The information volume of uncertain information: (4) Negation

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Abstract

Negation is an important operation on uncertainty information. Based on the information volume of mass function, a new negation of basic probability assignment is presented. The result show that the negation of mass function will achieve the information volume increasing. The convergence of negation is the situation when the Deng entropy is maximum, namely high order Deng entropy. If mass function is degenerated into probability distribution, the negation of probability distribution will also achieve the maximum information volume, where Shannon entropy is maximum. Another interesting results illustrate the situation in maximum Deng entropy has the same information volume as the whole uncertainty environment.

Keywords: Negation, Mass function, Information volume, Maximum entropy, Maximum Deng entropy, Maximum information volume

1. Introduction

Uncertainty plays an essential role in the real lief. In order to handle this issue, there are plenty of theories to express and deal with the uncertainty in the uncertain environment, for instance, probability theory [1], fuzzy set theory [2],

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Dempster-Shafer evidence theory (D-S theory) [3, 4], rough sets [5] and so on. D-S theory assigns probabilities to the power set of events, so as to better grasp the unknown and uncertainty of the problem [3, 4]. Based on the advantages of D-S theory, the paper mainly discusses the related contents of D-S theory.

In many cases, we can not easily come to the decision-making from the known information [6]. However, negation, namely the opposite side of known information, can give us a possible way of knowledge representation [7]. Zhade in his BISC blog raised the the negation of a probability distribution. Yager also proposed a new method to calculate the negation, and thought entropy can increase with negation [8].

The paper also proposed negation operation in D-S theory. Besides, what are there the changes after negation operation? What is the cause of these changes? In physics, some operations can the system more chaos. In our paper, we will use information volume based on Deng entropy to measure the changes after negation operation. Information volume can measure the more uncertainty than Deng entropy, which can be applied to absorb the complex imprecise (or unknown) phenomenon in the belief filed efficiently [9, 10]. Besides, information volume can show the existence of the fractal property [11].

The rest of this paper is organized as follows. In section **2**, some preliminaries are briefly reviewed. In section **3**, the proposed negation operation can be introduced. In section **4**, numerical examples are expounded to discuss the proposed method. In section **5**, we have a brief conclusion.

2. Preliminaries

Several preliminaries are briefly introduced in this section, including D-S theory, Deng entropy, information volume.

2.1. Dempster-Shafer evidence theory

Dempster-Shafer evidence theory[3, 4] can be used to deal with uncertainty. Besides, evidence theory satisfies the weaker conditions than the probability theory, which provides it with the ability to express uncertain information directly. Some basic conceptions of evidence theory are given as follows:

Definition 2.1. *(Frame of discernment)*
Let Θ be the set of mutually exclusive and collectively exhaustive events A_i [3, 4], namely

$$\Theta = \{A_1, A_2, \cdots, A_n\} \tag{1}$$

The power set of Θ composed of 2^N elements of is indicated by 2^Θ, namely [3, 4]:

$$2^\Theta = \{\phi, \{A_1\}, \{A_2\}, \cdots, \{A_1, A_2\}, \cdots, \Theta\} \tag{2}$$

Definition 2.2. *(Mass Function)*
For a frame of discernment $\Theta = \{A_1, A_2, \cdots, A_n\}$, the mass function m *is defined as a mapping of m from 0 to 1[3, 4], namely:*

$$m : 2^\Theta \to [0, 1] \tag{3}$$

which satifies

$$m(\phi) = 0 \tag{4}$$

$$\sum_{A \subseteq \Theta} m(A) = 1 \tag{5}$$

In D-S theory, a mass function is also called a basic probability assignment (BPA) or a piece of evidence or belief structure. The $m(A)$ measures the belief exactly assigned to A and represents how strongly the piece of evidence supports A.

2.2. Deng entropy

Deng proposed an *Deng Entropy*, which is an generalization of Shannon entropy [12].

Definition 2.3. *(Deng entropy)*

Given a BPA, Deng entropy can be defined as:

$$H_D = -\sum_{A\subseteq\Theta} m(A) log_2 \frac{m(A))}{2^{|A|}-1} \tag{6}$$

Through a simple transformation, Deng Entropy can be rewrite as follows:

$$H_D = \sum_{A\subseteq\Theta} m(A) log_2 (2^{|A|}-1) - \sum_{A\subseteq\Theta} m(A) log_2 m(A) \tag{7}$$

where m is a BPA defined on the frame of discernment Θ, and A is the focal element of m, $|A|$ is the cardinality of A. Besides, the term $\sum m(A) \times log_2(2^{|A|}-1)$ could be interpreted as a measure of total nonspecificity in the mass function m, and the term $-m(A) \times log_2 m(A)$ is the measure of discord of the mass function among various focal elements.

2.3. Information volume of mass function

Let the frame of discernment be $\Theta = \{\theta_1, \theta_2, \theta_3, \cdots, \theta_N\}$. Use index i to denote the times of this loop, and use $m(A_i)$ to denote different mass function of different loops. Based on Deng entropy, the information volume of mass function can be calculated by following steps [9]:

step 1: Input mass function $m(A_0)$.

step 2: Continuously separate the mass function of the element whose cardinal is larger than 1 until convergence. Concretely, repeat the loop from step 2-1 to step 2-3 until Deng entropy is convergent.

step 2-1: Focus on the element whose cardinal is larger than 1, namely, $|A_i| > 1$. And then, separate its mass function based on the proportion of Deng distribution:

$$m_D(A_i) = \frac{(2^{|A_i|}-1)}{\sum_{A_i\in 2^\Theta}(2^{|A_i|}-1)} \tag{8}$$

For example, given a focal element $A_{i-1} = \{\theta_x, \theta_y\}$ and its mass function $m(A_{i-1})$, the separating proportion is that $\frac{1}{5}$:

$\frac{1}{5}:\frac{3}{5}$. The ith times of separation divides $m(A_{i-1})$ and yields following new mass function: $m(X_i)$, $m(Y_i)$, $m(Z_i)$, where $X_i=\{\theta_x\}$, $Y_i=\{\theta_y\}$ and $Z_i=\{\theta_x,\theta_y\}$. In addition, they satisfy these equations:

$$m(X_i)+m(Y_i)+m(Z_i)=m(A_{i-1}) \tag{9}$$

$$m(X_i):m(Y_i):m(Z_i)=\frac{1}{5}:\frac{1}{5}:\frac{3}{5} \tag{10}$$

step 2-2: Based on Deng entropy, calculate the uncertainty of all the mass functions except for those who have been divided. The result is denoted as $H_i(m)$.

step 2-3: Calculate $\Delta_i=H_i(m)-H_{i-1}(m)$. When Δ_i satisfies following condition, jump out of this loop.

$$\Delta_i=H_i(m)-H_{i-1}(m)<\varepsilon \tag{11}$$

where ε is the allowable error.

step 3: Output $H_{IV-mass}(m)=H_i(m)$, which is the information volume of the mass function.

3. Proposed Negation Operation

Negation operation can provide us a new view to make decision from known information. Next, we will introduce the proposed negation operation.

Assume frame of discernment Θ has N elements, 2^Θcan be expressed as:

$$2^\Theta=\{A_1,A_2,\cdots,A_{2^N}\} \tag{12}$$

Especially, $A_1=\phi$,$A_{2^N}=\Theta$. Let m be a BPA. Assume that

$$\sum_{i=1}^{2^N} m_{A_i}=1 \tag{13}$$

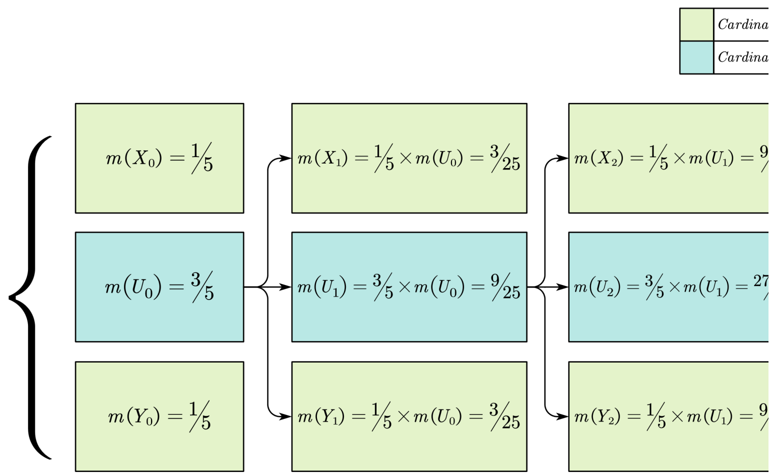

Figure 1: The procedure of from step 2-1 to step 2-3

Similarly, $\bar{m}_{A_i}$represent the negation of m_{A_i}, and the procedure of taking negation is listed as follows.

Step 1 : For each focal element A_i in the frame of discernment, we use $1 - m_{A_i}$ to represent the complementary probability of m_{A_i}.

$$\hat{m}_{A_i} = 1 - m_{A_i} \tag{14}$$

Step 2 : $\hat{m}_{A_i}$ can be divided into other elements without A_i as follow.

$$\bar{m}(A_j) = \hat{m}(A_i) \times \frac{2^{|A_j|} - 1}{\sum_{A_k \neq A_i}^{2^\Theta} 2^{|A_k|} - 1} \tag{15}$$

Step 3 : Calculate the sum λ of the negative belief of all the focal elements. Namely,

$$\lambda = \sum \bar{m}_{A_j} \tag{16}$$

Step 4 : The step is used to normalize the belief of all the negative focal elements.

$$\bar{m}_{A_i} = \frac{1 - m_{A_i}}{\lambda} \tag{17}$$

4. Numerical Examples and Discussion

Example 1 *Assume a frame of discernment* $\Theta = \{a, b, c\}$*, for a mass function which degenerate the probability* $m(a) = 0.1$*,* $m(b) = 0.2$*,* $m(c) = 0.7$ *the associated negation can be calculated as follows.*

$$\bar{m}(a) = 0.45$$

$$\bar{m}(b) = 0.4$$

$$\bar{m}(c) = 0.15$$

Next, we will discuss the final result after multi negation operations in Table **??**.

i	$\bar{m}_i(a)$	$\bar{m}_i$(b)	$\bar{m}_i(c)$
1	0.45	0.4	0.15
2	0.275	0.3	0.425
3	0.3625	0.35	0.2875
4	0.3188	0.325	0.3562
5	0.3406	0.3375	0.3219
6	0.3297	0.3313	0.3391
7	0.3352	0.3344	0.3305
8	0.3324	0.3328	0.3348
9	0.3338	0.3336	0.3326
10	0.3331	0.3332	0.3337

Table 1: The change of mass function after negation operation

In **??**, i represents the times of negation. It can be seen that the distribution tends to be more and more uniform distributed after multi negation operation. As we all known, in probability theory, uniform distribution represents the most chaotic system, namely the system has the biggest uncertainty. Besides, if we use the information volume to measure the uncertainty, what will change ? So, we use the information volume to calculate the change as Figure 2. It can be seen that the BPA have a trend of fluctuation after negation, however, the information volume can have a trend of increasing.

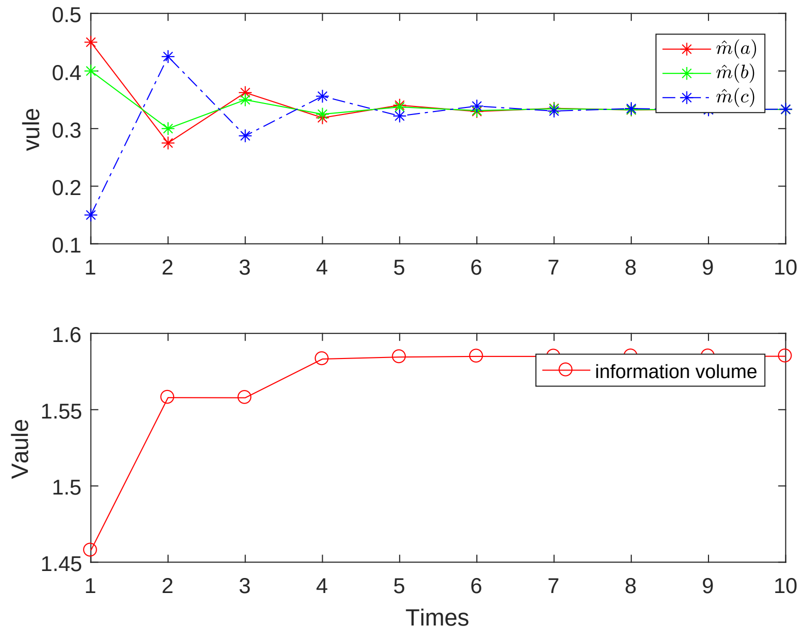

Figure 2: The trend of BPA and information volume after negation

Example 2 *Assume a frame of discernment* $\Theta = \{a, b\}$*, for a mass function which degenerate the probability* $m(a) = 0.5$*,* $m(b) = 0.4$*,* $m(a, b) = 0.1$ *the associated negation can be calculated as follows.*

$$\bar{m}(a) = 0.3$$

$$\bar{m}(b) = 0.2875$$

$$\bar{m}(a, b) = 0.4125$$

The specific process of negation calculation is as Figure 3.

Next, we will discuss the final result after multi negation operations in Table 2. From the Table 2, it can be seen that the information volume can increase with the negation operation.

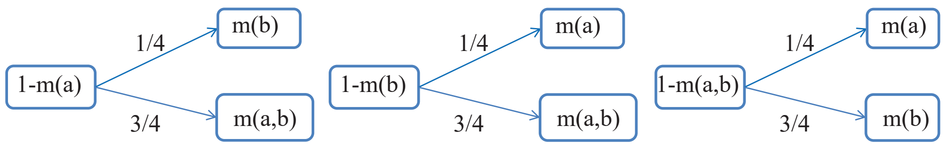

Figure 3: The specific process of negation calculation for Example 2

i	$\bar{m}_i(a)$	$\bar{m}_i$(b)	$\bar{m}_i(a,b)$	information volume
1	0.3	0.2875	0.4125	1.7026
2	0.2359	0.2343	0.5298	2.9779
3	0.2133	0.213	0.5737	3.2822
4	0.2050	0.2049	0.5902	3.3755
5	0.2018	0.2018	0.5963	3.4190
6	0.2000	0.2000	0.6000	3.4259
7	0.2000	0.2000	0.6000	3.4259
8	0.2000	0.2000	0.6000	3.4259
9	0.2000	0.2000	0.6000	3.4259
10	0.2000	0.2000	0.6000	3.4259

Table 2: The change of mass function after negation operation

5. Conclusion

In this paper, a negation method of mass function is presented based on the information volume. We show that the negation will achieve the information volume increasing, not only in the case of probability distribution but also mass function. If the input is probability distribution, the relative maximum information volume is Shannon entropy. While if the imput is mass function, the negation result is the Deng distribution relative to the maximum Deng entropy, also equals the information volume of total uncertainty environment.

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References

[1] P. Lee, Probability theory, Bulletin of the London Mathematical Society 12 (4) (1980) 318–319.

[2] L. A. Zadeh, Fuzzy sets, Information and control 8 (3) (1965) 338–353.

[3] A. P. Dempster, Upper and lower probabilities induced by a multivalued mapping, The Annals of Mathematical Statistics 38 (2) (1967) 325–339. doi:10.1214/aoms/1177698950.

[4] G. Shafer, A mathematical theory of evidence, Vol. 1, Princeton university press Princeton, 1976.

[5] Z. Pawlak, Rough sets, International journal of computer & information sciences 11 (5) (1982) 341–356.

[6] V. Huynh, Y. Nakamori, T. Ho, T. Murai, Multiple-attribute decision making under uncertainty: The evidential reasoning approach revisited, IEEE Transaction on Systems Man and Cybernetics Part A-Systems and Humans 36 (4) (2006) 804–822.

[7] L. Yin, X. Deng, Y. Deng, The negation of a basic probability assignment, IEEE Transactions on Fuzzy Systems 27 (1) (2018) 135–143.

[8] R. R. Yager, On the maximum entropy negation of a probability distribution, IEEE Transactions on Fuzzy Systems 23 (5) (2015) 1899–1902.

[9] Y. Deng, The information volume of uncertain informaion: (1) mass function, viXra preprint viXra (2020) 2006.0028.

[10] J. Deng, Y. Deng, The information volume of uncertain information: (2) fuzzy membership function, viXra preprint viXra (2020) 2006.0037.

[11] T. Wen, Y. Deng, The information volume of uncertain information: (3) information fractal dimension, viXra preprint viXra (2020) 2006.0035.

[12] Y. Deng, Deng entropy, Chaos, Solitons & Fractals 91 (2016) 549 – 553.