Representations of the Division by Zero Calculus by Means of Mean Values

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Abstract: In this paper, we will give simple and pleasant introductions of the division by zero calculus by means of mean values that give an essence of the division by zero. In particular, we will introduce a new mean value for real valued functions in connection with the Sato hyperfunction theory.

Recall that David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Meanwhile,

Oliver Heaviside: *Mathematics is an experimental science, and definitions do not come first, but later on.*

We would like to say:

GOD loves mean values.

Key Words: Division by zero, division by zero calculus, mean value, Sato hyperfunction, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0$, $[(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$.

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1 Background of division by zero calculus

We would like to consider some values for isolated singular points for analytic functions. The very typical problem is to consider some value of the function $W = 1/z$ at the origin. We found that its value is zero. When the result is written as

$$\frac{1}{0} = 0,$$

it will have a serious sense, because it looks like the division by zero that has a mysteriously long history ([1, 5, 22, 34, 35, 36]). However, note that $0 \times 0 \neq 1$. We showed that our result gave great impacts widely with over 1100 items. For example, look the papers cited in the reference.

The essence is stated as follows:

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n \tag{1.1}$$

we will define

$$f(a) = C_0. \tag{1.2}$$

For the correspondence (1.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (1.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

With this assumption, we can obtain many new results and new concepts. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problem. – In this viewpoint, **the division by zero calculus may be considered as an axiom.**

The essential properties of the division by zero calculus may be given as follows:

For any negative integer n and for the function, for any fixed a

$$f_n(z) = (z-a)^n$$

we assume that

$$f_n(a) = 0.$$

Of course, if the equation $f_n(z) = (z - a)^n = 0$ has a solution, then the solution has to be a. Indeed, we have considered to solve an equation by extending our concept in our mathematics.

We gave many reasons of those properties already as in stated in the papers in the reference. However, we found some very simple and interesting motivations of those fundamental properties. We will introduce a new type mean value for real variable functions. The essential idea for the new mean for real variable functions was discovered on September 6, 2020 by Ichiro Fujimoto from the viewpoint of the Sato hyperfunctions, however its essence is given simply independently of the Sato hyperfunctions.

In general, definitions and axioms are fundamental in mathematics and we find many related problems, for the present case, for example, another definitions of the division by zero calculus and their relations. In particular, we will have some interesting connection with the Sato hyperfunction theory. Therefore, in this paper, we will state only the new definitions (representation) of the division by zero calculus.

2 Complex variable functions case

For a function $f(z), z = x + iy$, we would like to see and realize the value $f(z)$ by

$$f(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta,$$

if the right hand side exists.

With this definition, we see immediately that the above fundamental properties are satisfied.

Of course, this mean value is very classical. However, note that we can introduce our division by zero calculus with this definition for some general complex variable functions with singularities.

3 Real variable functions case

For a real variable function $f(x)$, we would like to see and realize the value $f(x)$ by the mean value

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2} \left\{ f(x+\varepsilon) + f(x-\varepsilon) \right\},$$

if we can consider the right hand side.

With this definition, we will see the above fundamental case

$$f_n(x) = \frac{1}{x^n}.$$

Then, we have for odd n:

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2} \left\{ \frac{1}{(x+\varepsilon)^n} + \frac{1}{(x-\varepsilon)^n} \right\}.$$

$$= \begin{cases} 0 & \text{for } x = 0 \\ \frac{1}{x^n} & \text{for } x \neq 0. \end{cases}$$

This gives a good result for odd n.

However, for even n, the limit in the definition does not exist. However, for $x = 0$, from

$$\frac{1}{2} \left\{ \frac{1}{(x+\varepsilon)^n} + \frac{1}{(x-\varepsilon)^n} \right\}$$

$$= \frac{1}{\varepsilon^n},$$

if we consider the division by zero calculus for $\varepsilon = 0$, we have the desired result.

Therefore, we can give the representation (definition) as follows:

When there exists the limit

$$\lim_{\varepsilon \to 0} \frac{1}{2} \left\{ f(x+\varepsilon) + f(x-\varepsilon) \right\}$$

or

$$\frac{1}{2} \left\{ f(x+\varepsilon) + f(x-\varepsilon) \right\}$$

has the meaning for $\varepsilon = 0$ with the division by zero calculus, we define the value $f(x)$ with the limit and with the value in the sense of division by zero calculus, respectively.

We can obtain the following theorem, easily

Theorem:

(1) If $f(x)$ is differentiable at $x = 0$, then

$$\frac{f(x)}{x}|_{x=0} = f'(0).$$

(2) If n is even and $f(x)$ is odd with respect to $x = 0$, then

$$\frac{f(x)}{x^n}|_{x=0} = 0.$$

(3) If $n, n \geq 3$ is odd and $f(x)$ is even , then

$$\frac{f(x)}{x^n}|_{x=0} = 0.$$

(4) If $f(x)$ is real analytic around at $x = 0$, with the division by zero calculus,

$$\frac{f(x)}{x^n}|_{x=0} = \frac{f^{(n)}(0)}{n!}.$$

As we show the example of $y = \log x$ and for other reasons, we will introduce the new mean value by changing ε by $i\varepsilon$ as follows:

When there exists the limit

$$\lim_{\varepsilon \to 0} \frac{1}{2} \{f(x + i\varepsilon) + f(x - i\varepsilon)\}$$

or

$$\frac{1}{2} \{f(x + i\varepsilon) + f(x - i\varepsilon)\}$$

has the meaning for $\varepsilon = 0$ with the division by zero calculus, we define the value $f(x)$ with the limit and with the value in the sense of division by zero calculus, respectively.

Of course, in this definition we assume that the complex variable functions $f(x + i\varepsilon)$ and $f(x - i\varepsilon)$ are given around the real line.

Examples:

1. For

$$f(x) = \frac{|x|}{x^n}, \quad n \geq 1,$$

   we have

$$f(0) = 0.$$

   For

$$f(z) = \frac{|z|}{z^n}, \quad n \geq 1,$$

   we have

$$f(0) = 0.$$

2. For some functions that are not odd, we can obtain some reasonable values.

   For

$$f(x) = \log x,$$

   we have

$$f(0) = 0.$$

   Here, we consider the principal values on the domain $-\pi < \arg z < \pi$. Note that in this case, with the usual mean, we can not consider the value $\log 0$, but, by the complex mean value, we can consider the natural value $\log 0 = 0$ with the sense of the division by zero calculus.

   For $\log 0 = 0$, see [8].

3. For

$$f(x) = \frac{a^x}{x}, \quad a > 0$$

   we have

$$f(0) = \log a.$$

4. For the function

$$f(x) = e^{\frac{1}{x}},$$

we see that the limit in the definition does not exist, however, from the identity, for $x = 0$

$$\frac{1}{2}\{f(x + i\varepsilon) + f(x - i\varepsilon)\} = \cos\frac{1}{\varepsilon},$$

we have $f(0) = 1$ with the sense of the division by zero calculus. With the real mean value, we have the similar result.

For the complex valued function

$$f(z) = e^{\frac{1}{z}},$$

we have that

$$f(0) = 1$$

with the division by zero calculus.

5. For the function

$$f(x) = x \sin\frac{1}{x}$$

we have $f(0) = 0$ with the mean value method, however, by the division by zero calculus, we have $f(0) = 1$.

6. For the function

$$f(x) = \frac{e^{(1/x)}}{e^{(1/x)} - e^{(-1/x)}}$$

we have

$$\lim_{x \to +0} f(x) = 1$$

and

$$\lim_{x \to -0} f(x) = 0.$$

However, by the division by zero calculus and by the mean value method, we have $f(0) = 1/2$.

4 From the viewpoint of the Sato hyperfunction theory

The form in the representation of real variable function may be looked as in the Sato hyperfunction, indeed as in the definition of the finite part of Hadamard (page 11, [3]). Therefore, we would like to consider the elementary relation of real valued functions, the division by zero and the Sato hyperfunction theory.

For an analytic function $f(z)$ around $z = x$, we will consider the Cauchy integral representation. However, following the basic idea of Sato, we will consider it as in the following way. Let γ_- be an analytic curve in the lower complex plane whose start point is $x-E$ and ends at the point $x+E$ ($E > 0$). Let γ_+ be an analytic curve in the upper complex plane whose start point is $x - E$ and ends at the point $x + E$. Here, we are considering that both curves γ_- and γ_+ are near to the real line and they tends to the real line in some sense. Of course they are in some neighborhood of analytic domain of the function $f(z)$. Then, the Cauchy integral representation is as follows:

$$f(x) = \int_{\gamma_+} -\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - x}d\zeta - \int_{\gamma_-} -\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - x}d\zeta$$

$$= \int_{-E}^{E} \left(-\frac{1}{2\pi i}\right)\left(\frac{1}{\xi - x + i0} - \frac{1}{\xi - x - i0}\right) f(\xi)d\xi.$$

By setting

$$\delta(\xi - x) = \left(-\frac{1}{2\pi i}\right)\left(\frac{1}{\xi - x + i0} - \frac{1}{\xi - x - i0}\right),$$

we have the identity

$$f(x) = \int_{-E}^{E} \delta(\xi - x)f(\xi)d\xi.$$

This idea is the basic idea of the Sato hyperfunction theory. For an analytic function $f(z)$ we can consider one representation of the division by zero calculus, because we used the Cauchy integral representation. The serious problem is on the above identities and their interpretation. The curves γ_- and γ_+ approach to the real line and its result may be represented as in the above. In particular, the convergence is an essential problem in the Sato hyperfunction theory. Indeed, for some general functions, we can consider such limits. See the elementary facts, for example, [3, 4].

In order to see the notation, we will write

$$\delta(x) = \left(-\frac{1}{2\pi i}\right)\left(\frac{1}{x+i0} - \frac{1}{x-i0}\right)$$

$$= \left[-\frac{1}{2\pi i}\frac{1}{z}\right]$$

and the function

$$-\frac{1}{2\pi i}\frac{1}{z}$$

is a defining function of the generalized function $\delta(x)$.

In general, for a meromorphic function $F(z)$ around a part of the real line, we define a finite part of the function $F(x)$ in the sense of Hadamard by

$$f.p.F(x) = \frac{1}{2}(F(x+i0) + F(x-i0)).$$

Then, we see that

$$f.p.\frac{1}{x} = \frac{1}{2}\left(\frac{1}{x+i0} + \frac{1}{x-i0}\right).$$

In this case, we found interestingly that

$$\frac{1}{x} = \lim_{\varepsilon\to 0}\frac{1}{2}\left(\frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon}\right).$$

Since this representation shows the zero property of the function $y = 1/x$ at the origin, we see that the very interesting property of singular points of analytic functions and the theory of the Sato hyperfunction theory.

In particular, note the important property that for a continuous function $f(x)$ except for an isolated point, its defining analytic function $F(z)$ may be represented by the Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_R f(\xi)\frac{1}{\xi - z}d\xi,$$

if there exists the integral.

In general, apparently, we can not connect with the finite part of Hadamard and the mean value. In order to see some details, we will consider the prototype case of the function $1/x^2$.

We have the identity

$$\frac{1}{x^2}|_{x=0} = \int_{\gamma_+} -\frac{1}{2\pi i}\frac{1}{\zeta^3}d\zeta - \int_{\gamma_-} -\frac{1}{2\pi i}\frac{1}{\zeta^3}d\zeta$$
$$= \frac{1}{4\pi i}\left[\frac{1}{\zeta^2}\right]_{\gamma_+} - \frac{1}{4\pi i}\left[\frac{1}{\zeta^2}\right]_{\gamma_-}.$$

This can be considered as a mean value of the function $1/x^2$ around the origin. However, we can not connect with the mean value as in the real variable function and also, a finite part of Hadamard.

As a typical example in A. Kaneko ([3], page 11) in the theory of hyperfunction theory we see that for non-integers λ, we have

$$x_+^\lambda = \left[\frac{-(-z)^\lambda}{2i\sin\pi\lambda}\right] = \frac{1}{2i\sin\pi\lambda}\{(-x+i0)^\lambda - (-x-i0)^\lambda\}$$

where the left hand side is a Sato hyperfunction and the middle term is the representative analytic function whose meaning is given by the last term. For an integer n, Kaneko derived that

$$x_+^n = \left[-\frac{z^n}{2\pi i}\log(-z)\right],$$

where log is a principal value on $\{-\pi < \arg z < +\pi\}$. Kaneko stated there that by taking a finite part (C_0) of the Laurent expansion, the formula is derived. Indeed, we have the expansion, around an integer n,

$$\frac{-(-z)^\lambda}{2i\sin\pi\lambda}$$
$$= \frac{-z^n}{2\pi i}\frac{1}{\lambda - n} - \frac{z^n}{2\pi i}\log(-z)$$
$$-\left(\frac{\log^2(-z)z^n}{2\pi i\cdot 2!} + \frac{\pi z^n}{2i\cdot 3!}\right)(\lambda - n) + ...$$

([3], page 220).

By the division by zero calculus, however, we can derive this result from the Laurent expansion, immediately.

Meanwhile, M. Morimoto derived this result by using the Gamma function with the elementary means in [11], pages 60-62.

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