(x) = 0 \ (32)$$

Conclusions

As claimed the analysis of the Delta function brings out unacceptable features in relation to the conventional law in regarding its derivatives.

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