# A Note on Lattice Theory

Henry Wong

**Abstract**

Lattice is a partially ordered set with two operations defined on it that satisfy certain conditions. Lattice theory itself is a branch of abstract algebra. In this paper we present solutions to three classical problems in lattice theory. The solutions are by no means novel.

**Theorem 1**
Let $L$ be a lattice. For $x, y, z$ in $L$,

$$x \cap (y \cup z) \geqq (x \cap y) \cup (x \cap z) \tag{1}$$

$$(x \cup y) \cap (x \cup z) \geqq x \cup (y \cap z) \tag{2}$$

**Theorem 2**
Let $L$ be a modular lattice such that $a \geqq b,\ a \cup x = b \cup x,\ a \cap x = b \cap x$. Then $a = b$.

**Theorem 3**
Let $L$ be a modular lattice. Suppose

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z) \tag{3}$$

for some $x, y, z$ in $L$. Then $z \cap (x \cup y) = (z \cap x) \cup (z \cap y)$.
*Proof.*
The outline of this proof can be found in [1]. Since

$$z \cap (x \cup y) \geqq (z \cap x) \cup (z \cap y) \tag{4}$$

by (1), it follows that

$$[z \cap (x \cup y)] \cup x \geqq [(z \cap x) \cup (z \cap y)] \cup x \tag{5}$$

and

$$[z \cap (x \cup y)] \cap x \geqq [(z \cap x) \cup (z \cap y)] \cap x. \tag{6}$$

Moreover,

$$\begin{aligned}
[(z \cap x) \cup (z \cap y)] \cup x &\geqq (z \cap y) \cup x \\
&= (z \cap y) \cup (x \cup x) \\
&= [(z \cap y) \cup x] \cup x \\
&= [x \cup (z \cap y)] \cup x \\
&= [x \cup (y \cap z)] \cup x \\
&= [(x \cup y) \cap (x \cup z)] \cup x \qquad \text{since (3)} \\
&\geqq [(x \cup y) \cap z] \cup x \\
&= [z \cap (x \cup y)] \cup x.
\end{aligned}$$

Therefore

$$[(z \cap x) \cup (z \cap y)] \cup x \geqq [z \cap (x \cup y)] \cup x. \tag{7}$$

On the other hand,

$$\begin{aligned}[(z \cap x) \cup (z \cap y)] \cap x &\geqq (z \cap x) \cap x \\ &= z \cap (x \cap x) \\ &= z \cap x \\ &= z \cap [x \cap (x \cup y)] \\ &= z \cap [(x \cup y) \cap x] \\ &= [z \cap (x \cup y)] \cap x.\end{aligned}$$

Therefore

$$[(z \cap x) \cup (z \cap y)] \cap x \geqq [z \cap (x \cup y)] \cap x. \tag{8}$$

From (5) and (7),

$$[z \cap (x \cup y)] \cup x = [(z \cap x) \cup (z \cap y)] \cup x. \tag{9}$$

Similarly, from (6) and (8),

$$[z \cap (x \cup y)] \cap x = [(z \cap x) \cup (z \cap y)] \cap x. \tag{10}$$

To conclude, from (4), (9), (10),

$$z \cap (x \cup y) = (z \cap x) \cup (z \cap y).$$

by Theorem 2.

**Theorem 4**
For $x, y, z$ in $L$, $[(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)] = x \cap y$.
*Proof.*
Since $(x \cap y) \cup (x \cap z) \geqq x \cap y$ and $(x \cap y) \cup (y \cap z) \geqq x \cap y$,

$$\begin{aligned}[(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)] &\geqq (x \cap y) \cap (x \cap y). \\ &= x \cap y.\end{aligned}$$

Thus

$$[(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)] \geqq x \cap y. \tag{11}$$

Moreover,

$$\begin{aligned}x &\geqq x \cap [(x \cap y) \cup z] \\ &= [x \cup (x \cap y)] \cap [(x \cap y) \cup z] \\ &= [(x \cap y) \cup x] \cap [(x \cap y) \cup z] \\ &\geqq (x \cap y) \cup (x \cap z) && \text{by (2)}\end{aligned}$$

and hence

$$x \geqq (x \cap y) \cup (x \cap z). \tag{12}$$

Furthermore,

$$\begin{aligned}y &\geqq y \cap [(x \cap y) \cup z] \\ &= [y \cup (y \cap x)] \cap [(x \cap y) \cup z] \\ &= [(y \cap x) \cup y] \cap [(x \cap y) \cup z] \\ &= [(x \cap y) \cup y] \cap [(x \cap y) \cup z] \\ &\geqq (x \cap y) \cup (y \cap z) && \text{by (2)}\end{aligned}$$

and hence

$$y \geqq (x \cap y) \cup (y \cap z). \tag{13}$$

From (12) and (13),

$$x \cap y \geqq [(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)]. \tag{14}$$

From (11) and (14),

$$[(x \cap y) \cup (x \cap z)] \cap [(x \cap y) \cup (y \cap z)] = x \cap y.$$

**Theorem 5**
Let $L$ be a lattice. If

$$(x \cup y) \cap [z \cup (x \cap y)] = (x \cap y) \cup (y \cap z) \cup (z \cap x) \tag{15}$$

for all $x, y, z$ in $L$, then $L$ is distributive.
*Proof.*
Suppose $x \geqq y$. Then

$$\begin{aligned} x \cap (z \cup y) &= (x \cup y) \cap [z \cup (x \cap y)] \\ &= (x \cap y) \cup (y \cap z) \cup (z \cap x) \qquad \text{by (15)} \\ &= y \cup (y \cap z) \cup (z \cap x) \\ &= [y \cup (y \cap z)] \cup (z \cap x) \\ &= y \cup (z \cap x). \end{aligned}$$

Therefore

$$x \cap (z \cup y) = y \cup (z \cap x) \quad \text{whenever} \quad x \geqq y. \tag{16}$$

To conclude

$$\begin{aligned} (x \cup y) \cap (x \cup z) &= (x \cup y) \cap (z \cup x) \\ &= (x \cup y) \cap \{z \cup [x \cup (x \cap y)]\} \\ &= (x \cup y) \cap \{z \cup [(x \cap y) \cup x]\} \\ &= (x \cup y) \cap \{[z \cup (x \cap y)] \cup x\} \\ &= x \cup \{[z \cup (x \cap y)] \cap (x \cup y)\} \qquad \text{by (16)} \\ &= x \cup \{(x \cup y) \cap [z \cup (x \cap y)]\} \\ &= x \cup [(x \cap y) \cup (y \cap z) \cup (z \cap x)] \qquad \text{since (15)} \\ &= x \cup \{(x \cap y) \cup [(y \cap z) \cup (z \cap x)]\} \\ &= x \cup \{(x \cap y) \cup [(z \cap x) \cup (y \cap z)]\} \\ &= x \cup \{[(x \cap y) \cup (z \cap x)] \cup (y \cap z)\} \\ &= \{x \cup [(x \cap y) \cup (z \cap x)]\} \cup (y \cap z) \\ &= \{[x \cup (x \cap y)] \cup (z \cap x)\} \cup (y \cap z) \\ &= [x \cup (z \cap x)] \cup (y \cap z) \\ &= [x \cup (x \cap z)] \cup (y \cap z) \\ &= x \cup (y \cap z). \end{aligned}$$

**References**

[1] T. S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer-Verlag, London, 2005.

[2] D. E. Rutherford, *Introduction to Lattice Theory*, Oliver and Boyd Ltd., Edinburgh, 1965.