The Secret Geometry of the Quartic Equation

How we are led to the solution from common concepts of geometry

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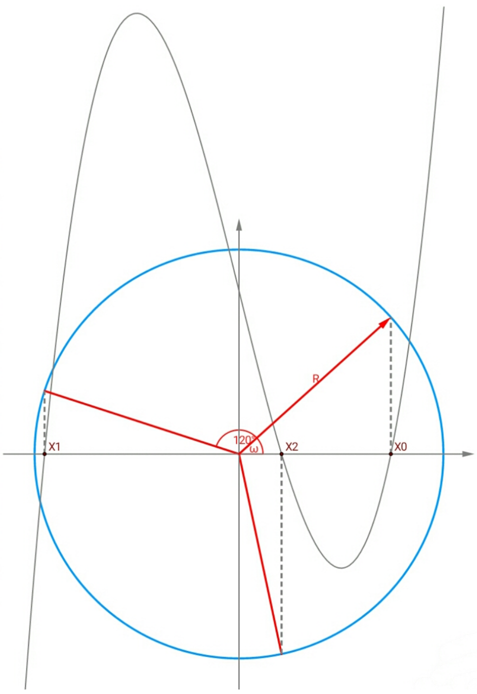
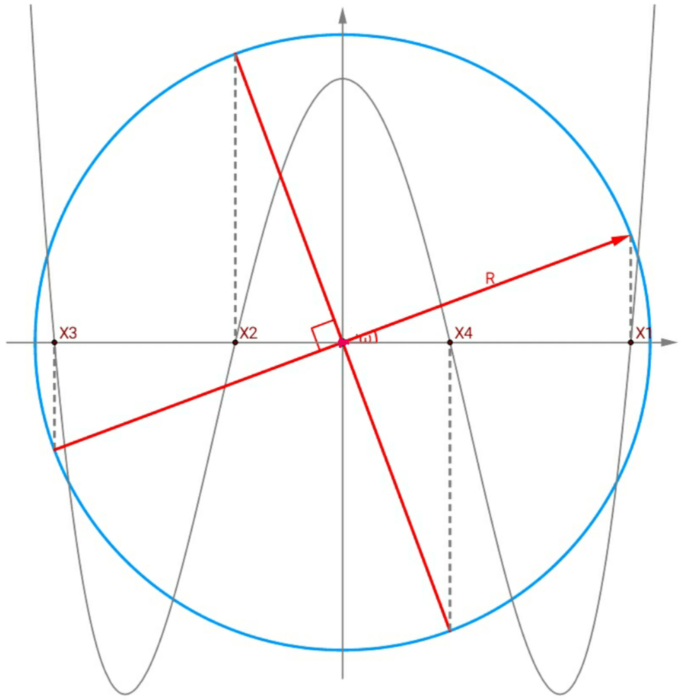
Abstract

Rene Descartes and Francois Viete were the first to deal substantially with the modeling of polynomial equations, but to date no one saw in these works anything more than remarkable constructions. The model presented here briefly is the natural continuation of the model discovered by Viete for the cubic equation.



According to the construction of Viete, the model of the quartic equation should look something like this:



$$x^4 + cx^2 + e = 0$$

$$0 \le e \le \frac{c^2}{4}, c < 0$$

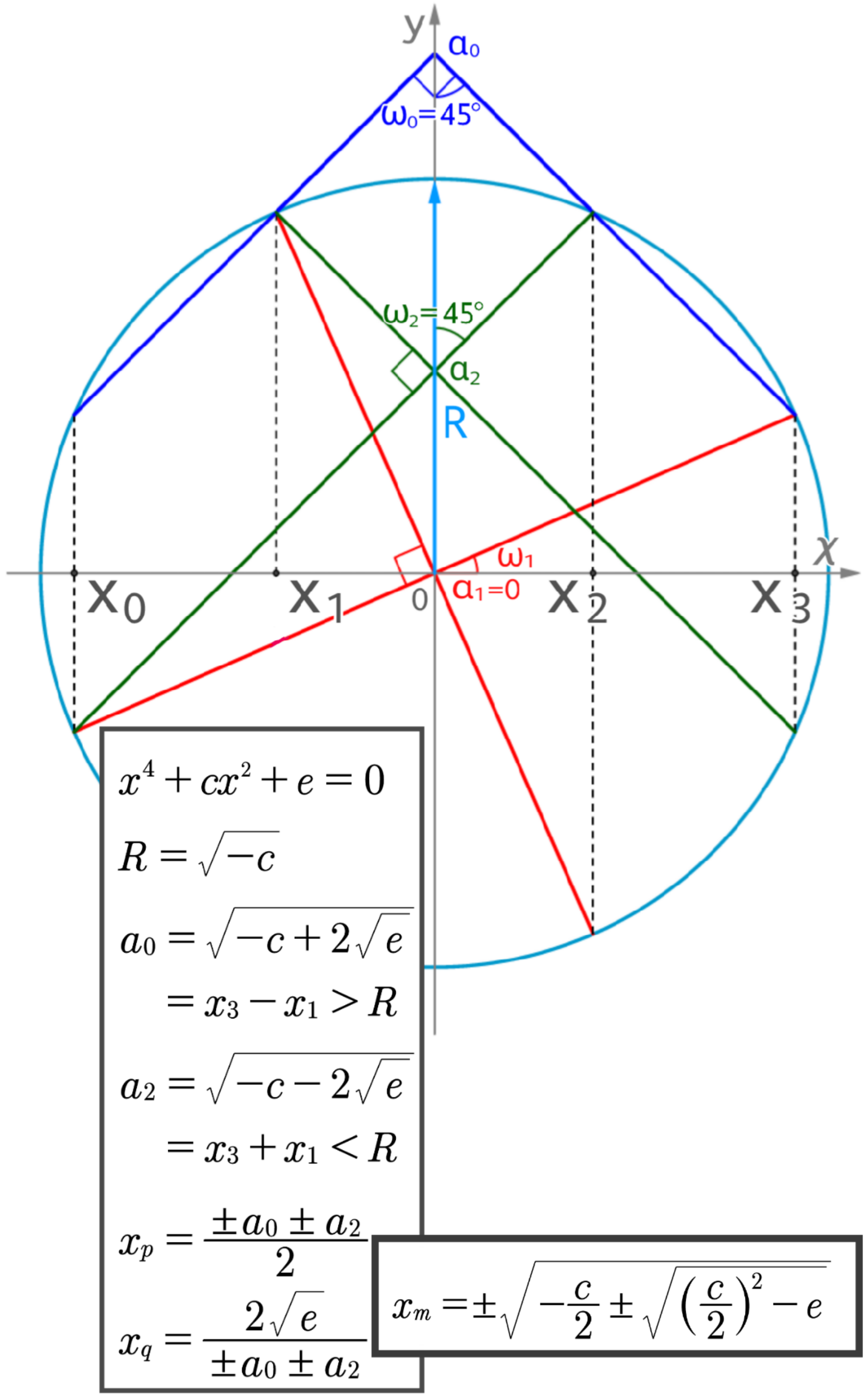
$$u = c = \frac{c^2}{4}, R = \sqrt{-c}, r = e - \frac{u}{2}$$

$$\omega = \frac{1}{4} \cos^{-1}\left(\frac{-2r}{u}\right)$$

$$x_n = R \cos(\omega + 90° n), n = 0, 1, 2, 3$$

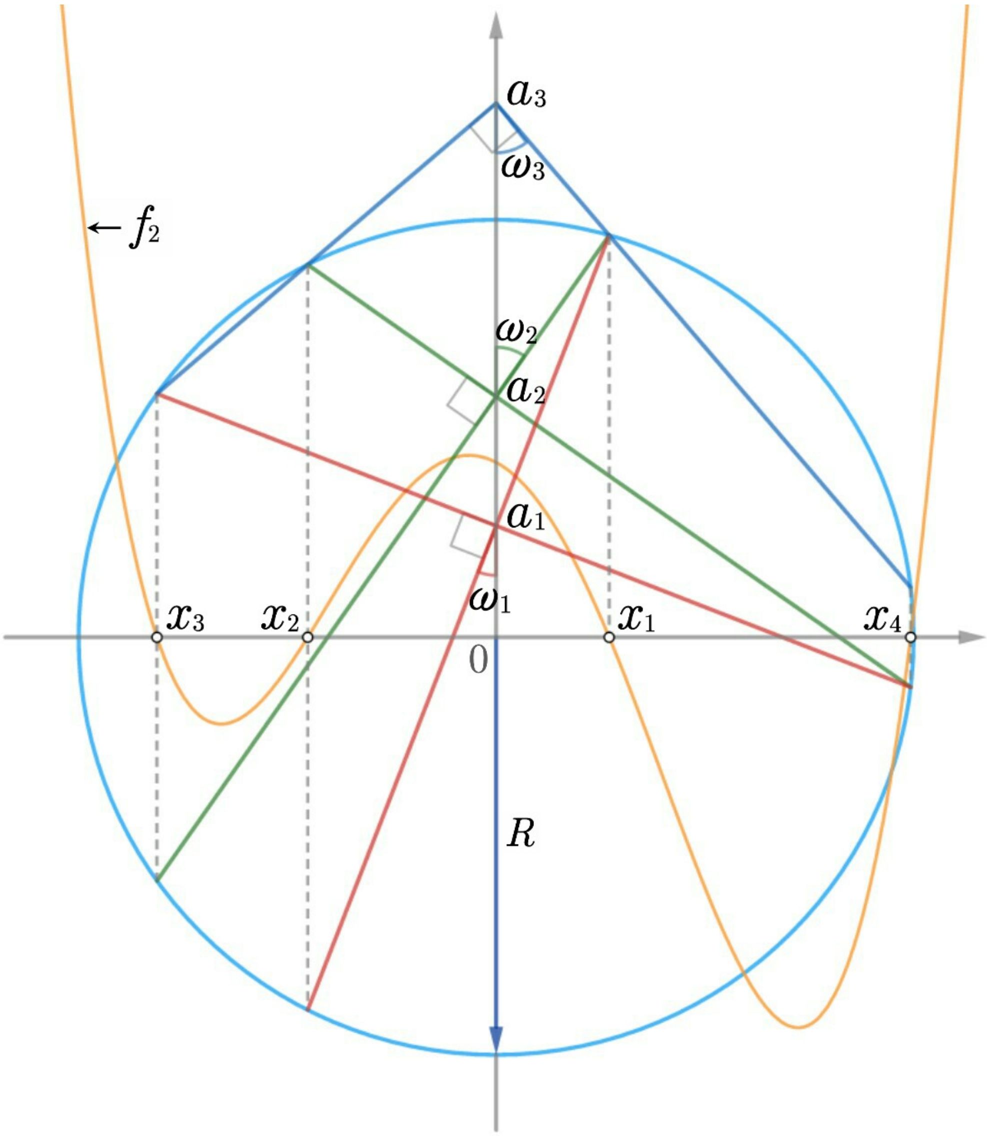
As is clear from the figure, this concerns the biquadratic equation, whose roots are symmetrically arranged with respect to the y-axis. Next to it is the solution to which the geometric analysis of the figure leads. Intuition tells us that good models lead to simple and elegant solutions, but this solution is neither simple nor elegant. This may mean that our model is not complete and must be supplemented by additional geometric data.

To find this data, we first observe that the red cross intersects the circle at four points. These points locate the roots. We believe that there are additional geometric systems associated with the roots. So we draw four lines that are perpendicular to the x-axis and pass through the roots. These lines will intersect the circle at eight points. Then we draw all the lines that connect these points in pairs and rotate the red cross to see which structures remain unchanged. With this heuristic method we will come up with the following figure.



As the red cross rotates around the fixed point a_1 at an angle ω_1, the remaining crosses move vertically because the values of a_0, a_2 change, keeping the angles ω_0, ω_2 constant. The formulas within the big box come from the geometric analysis of the figure. For the time being we cannot extract the well-known formula for solving the biquadratic equation which is printed in the small box.

Now we are ready for the big leap. If the quartic equation model exists, then the above figure will be a special case. To this end, we observe that all three crosses are defined by two parameters: (R,a_0), (R,ω_1), (R,a_2). We can see that if we combine the vertical with the rotational motion, then each of the crosses will receive three degrees of freedom (R,ω_K,a_K). Now our model takes the following form.



Using elementary trigonometry we can find that the sum of the roots x1, x2, x3, x4 is equal to zero. Therefore, this model identifies the roots of the equation $x^4+cx^2+dx+e=0$. The higher the value of the coefficient d, the more asymmetric the graph of the quartic function becomes. The same goes for the roots. Later we will see that the radius R of the circle is equal to

$$R = \sqrt{\frac{d^2}{4e} - c}$$

This relationship is valid when two of the roots are positive and the other two are negative, all real. However, with analytical methods it can be shown that

$$c, d, e \in \mathbb{R}$$

$$x_m \in \mathbb{R}, e > 0 \Rightarrow |\, x_m \,| \leq \sqrt{\frac{d^2}{4e} - c}$$

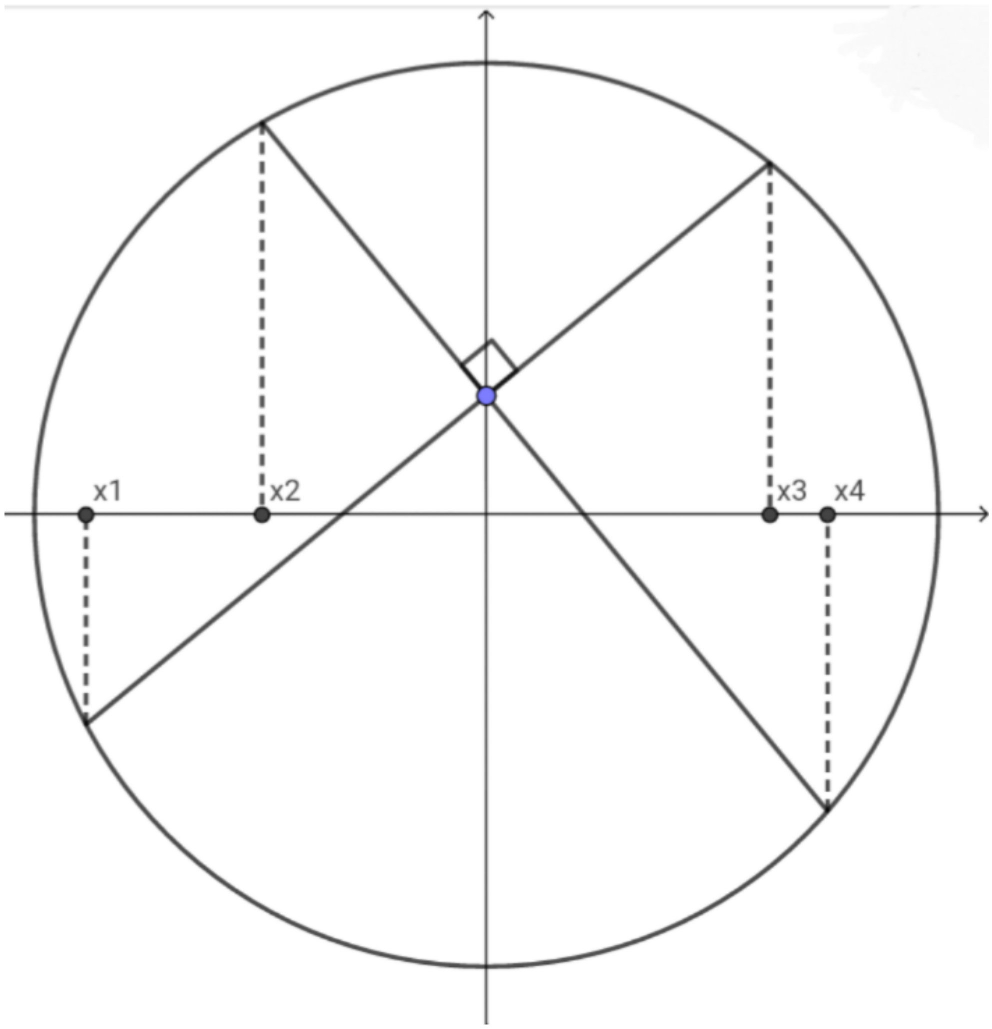
$$x_m \in \mathbb{R}, e < 0 \Rightarrow |\, x_m \,| \leq \sqrt{\frac{27}{34}} \sqrt{-\frac{d^2}{4e} - c}$$

$$x_m \in \mathbb{C} \Rightarrow |\, \mathrm{Re}(x_m) \,| \leq \sqrt{\frac{d^2}{4\,|\, e \,|} - c}$$

The root factor 27/34 is not arbitrary but a product of analysis.

Based on what has been said so far, draw a circle with a random radius, with the center at the beginning of the axes. Select a random point α on the y-axis and within the circle. Draw a random chord passing through α. Now draw a chord that passes through α and is perpendicular to the previous chord. The chords intersect the circle at four points. From these points draw lines perpendicular to the x-axis. Then these points on the x-axis will have a sum of zero and will be roots of the equation $x^4 + cx^2 + dx + e = 0$. The radius of the circle will be equal to

$$R = \sqrt{\frac{d^2}{4e} - c}$$



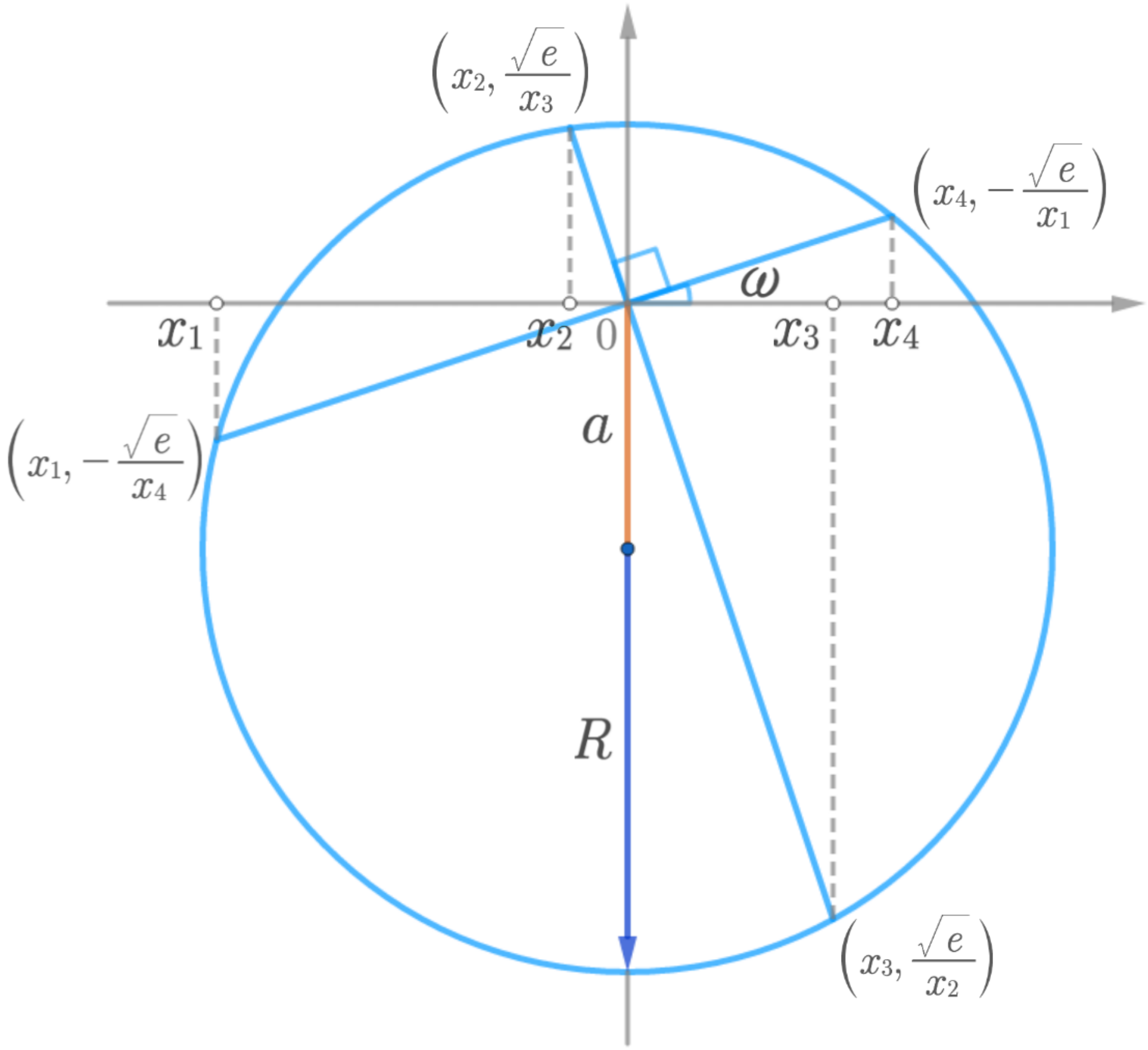
Because the term bx^3 of the equation is zero, you can use the following relationships to calculate the coefficients:

$$c = \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{-2}$$

$$d = \frac{x_1^3 + x_2^3 + x_3^3 + x_4^3}{-3}$$

$$e = x_1 x_2 x_3 x_4$$

The following is a more detailed example.



$$x^4 + bx^3 + cx^2 + dx + e = 0$$

$$-b = x_1 + x_2 + x_3 + x_4 = 0$$

$$R = \sqrt{\frac{d^2}{4e} - c}$$

$$c = \frac{{x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2}{-2}$$

$$\cos\omega = \frac{1}{\sqrt{1 + \frac{x_2 x_3}{x_1 x_4}}}$$

$$d = \frac{{x_1}^3 + {x_2}^3 + {x_3}^3 + {x_4}^3}{-3}$$

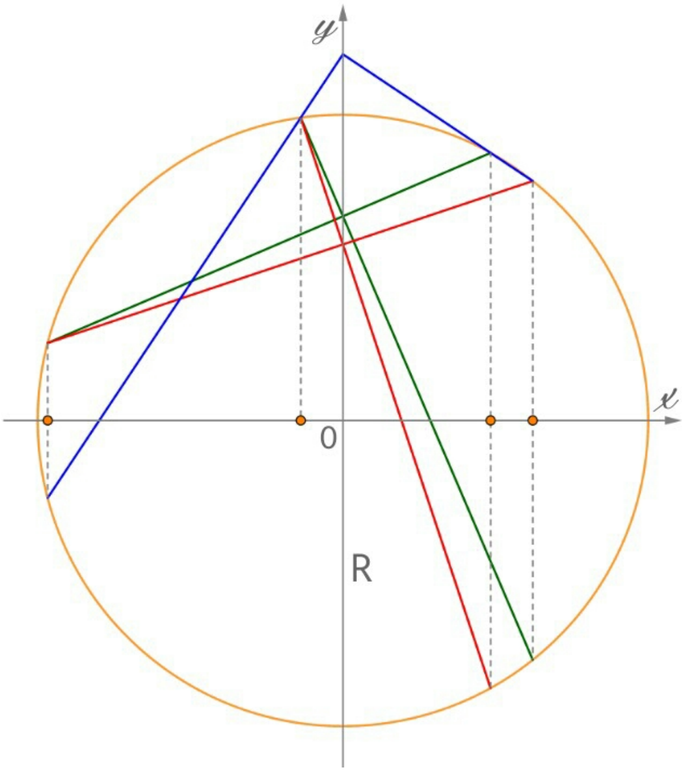
$$a = \frac{\sqrt{e}}{2}\left(-\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{1}{x_4}\right)$$

$$e = x_1 x_2 x_3 x_4$$

In this example the x-axis passes through the center of the cross. This option simplifies the various relationships we can derive from the geometric analysis of the model. Immediately now several such relationships will be given for the equation

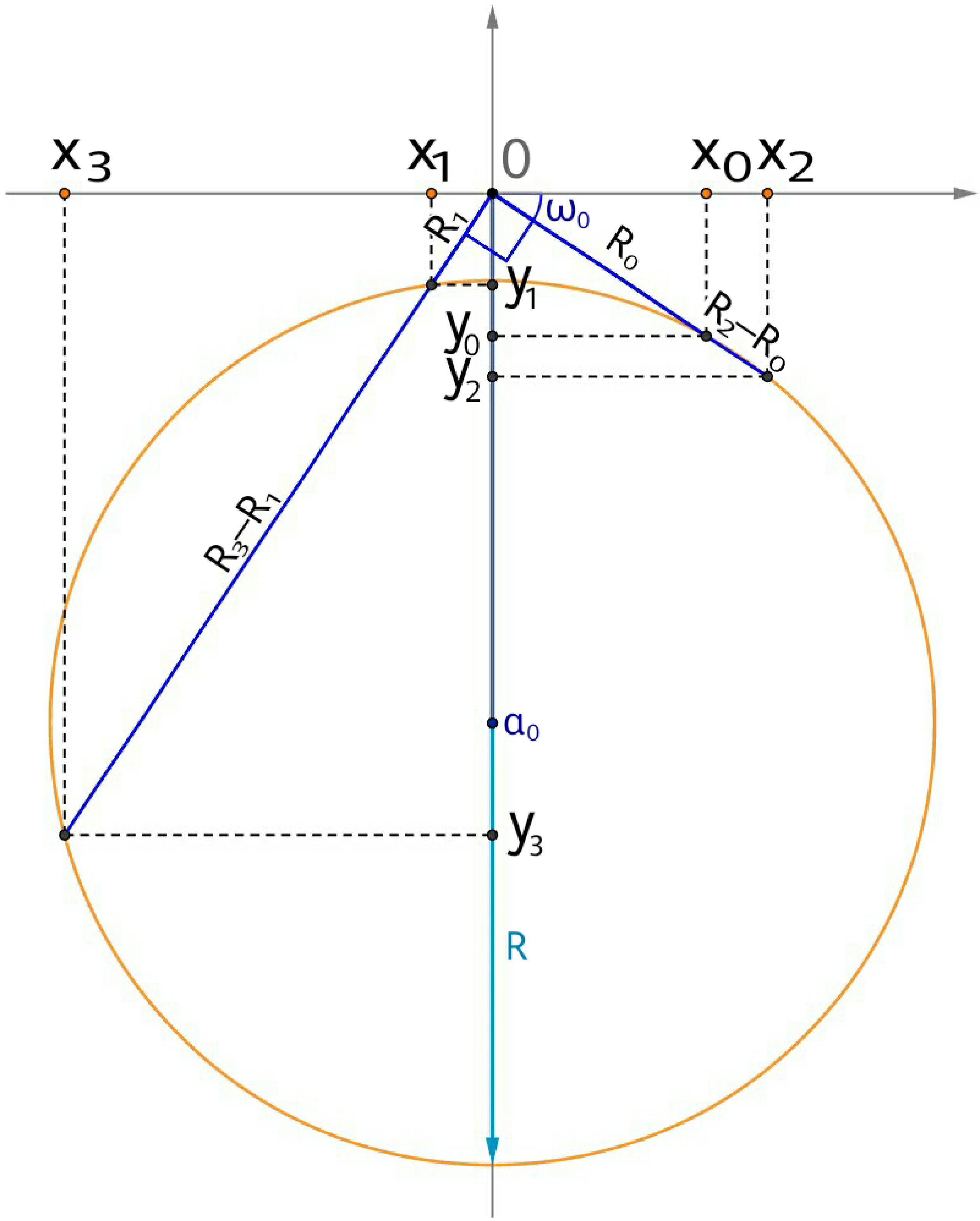
$$x^4 \underbrace{-1485x^2}_{c} + \underbrace{15120x}_{d} + \underbrace{142884}_{e} = 0$$

This equation is zeroed when x is equal to -42, -6, 21 and 27. The radius *R* is approximately equal to 43.42. The equation model has the following appearance.

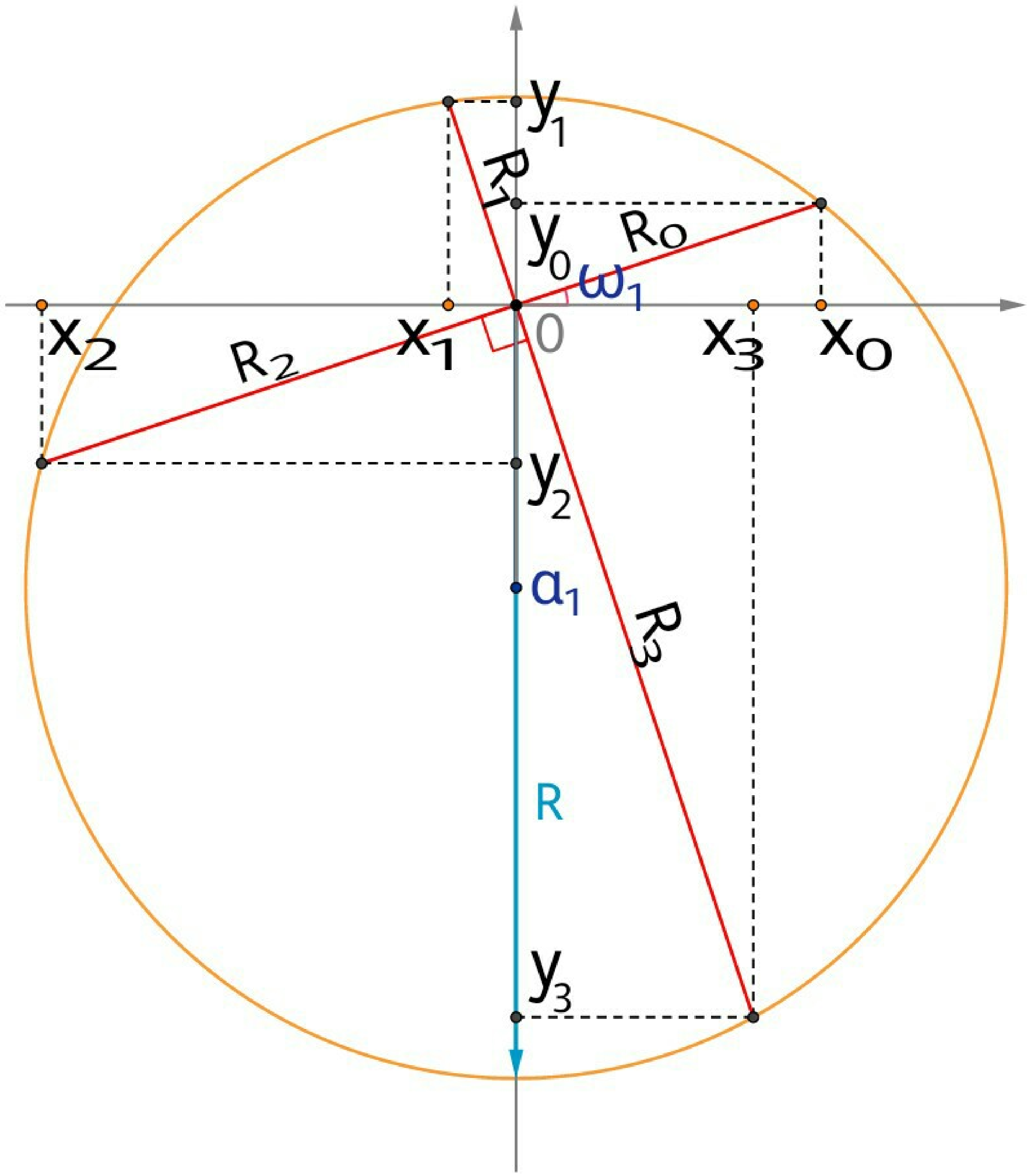


In the following figures, the three subsystems of the model (red, blue and green cross) are analyzed separately. Note that the angles ω_0, ω_1 and ω_2 are the smallest of the complementary angles between two successive rays of the cross. That is, ω_K is less than or equal to 45°.

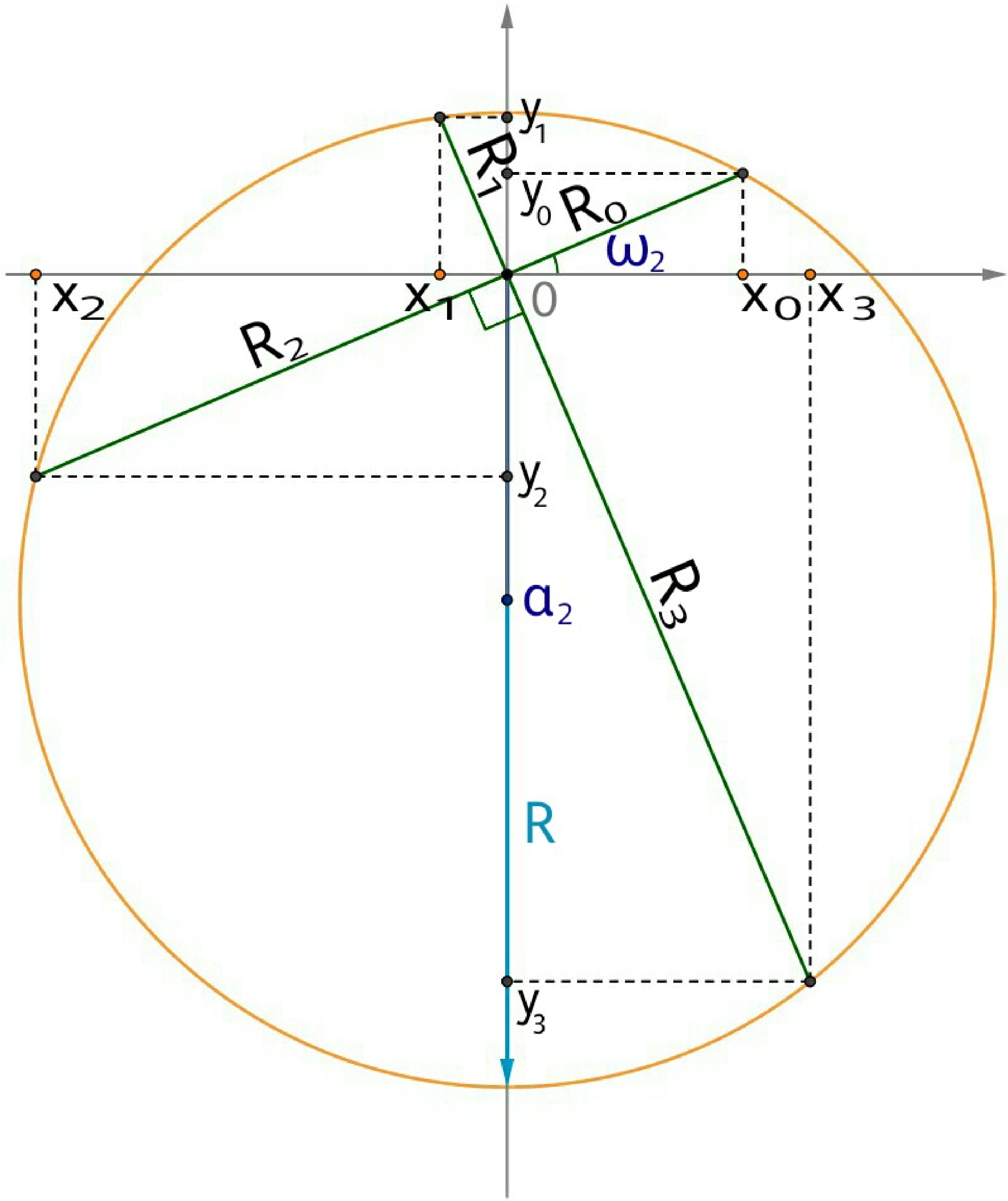
Subsystem 0



Subsystem 1



Subsystem 2



The following are various relationships that apply to all three of the above subsystems.

$$c = -1485,\, d = 15120,\, e = 142884$$
$$R = 43.42$$
$$z_0 = 819,\, z_1 = -1260,\, z_2 = -1044$$

$$\begin{array}{llll} x_0 = 21 & y_0 = -14 & & \\ x_1 = -6 & y_1 = -9 & & a_0 = -52 \\ x_2 = 27 & y_2 = -18 & & \omega_0 = 33.69^\circ \\ x_3 = -42 & y_3 = -63 & & \end{array}$$

$$\begin{array}{llll} x_0 = 27 & y_0 = 9 & & \\ x_1 = -6 & y_1 = 18 & & a_1 = -25 \\ x_2 = -42 & y_2 = -14 & & \omega_1 = 18.43^\circ \\ x_3 = 21 & y_3 = -63 & & \end{array}$$

$$\begin{array}{llll} x_0 = 21 & y_0 = 9 & & \\ x_1 = -6 & y_1 = 14 & & a_2 = -29 \\ x_2 = -42 & y_2 = -18 & & \omega_2 = 23.20^\circ \\ x_3 = 27 & y_3 = -63 & & \end{array}$$

The next two pages are dedicated to the relationships between roots, coefficients and parameters of the model.

$$x^4+cx^2+dx+e=0$$

$$c=\frac{x_0^2+x_1^2+x_2^2+x_3^2}{-2}$$

$$d=\frac{x_0^3+x_1^3+x_2^3+x_3^3}{-3}$$

$$e=x_0x_1x_2x_3$$

$$R=\sqrt{\frac{d^2}{4e}-c}$$

$$y_0=-\frac{\sqrt{e}}{x_2},\ y_1=\frac{\sqrt{e}}{x_3},\ y_2=-\frac{\sqrt{e}}{x_0},\ y_3=\frac{\sqrt{e}}{x_1}$$

$$y_0y_1y_2y_3=x_0x_1x_2x_3=e$$

$$\begin{aligned}a_n&=y_0+y_2-\sqrt{R^2+c}\\&=y_1+y_3+\sqrt{R^2+c}\\&=\frac{(x_1+x_3)(x_1x_3+x_0x_2)}{2\sqrt{e}}\\&=\frac{(x_1+x_3)\left((x_1+x_3)^2+c\right)}{2\sqrt{e}}\end{aligned}$$

$$\begin{aligned}2a_n&=y_0+y_1+y_2+y_3\\&=2\left(y_0+y_2\right)-\frac{d}{\sqrt{e}}\\&=2\left(y_1+y_3\right)+\frac{d}{\sqrt{e}}\\&=\left(R_p^2-R_q^2\right)\left(y_p+y_q\right),\ p\neq q\end{aligned}$$

$$\cos\omega_n=\frac{1}{\sqrt{1+\frac{x_1x_3}{x_0x_2}}},\ \sin\omega_n=\frac{1}{\sqrt{1+\frac{x_0x_2}{x_1x_3}}}$$

$$\left|\frac{1}{y_0}\right|-\left|\frac{1}{y_1}\right|-\left|\frac{1}{y_2}\right|+\left|\frac{1}{y_3}\right|=0$$

$$y_0y_1+y_0y_2+y_0y_3+y_1y_2+y_1y_3+y_2y_3=2\left(a_n^2-R^2\right)-c$$

$$y_0^2+y_1^2+y_2^2+y_3^2=\frac{d^2}{e}-2c$$

$$R_p\perp R_q\Rightarrow y_py_q=-x_px_q$$

$$R_p\parallel R_q\Rightarrow y_py_q=+x_px_q$$

$$\begin{aligned}z_n&=a_n^2-R^2\\&=x_1x_3+x_0x_2\\&=(x_1+x_3)^2+c=(x_0+x_2)^2+c\\&=\pm R_1R_3=\pm R_0R_2\end{aligned}$$

$$R^{-2}=-z_0^{-1}-z_1^{-1}-z_2^{-1}$$

$$\cos\omega_0\cos\omega_1\cos\omega_2\sin\omega_0\sin\omega_1\sin\omega_2=\frac{\sqrt{e}}{4R^2}$$

$$z_n\cos\omega_n\sin\omega_n=\pm\sqrt{e}$$

$$a_0^2+a_1^2+a_2^2=3R^2+c$$

Solution methods

We first find the resolvent cubic.

We set

$$z_0 = a_0^2 - R^2$$

$$= \left(\frac{(x_0 + x_3)(x_0 x_3 + x_1 x_2)}{2\sqrt{e}} \right)^2 - R^2$$

$$= \frac{(x_0 + x_3)^2 (x_0 x_3 + x_1 x_2)^2}{4e} - R^2 \quad (I)$$

Because they apply

$$(x_0 + x_3)^2 = z_0 - c$$

$$x_0 x_3 + x_1 x_2 = z_0$$

we will have

$$z_0 = \frac{(z_0 - c) z_0^2}{4e} - R^2 \ \Rightarrow$$

$$z_0^3 - c z_0^2 - 4e z_0 - 4eR^2 = 0 \ \Rightarrow$$

$$z_0^3 - c z_0^2 - 4e z_0 + 4ec - d^2 = 0$$

Therefore the resolvent cubic is

$$z^3 - cz^2 - 4ez + 4ec - d^2 = 0$$

This can be solved as follows.

$$r = \frac{2}{3}\sqrt{c^2 + 12e}$$

$$\varphi = \frac{1}{3}\cos^{-1}\left(\frac{8c^3 - 288ce + 108d^2}{27r^3}\right)$$

$$z_n = \frac{c}{3} + r\cos\left(\varphi + 120°n\right),\, n = 0, 1, 2$$

$$d = 0 \Rightarrow z_0 = 2\sqrt{e},\, z_1 = c,\, z_2 = -2\sqrt{e}$$

Since this equation does not recognize a sign in d, we must in all methods of solving the quartic equation multiply the derived roots by $d/|d|$ in order to receive the correct sign. Also, because the roots of the resolvent cubic are predetermined when $d=0$, we can be led to all known or unknown formulas for solving the biquadratic equation by replacing and simplifying each of the presented methods.

General method

$$t_n = \sqrt{z_n - c},\, n = 0, 1, 2$$

$$x_0 = \frac{-t_0 - t_1 - t_2}{2} \cdot \frac{d}{|d|}$$

$$x_1 = \frac{+t_0 + t_1 - t_2}{2} \cdot \frac{d}{|d|}$$

$$x_2 = \frac{+t_0 - t_1 + t_2}{2} \cdot \frac{d}{|d|}$$

$$x_3 = \frac{-t_0 + t_1 + t_2}{2} \cdot \frac{d}{|d|}$$

Eccentric method

$$a_n = -\sqrt{z_n + R^2}$$

$$t_0 = -a_0 - a_1 - a_2$$

$$t_1 = -a_0 + a_1 + a_2$$

$$t_2 = +a_0 - a_1 + a_2$$

$$t_3 = +a_0 + a_1 - a_2$$

$$x_m = \frac{-4e}{\frac{d}{|d|} t_m 2\sqrt{e} + d}$$

Trigonometric method

This method, although a bit complicated, provides the best geometric interpretation of the solutions. Typically, the trigonometric method is applied when the equation has two positive and two negative real roots. Otherwise its trigonometric functions may express corresponding hyperbolics. For example, when three of the roots of the quartic equation have the same sign and the other has the opposite sign, then its model is not drawable as the angles ω_K become complex. However, the trigonometric method can be applied.

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$$\omega_n = \min(\omega_n, 90^\circ - \omega_n)\,(\leq 45^\circ),\ n = 0,1,2$$

$$\cos\omega_n = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{e}{z_n^2}}}$$

$$\sin\omega_n = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{e}{z_n^2}}}$$

$$x_0 = -\,2R\sin\omega_0\,\sin\omega_1\,\sin\omega_2 \cdot d/\,|\,d\,|$$

$$x_1 = -\,2R\sin\omega_0\cos\omega_1\cos\omega_2 \cdot d/\,|\,d\,|$$

$$x_2 = +\,2R\cos\omega_0\sin\omega_1\cos\omega_2 \cdot d/\,|\,d\,|$$

$$x_3 = +\,2R\cos\omega_0\cos\omega_1\sin\omega_2 \cdot d/\,|\,d\,|$$

$$x_0 = \frac{-\sqrt{e}}{2R\cos\omega_0\cos\omega_1\cos\omega_2} \cdot d/\,|\,d\,|$$

$$x_1 = \frac{-\sqrt{e}}{2R\cos\omega_0\,\sin\omega_1\,\sin\omega_2} \cdot d/\,|\,d\,|$$

$$x_2 = \frac{\sqrt{e}}{2R\ \sin\omega_0\cos\omega_1\,\sin\omega_2} \cdot d/\,|\,d\,|$$

$$x_3 = \frac{\sqrt{e}}{2R\ \sin\omega_0\,\sin\omega_1\cos\omega_2} \cdot d/\,|\,d\,|$$

$$x_0 = R\sin(-\,\omega_0 - \omega_1 - \omega_2) \cdot d/\,|\,d\,|$$

$$x_1 = R\sin(+\,\omega_0 - \omega_1 - \omega_2) \cdot d/\,|\,d\,|$$

$$x_2 = R\sin(-\,\omega_0 + \omega_1 - \omega_2) \cdot d/\,|\,d\,|$$

$$x_3 = R\sin(-\,\omega_0 - \omega_1 + \omega_2) \cdot d/\,|\,d\,|$$

$$x_m = \frac{\pm\sqrt{e}}{a_n \pm R\cos(\pm\,\omega_0 \pm \omega_1 \pm \omega_2)} \cdot d/\,|\,d\,|$$

Special example 1

$$c = -55,\ d = 210,\ e = -216$$

$$z_0 = -6,\ z_2 = -30,\ z_3 = -19$$

$$c_0 = \cosh\omega_0 = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{-216}{(-6)^2}}} = \sqrt{3}$$

$$s_0 = \sinh\omega_0 = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{-216}{(-6)^2}}} = i\sqrt{2}$$

$$c_1 = \cosh\omega_1 = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{-216}{(-30)^2}}} = \frac{6}{\sqrt{30}}$$

$$s_1 = \sinh\omega_1 = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{-216}{(-30)^2}}} = \frac{i}{\sqrt{5}}$$

$$c_2 = \cosh\omega_2 = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{-216}{(-19)^2}}} = 3\sqrt{\frac{3}{19}}$$

$$s_2 = \sinh\omega_2 = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{-216}{(-19)^2}}} = 2i\sqrt{\frac{2}{19}}$$

$$R = \sqrt{\frac{210^2}{4\cdot(-216)} - (-55)} = \sqrt{\frac{95}{24}}$$

$$x_0 = -2Rc_0c_1c_2 = -2\sqrt{95/24} \cdot \sqrt{3} \cdot 6/\sqrt{30} \cdot 3\sqrt{3/19} = -9$$

$$x_1 = -2Rc_0s_1s_2 = -2\sqrt{95/24} \cdot \sqrt{3} \cdot i/\sqrt{5} \cdot 2i\sqrt{2/19} = 2$$

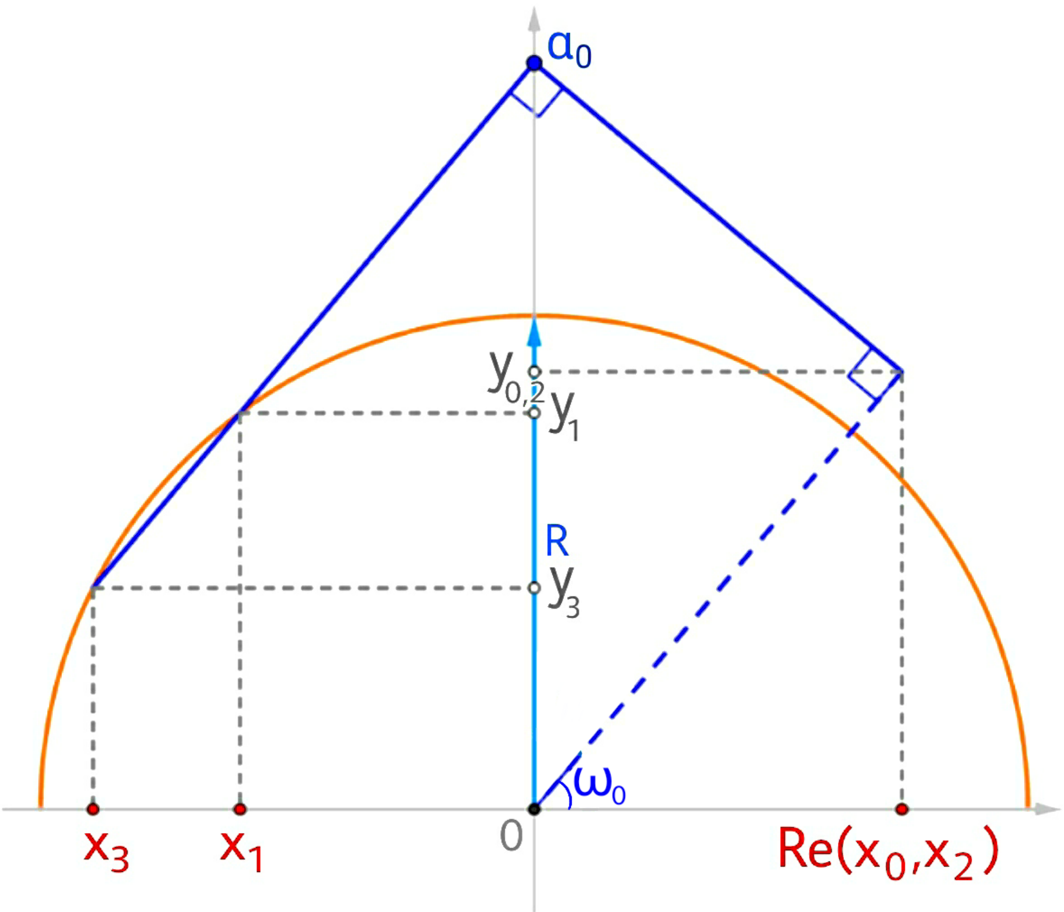
$$x_2 = -2Rs_0c_1s_2 = -2\sqrt{95/24} \cdot i\sqrt{2} \cdot 6/\sqrt{30} \cdot 2i\sqrt{2/19} = 4$$

$$x_3 = -2Rs_0s_1c_2 = -2\sqrt{95/24} \cdot i\sqrt{2} \cdot i/\sqrt{5} \cdot 3\sqrt{3/19} = 3$$

Identify real parts of complex roots

If we rotate the outer cross of the model (the outer subsystem), at some point its arm will touch the circle. At the same time, the two internal subsystems will overlap. Then, when the arm of the outer subsystem is detached from the circle, the inner crosses will disappear as their angles acquire complex values. Then, the detachable arm of the external subsystem will locate on the x-axis the real parts of the complex roots of the corresponding equation. This is done in the way shown in the following example.

Example 1



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$c = -1050, d = 12500, e = 510000$

$z_0 = 1450, z_1 = -1250 - 15i, z_2 = -1250 + 15i$

$$R = \sqrt{\frac{d^2}{4e} - c} = 33.56$$

$$a_0 = \sqrt{z_0 + R^2} = 50.76$$

$$\omega_0 = \cos^{-1}\left(\sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{e}{z_0^2}}}\right) = 49.96°$$

$x_0 = 25 + 15i, x_1 = -20, x_2 = 25 - 15i, x_3 = -30$

$$y_0 = \mathrm{Re}\left(a_0 - \frac{\sqrt{e}}{x_2}\right) = 29.8$$

$$y_1 = a_0 + \frac{\sqrt{e}}{x_3} = 27.0$$

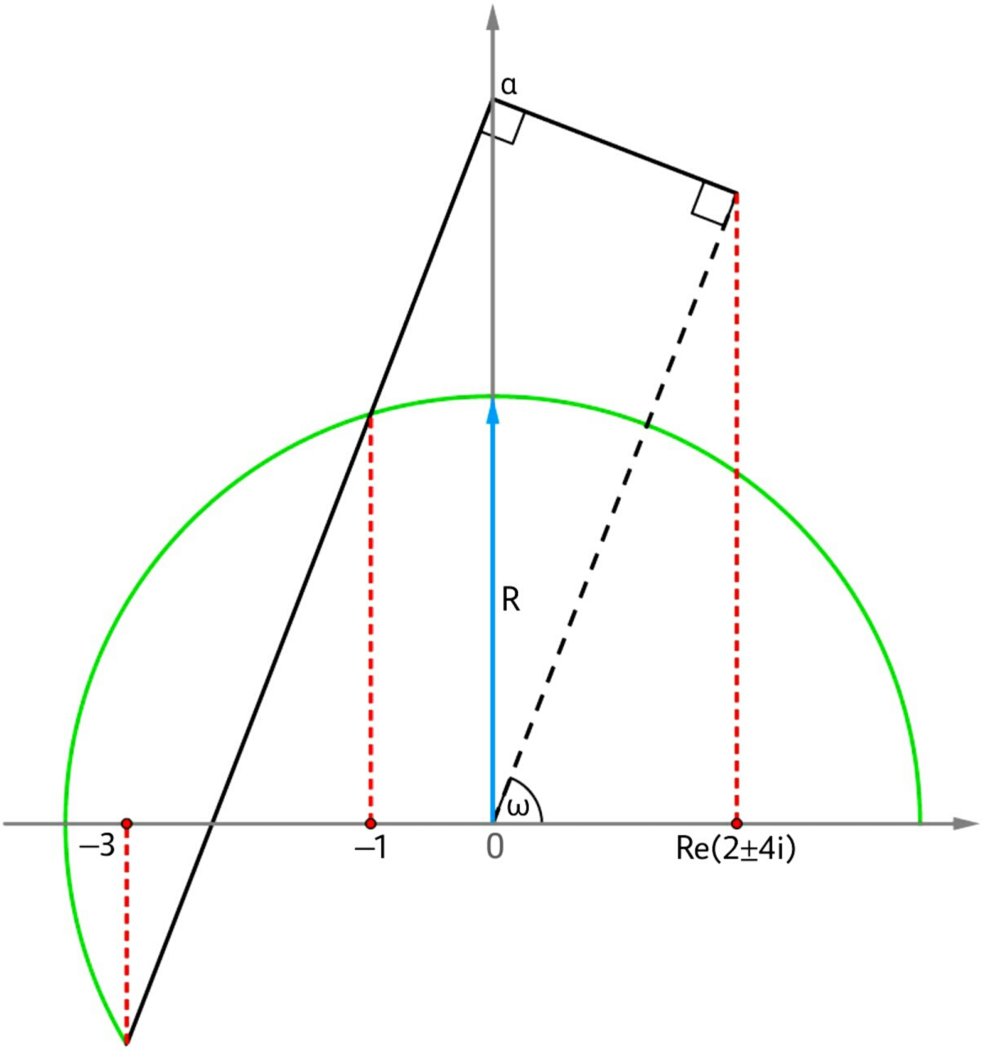
$$y_2 = \mathrm{Re}\left(a_0 - \frac{\sqrt{e}}{x_0}\right) = 29.8$$

$$y_3 = a_0 + \frac{\sqrt{e}}{x_1} = 15.1$$

$$\omega_0 = \cos^{-1}\left(\frac{1}{\sqrt{1 + \frac{x_0 x_2}{x_1 x_3}}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1 + \frac{25^2 + 15^2}{(-20)(-30)}}}\right) = 49.96°$$

$$(x_0 x_2 = (25 - 15i)(25 + 15i) = 25^2 + 15^2)$$

Example 2



$c = 7, d = 68, e = 60$

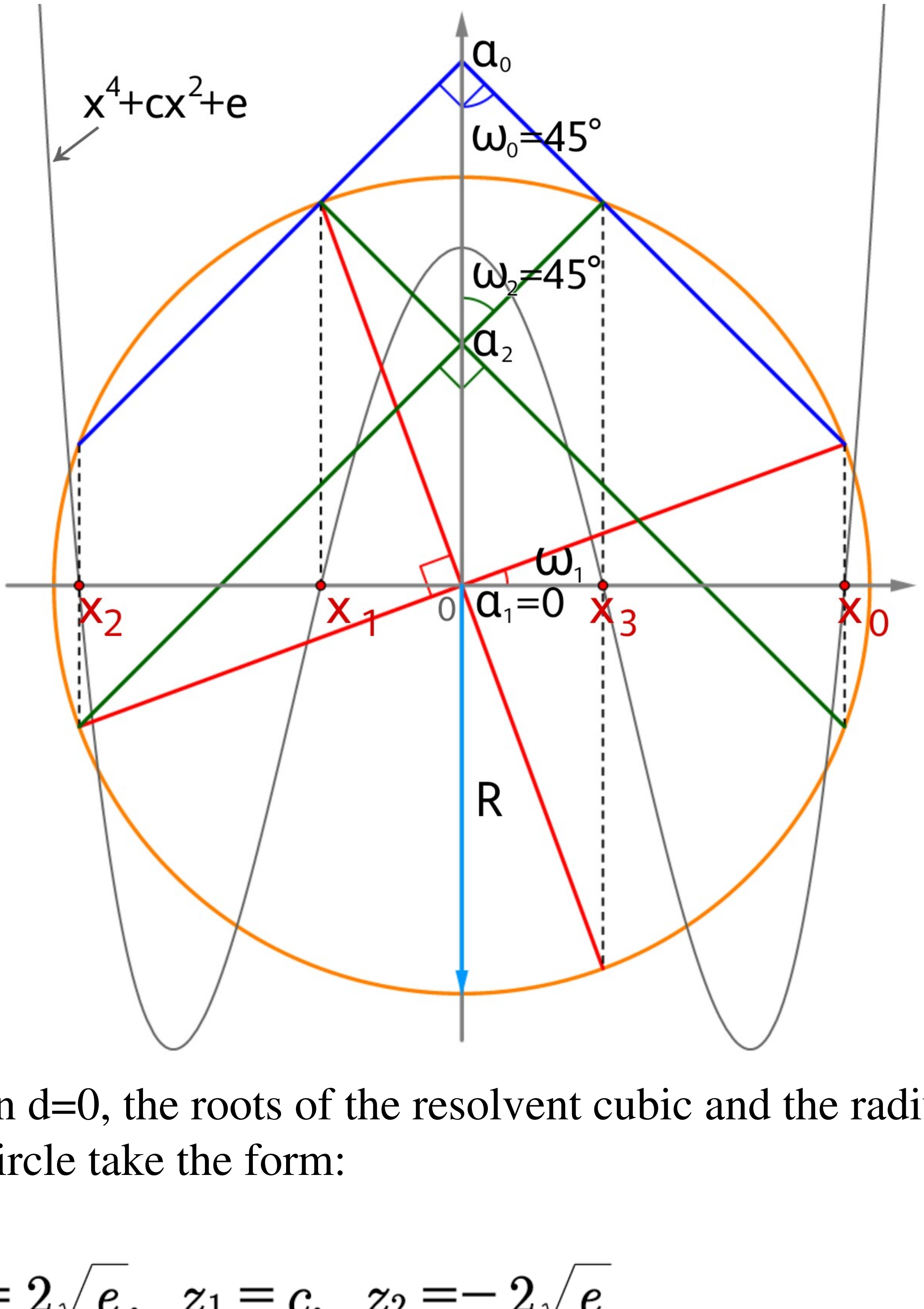
$z = 23$

$$R = \sqrt{\frac{68^2}{4 \cdot 60} - 7} = 3.5$$

$$a = \sqrt{z + R^2} = 5.94$$

$$\omega = \cos^{-1}\left(\sqrt{1/2 - \sqrt{1/4 - 60/23^2}}\right) = 68.83^\circ$$

Geometric interpretation of the biquadratic equation



When d=0, the roots of the resolvent cubic and the radius of the circle take the form:

$$z_0 = 2\sqrt{e}, \quad z_1 = c, \quad z_2 = -2\sqrt{e}$$

$$R = \sqrt{-c}$$

By replacing and simplifying these relationships in the trigonometric method, we get

$$\omega_0 = \omega_2 = 45^\circ \Rightarrow$$
$$\cos\omega_0 = \cos\omega_2 = \sin\omega_0 = \sin\omega_2 = 1/\sqrt{2}$$
$$\cos\omega_1 = \sqrt{1/2+\sqrt{1/4-e/c^2}}$$
$$\sin\omega_1 = \sqrt{1/2-\sqrt{1/4-e/c^2}} \Rightarrow$$
$$\begin{aligned} x_0 &= -2R\sin\omega_0\sin\omega_1\sin\omega_2 \\ &= -2\sqrt{-c}\,(1/\sqrt{2})\sin\omega_1\,(1/\sqrt{2}) \\ &= -\sqrt{-c}\sqrt{1/2-\sqrt{1/4-e/c^2}} \\ &= -\sqrt{-c/2-\sqrt{c^2/4-e}} \end{aligned}$$

Doing the same for the other roots of the biquadratic equation we will get the standard formula

$$x_k = \pm\sqrt{-c/2 \pm \sqrt{c^2/4-e}}$$

How the relationship for the radius of the circle arises

We start with the obvious relationship

$$R^2 = (y_0 - a)^2 + x_0^2$$

Replace y_0 and α with

$$y_0 = -\frac{\sqrt{e}}{x_2},\ \ a = \frac{(x_1+x_3)(x_1x_3+x_0x_2)}{2\sqrt{e}}$$

We first analyze the y_0-a with the help of the relationships between the roots and the coefficients:

$$y_0 - a = -\frac{\sqrt{e}}{x_2} - \frac{(x_1 + x_3)(x_1x_3 + x_0x_2)}{2\sqrt{e}}$$

$$= \frac{-2e - x_1^2x_2x_3 - x_0x_1x_2^2 - x_1x_2x_3^2 - x_0x_2^2x_3}{2x_2\sqrt{e}}$$

$$= \frac{-2x_0x_1\cancel{x}_2x_3 - \cancel{x}_2(x_1^2x_3 - x_0x_1x_2 - x_1x_3^2 - x_0x_2x_3)}{2\cancel{x}_2\sqrt{e}}$$

$$\Rightarrow 2(y_0 - a)\sqrt{e} = -x_0x_1x_2 - x_0x_2x_3 - 2x_0x_1x_3 - x_1^2x_3 - x_1x_3^2$$

We add $x_1x_2x_3$-$x_1x_2x_3$ (=0) to the last representation to get d:

$$\Rightarrow 2(y_0 - a)\sqrt{e} = -x_0x_1x_2 - x_0x_2x_3 - x_0x_1x_3 - x_0x_2x_3 + (x_1x_2x_3 - x_1x_2x_3) - x_1^2x_3 - x_1x_3^2$$

$$= d - x_0x_1x_3 + x_1x_2x_3 - x_1^2x_3 - x_1x_3^2$$

$$= d + x_1x_3(x_2x_3 - x_0 - x_1 - x_3)$$

$$= d + x_1x_3(x_2 + x_2)$$

$$= d + 2x_1x_2x_3$$

$$= d + \frac{2e}{x_0}$$

$$\Rightarrow y_0 - a = \frac{d + \frac{2e}{x_0}}{2\sqrt{e}} = \frac{d}{2\sqrt{e}} + \frac{e}{x_0\sqrt{e}} = \frac{d}{2\sqrt{e}} + \frac{\sqrt{e}}{x_0}$$

$$\Rightarrow R^2 = \left(\frac{d}{2\sqrt{e}} + \frac{\sqrt{e}}{x_0}\right)^2 + x_0^2 = \frac{d^2}{4e} + \frac{e}{x_0^2} + x_0^2$$

$$\Rightarrow R^2 - \frac{d^2}{4e} = \frac{\cancel{x}_0x_1x_2x_3}{\cancel{x}_0x_0} + \frac{-\cancel{x}_0x_1x_2 - \cancel{x}_0x_1x_3 - \cancel{x}_0x_2x_3}{\cancel{x}_0} - \frac{x_1x_2x_3}{x_0} + x_0(-x_1 - x_2 - x_3)$$

$$= \cancel{\frac{x_1x_2x_3}{x_0}} - x_1x_2 - x_1x_3 - x_2x_3 - \cancel{\frac{x_1x_2x_3}{x_0}} - x_0x_1 - x_0x_2 - x_0x_3 = c$$

$$\Rightarrow R^2 = \frac{d^2}{4e} - c$$

About the existence of similar models for polynomial equations of degree n with sum of roots equal to zero

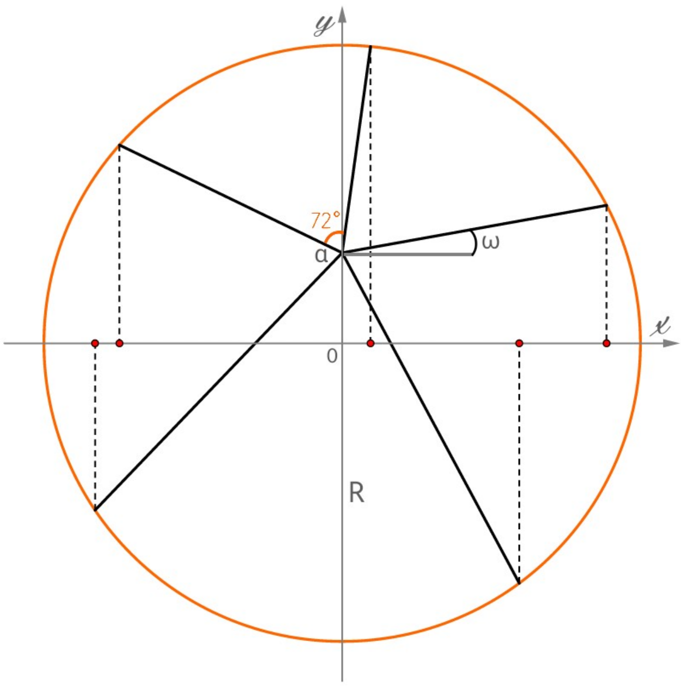
The following result shows whether it would be possible for such an equation to have a similar model with parameters R, α, ω.

If R,α and n have constant values with $R>|\alpha|>-1$ and $n>2$, then for every angle ω will be valid:

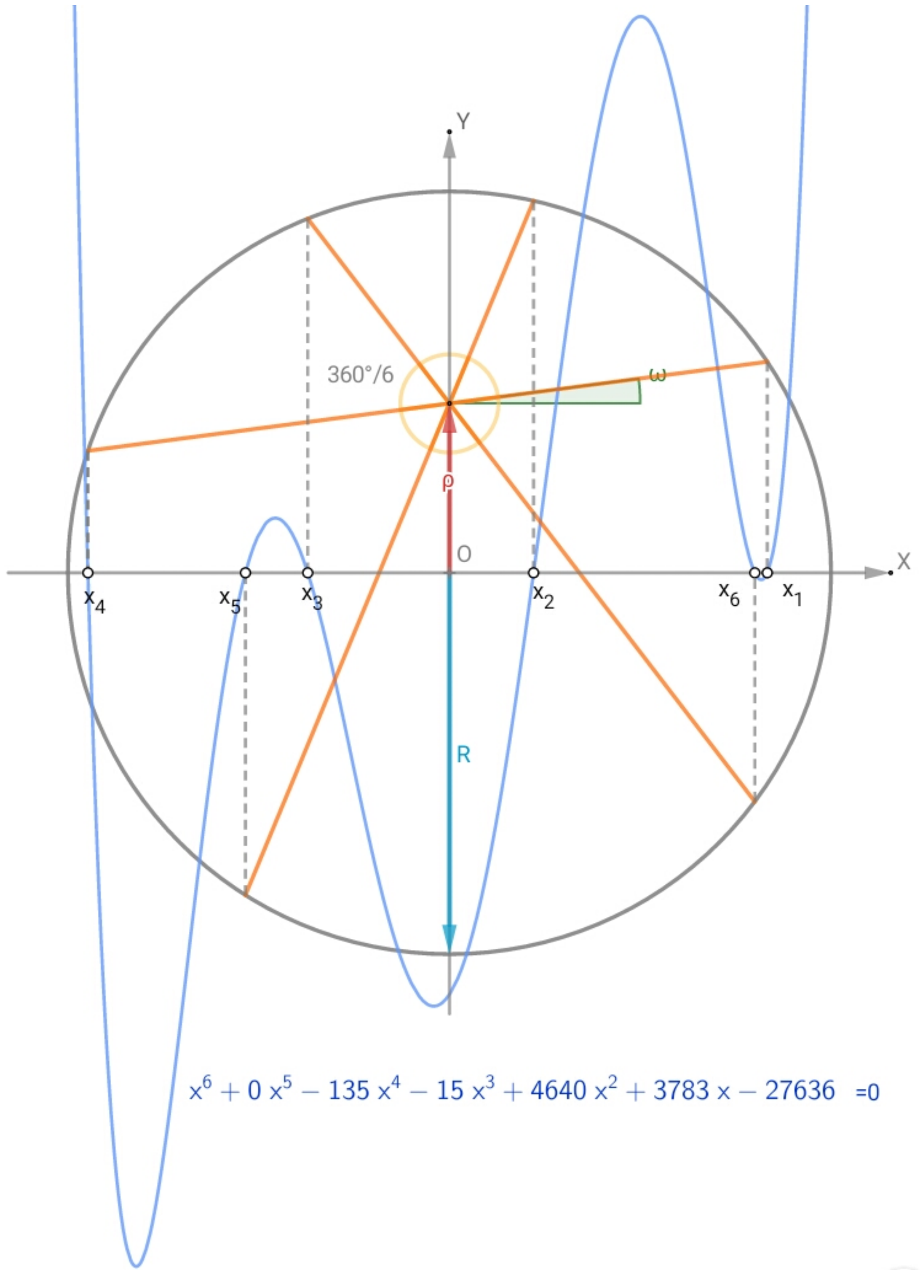
$x_1+x_2+x_3+\ldots+x_n=\delta\cos(n\omega)$

where δ is a constant. Therefore δ will be equal to the sum of the roots for $\omega=0°$. When n is an odd number it will be $\delta=0$ only if $\alpha=0$, while when n is an even number it will always be $\delta=0$ due to the symmetry that these systems have with respect to the y-axis for $\omega=0$.

examples



Inappropriate model



Suitable model

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