

[A] High Power Partial Fraction Decomposition Theorem

Pagano Federico

(03/2016)

Chemistry Engineer (UTN), chemistry technician, and member of the reverse osmosis & arsenic removal research team (UTN)

e-mail: ludhorr@gmail.com // ludhor@hotmail.com

Abstract

[T]he problem of fraction decomposition it's easy to solve by using the cover up method, when there are no repeated linear factors in the denominator. Nevertheless it could turn into a hard work if these factors are raised to a high power, where the cover up method doesn't work. This technique shows how to calculate these coefficients without solving large systems of equations.

Keywords: [P]artial fractions, Horner's method, partial fraction decomposition, Heaviside decomposition, cover-up method, [] theorem.

1 hypothesis: Let $P(x)$ be a polynomial of degree less than $m+n$, which it's not divisible by $(x-a)^n$ or $(x-b)^m$.

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=1}^n \frac{\alpha_k}{(x-a)^k} + \sum_{j=1}^m \frac{\beta_j}{(x-b)^j}$$

$$A(x) = \frac{P(x)}{(x-b)^m}, B(x) = \frac{P(x)}{(x-a)^n}$$

2 thesis

$$\alpha_k = \frac{1}{(n-k)!} D^{n-k} A(a)$$

$$\beta_j = \frac{1}{(m-j)!} D^{m-j} B(b)$$

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=1}^n \frac{1}{(n-k)!} \frac{D^{n-k} A(a)}{(x-a)^k} + \sum_{j=1}^m \frac{1}{(m-j)!} \frac{D^{m-j} B(b)}{(x-b)^j}$$

(Note: Corrections on 1st page are made by viXra Admin to conform with scholarly norm)

3 demonstration

3.1 α_k derivation

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^1 \frac{\alpha_k}{(x-a)^k} + \sum_{j=m}^1 \frac{\beta_j}{(x-b)^j}$$

$$\frac{P(x)}{(x-b)^m} = \sum_{k=n}^1 \frac{\alpha_k(x-a)^n}{(x-a)^k} + \sum_{j=m}^1 \frac{\beta_j(x-a)^n}{(x-b)^j}$$

$$A(x) = \sum_{k=n}^1 \frac{\alpha_k(x-a)^n}{(x-a)^k} + \sum_{j=m}^1 \frac{\beta_j(x-a)^n}{(x-b)^j}$$

$$A(x) = \sum_{k=n}^1 \alpha_k(x-a)^{n-k} + \sum_{j=m}^1 \beta_j \frac{(x-a)^n}{(x-b)^j}$$

$$A(x) = \alpha_n + \alpha_{n-1}(x-a) + \alpha_{n-2}(x-a)^2 + \alpha_{n-3}(x-a)^3 + \dots + \alpha_1(x-a)^{n-1} + \sum_{j=m}^1 \frac{\beta_j(x-a)^n}{(x-b)^j}$$

$$DA(x) = \alpha_{n-1} + 2\alpha_{n-2}(x-a) + 3\alpha_{n-3}(x-a)^2 + \dots + (n-1)\alpha_1(x-a)^{n-2} + \sum_{j=m}^1 \frac{\beta_j n(x-a)^{n-1}}{(x-b)^j}$$

$$D^2A(x) = 2\alpha_{n-2} + 3 \cdot 2 \cdot \alpha_{n-3}(x-a) + \dots + (n-1)(n-2)\alpha_1(x-a)^{n-3} + \sum_{j=m}^1 \frac{\beta_j n(n-1)(x-a)^{n-2}}{(x-b)^j}$$

$$D^3A(x) = 3 \cdot 2 \cdot \alpha_{n-3} + \dots + (n-1)(n-2)(n-3)\alpha_1(x-a)^{n-4} + \sum_{j=m}^1 \frac{\beta_j n(n-1)(n-2)(x-a)^{n-3}}{(x-b)^j}$$

$D^r A(a) = r! \cdot \alpha_{n-r}$, substitution $r=n-k$,

$$\alpha_k = \frac{1}{(n-k)!} D^{n-k} A(a)$$

3.2 β_j derivation

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^1 \frac{\alpha_k}{(x-a)^k} + \sum_{j=m}^1 \frac{\beta_j}{(x-b)^j}$$

$$\frac{P(x)}{(x-a)^n} = \sum_{k=n}^1 \frac{\alpha_k(x-b)^m}{(x-a)^k} + \sum_{j=m}^1 \frac{\beta_j(x-b)^m}{(x-b)^j}$$

$$B(x) = \sum_{k=n}^1 \frac{\alpha_k(x-b)^m}{(x-a)^k} + \sum_{j=m}^1 \beta_j(x-b)^{m-j}$$

$$B(x) = \sum_{k=n}^1 \frac{\alpha_k(x-b)^m}{(x-a)^k} + \beta_m + \beta_{m-1}(x-b) + \beta_{m-2}(x-b)^2 + \beta_{m-3}(x-b)^3 + \dots + \beta_1(x-b)^{m-1}$$

$$D B(x) = \sum_{k=n}^1 \frac{m\alpha_k(x-b)^{m-1}}{(x-a)^k} + \beta_{m-1} + 2\beta_{m-2}(x-b) + 3.\beta_{m-3}(x-b)^2 + \dots + (m-1)\beta_1(x-b)^{m-2}$$

$$D^2 B(x) = \sum_{k=n}^1 \frac{m(m-1)\alpha_k(x-b)^{m-2}}{(x-a)^k} + 2\beta_{m-2} + 3.2.\beta_{m-3}(x-b) + \dots + (m-1)(m-2)\beta_1(x-b)^{m-3}$$

$$D^3 B(x) = \sum_{k=n}^1 \frac{m(m-1)(m-2)\alpha_k(x-b)^{m-3}}{(x-a)^k} + 3.2.\beta_{m-3} + \dots + (m-1)(m-2)(m-3)\beta_1(x-b)^{m-4}$$

$$D^t B(b) = t! . \beta_{m-t} \text{ substitution } j=m-t$$

$$\beta_j = \frac{1}{(m-j)!} D^{m-j} B(b)$$

remark: what we should notice about this formula :

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^1 \frac{1}{(n-k)!} \frac{D^{n-k}A(a)}{(x-a)^k} + \sum_{j=m}^1 \frac{1}{(m-j)!} \frac{D^{m-j}B(b)}{(x-b)^j}$$

is that we are changing the hard work, instead of solving a $(m+n) \times (m+n)$ system of equations, we could just take “simple” derivatives but when we go back to the definition to see who $A(x)$ and $B(x)$ are, we find that both of them are rational functions and it doesn't sound “so easy” to make high order derivatives of a rational function. The following examples show how to “avoid” this problem with the help of Horner's rule and a clever rearrangement of the expressions $A(x)$ and $B(x)$.

4 Examples

Example 4.1

$$\frac{x^3 - 2x^2 + 7x - 1}{(x-2)^4(x-3)^3}$$

remark: first of all, we must expand the numerator in terms of the right factor: $(x-3)$ applying Horner's method

	x^3	x^2	x^1	x^0
	1	-2	7	-1
3		3	3	30
	1	1	10	29
3		3	12	
	1	4	22	
3		3		
	1	7		

$$\frac{1(x-3)^3 + 7(x-3)^2 + 22(x-3) + 29}{(x-2)^4(x-3)^3} = \frac{1 + 7(x-3)^{-1} + 22(x-3)^{-2} + 29(x-3)^{-3}}{(x-2)^4} = \frac{A(x)}{(x-2)^4}$$

we could do the same work for the left factor $(x-2)$

	x^3	x^2	x^1	x^0
	1	-2	7	-1
2		2	0	14
	1	0	7	13
2		2	4	
	1	2	11	
2		2		
	1	4		

$$\frac{1(x-2)^3 + 4(x-2)^2 + 11(x-2) + 13}{(x-2)^4(x-3)^3} = \frac{1(x-2)^{-1} + 4(x-2)^{-2} + 11(x-2)^{-3} + 13(x-2)^{-4}}{(x-3)^3} = \frac{B(x)}{(x-3)^3}$$

$$\frac{P(x)}{(x-a)^n(x-b)^m} = \sum_{k=n}^1 \frac{1}{(n-k)!} \frac{D^{n-k}A(a)}{(x-a)^k} + \sum_{j=m}^1 \frac{1}{(m-j)!} \frac{D^{m-j}B(b)}{(x-b)^j}$$

$$\frac{x^3 - 2x^2 + 7x - 1}{(x-2)^4(x-3)^3} = \sum_{k=4}^1 \frac{1}{(4-k)!} \frac{D^{4-k}A(2)}{(x-2)^k} + \sum_{j=3}^1 \frac{1}{(3-j)!} \frac{D^{3-j}B(3)}{(x-3)^j}$$

$$\frac{A(2)}{(x-2)^4} + \frac{D^1A(2)}{(x-2)^3} + \frac{1}{2!} \frac{D^2A(2)}{(x-2)^2} + \frac{1}{3!} \frac{D^3A(2)}{(x-2)} + \frac{B(3)}{(x-3)^3} + \frac{D^1B(3)}{(x-3)^2} + \frac{1}{2!} \frac{D^2B(3)}{(x-3)}$$

$A(x)$	$1 + 7(x-3)^{-1} + 22(x-3)^{-2} + 29(x-3)^{-3}$	$A(2) = -13$
$D^1A(x)$	$-7(x-3)^{-2} - 44(x-3)^{-3} - 87(x-3)^{-4}$	$D^1A(2) = -50$
$D^2A(x)$	$14(x-3)^{-3} + 132(x-3)^{-4} + 348(x-3)^{-5}$	$D^2A(2) = -230$
$D^3A(x)$	$-42(x-3)^{-4} - 528(x-3)^{-5} - 1740(x-3)^{-6}$	$D^3A(2) = -1254$

$B(x)$	$1(x-2)^{-1} + 4(x-2)^{-2} + 11(x-2)^{-3} + 13(x-2)^{-4}$	$B(3) = 29$
$D^1B(x)$	$-1(x-2)^{-2} - 8(x-2)^{-3} - 33(x-2)^{-4} - 52(x-2)^{-5}$	$D^1B(3) = -94$
$D^2B(x)$	$2(x-2)^{-3} + 24(x-2)^{-4} + 132(x-2)^{-5} + 260(x-2)^{-6}$	$D^2B(3) = 418$

$$\frac{x^3 - 2x^2 + 7x - 1}{(x-2)^4(x-3)^3} = \frac{A(2)}{(x-2)^4} + \frac{D^1A(2)}{(x-2)^3} + \frac{1}{2!} \frac{D^2A(2)}{(x-2)^2} + \frac{1}{3!} \frac{D^3A(2)}{(x-2)} + \frac{B(3)}{(x-3)^3} + \frac{D^1B(3)}{(x-3)^2} + \frac{1}{2!} \frac{D^2B(3)}{(x-3)}$$

$$\frac{x^3 - 2x^2 + 7x - 1}{(x-2)^4(x-3)^3} = \frac{-13}{(x-2)^4} - \frac{50}{(x-2)^3} - \frac{115}{(x-2)^2} - \frac{209}{(x-2)} + \frac{29}{(x-3)^3} - \frac{94}{(x-3)^2} + \frac{209}{(x-3)}$$

Example 4.2

$$\frac{1}{(x-1)^4(x-2)^2}$$

$$\frac{1}{(x-1)^4(x-2)^2} = \frac{A(1)}{(x-1)^4} + \frac{D^1A(1)}{(x-1)^3} + \frac{1}{2!} \frac{D^2A(1)}{(x-1)^2} + \frac{1}{3!} \frac{D^3A(1)}{(x-1)} + \frac{B(2)}{(x-2)^2} + \frac{D^1B(2)}{(x-2)}$$

$$A(x) = (x - 2)^{-2}, B(x) = (x - 1)^{-4}$$

$A(x)$	$(x - 2)^{-2}$	$A(1) = 1$
$D^1A(x)$	$-2(x - 2)^{-3}$	$D^1A(1) = 2$
$D^2A(x)$	$6(x - 2)^{-4}$	$D^2A(1) = 6$
$D^3A(x)$	$-24(x - 2)^{-5}$	$D^3A(1) = 24$

$B(x)$	$(x - 1)^{-4}$	$B(2) = 1$
$D^1B(x)$	$-4(x - 1)^{-5}$	$D^1B(2) = -4$

$$\frac{1}{(x-1)^4(x-2)^2} = \frac{1}{(x-1)^4} + \frac{2}{(x-1)^3} + \frac{3}{(x-1)^2} + \frac{4}{(x-1)} + \frac{1}{(x-2)^2} - \frac{4}{(x-2)}$$

5 Conclusion: this method is helpful to turn rational functions into a partial fraction decomposition, avoiding any kind of large system of equations, and if we apply it with the clever rearrangement of the numerator, we could just be able to face higher powers of the denominator, because this kind of numerator rearrangement allows always to find an easy pattern for the n-derivatives which involves of course factorials.