

ON THE JOURNEY TO THE PIERCE-BIRKHOFF CONJECTURE VIA EXPANSION INEQUALITIES

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ABSTRACT. This paper initializes the study of the Pierce-Birkhoff conjecture. We start by introducing the notion of the area and volume induced by a multivariate expansion and develop some inequalities for our next studies. In particular we obtain the inequality

$$\sum_{\substack{i,j \in [1,n] \\ a_{i\sigma(s)} < a_{j\sigma(s)} \\ s \in [1,l] \\ v \neq i,j \\ v \in [1,n]}} \left\| \vec{a}_i \diamond \vec{a}_j \diamond \cdots \diamond \vec{a}_v \right\| \sum_{k=1}^n \int_{a_{i\sigma(l)}}^{a_{j\sigma(l)}} \int_{a_{i\sigma(l-1)}}^{a_{j\sigma(l-1)}} \cdots \int_{a_{i\sigma(1)}}^{a_{j\sigma(1)}} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \\ \leq 2C \times \binom{n}{2} \times \sqrt{n} \times \\ \times \int_{a_{i\sigma(l)}}^{a_{j\sigma(l)}} \int_{a_{i\sigma(l-1)}}^{a_{j\sigma(l-1)}} \cdots \int_{a_{i\sigma(1)}}^{a_{j\sigma(1)}} \sqrt{\left(\sum_{k=1}^n (\max(g_k))^2 \right)} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}$$

for some constant $C > 0$, where $\sigma : \{1, 2, \dots, l\} \rightarrow \{1, 2, \dots, l\}$ is a permutation for $g_k \in \mathbb{R}[x_1, x_2, \dots, x_l]$ and $\vec{a}_i \diamond \vec{a}_j \diamond \cdots \diamond \vec{a}_k \diamond \vec{a}_v$ is the cross product of any of the $n - 1$ fixed spots in \mathbb{R}^l including the spots \vec{a}_i, \vec{a}_j .

1. Introduction

By an expansion in the direction $[x_i]$ we mean the map

Definition 1.1. Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^\infty$ be a collection of tuples of polynomials $f_k \in \mathbb{C}[x_1, x_2, \dots, x_n]$. Then by an expansion on $\mathcal{S} \in \mathcal{F} := \{\mathcal{S}_i\}_{i=1}^\infty$ in the direction x_i for $1 \leq i \leq n$, we mean the composite map

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} : \mathcal{F} \rightarrow \mathcal{F}$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

with

$$\nabla_{[x_i]} = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \dots, \frac{\partial f_n}{\partial x_i} \right).$$

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The value of the l th expansion at a given value a of x_i is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[\sigma(x_i)]}^l(\mathcal{S}) \in \mathbb{C}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n].$$

Similarly by an expansion in the mixed direction $\otimes_{i=1}^l [x_{\sigma(i)}]$ we mean

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=2}^l [x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S})$$

for some permutation $\sigma : \{1, 2, \dots, l\} \longrightarrow \{1, 2, \dots, l\}$. The value of this expansion on a given value a_i of $x_{\sigma(i)}$ for all $i \in [\sigma(1), \sigma(l)]$ is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(a_i)(\mathcal{S}) \in \mathbb{C}.$$

In the first part of our extension program we show that an expansion is a commutative map and established among other things the notion of the **Doppler effect and destabilization** of an expansion. This notion is an extension of the single variable expansivity theory developed by the author [1]. The current paper develops some inequalities to study the Pierce-Birkhoff conjecture by exploiting the notion of the **area** and **volume** of a multivariate expansion.

2. The area of an expansion

In this section we introduce the notion of the **area** of a multivariate expansion. This is an extension of the area of an expansion under the single variable theory. We exploit some applications in this context.

Definition 2.1. Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^\infty$ be a collection of tuples of the ring $\mathbb{C}[x_1, x_2, \dots, x_l]$. Then by the **area** induced by the expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S}),$$

denoted $\mathcal{A}_{\vec{a}, \vec{b}}(\mathcal{S})$, from the point \vec{a} to the point \vec{b} we mean the quantity

$$\begin{aligned} \mathcal{A}_{\vec{a}, \vec{b}}(\mathcal{S}) &:= \overrightarrow{O\Delta_{\vec{a}}^{\vec{b}} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S}) \right]} \cdot \overrightarrow{OS_e} \\ &= \sum_{j=1}^n \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \end{aligned}$$

where

$$\begin{aligned} \Delta_{\vec{a}}^{\vec{b}} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S}) \right] &= \left(\int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_1 dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}, \dots, \right. \\ &\quad \left. \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_n dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right) \end{aligned}$$

and $\mathcal{S}_e = (1, 1, \dots, 1)$ is the unit tuple and $g_i \in \mathbb{C}[x_1, x_2, \dots, x_l]$ for $1 \leq i \leq n$.

Next we examine some properties of the area induced by an expansion between any two points in space.

Proposition 2.2. *The area of an expansion between points in space is linear.*

Proof. Let $\omega, \mu \in \mathbb{R}$ and $\mathcal{S}_1, \mathcal{S}_2$ be a tuple of polynomials whose entry belongs to the ring $\mathbb{C}[x_1, x_2, \dots, x_n]$ with $\mathcal{S}_1 = (f_1, f_2, \dots, f_n)$ and $\mathcal{S}_2 = (g_1, g_2, \dots, g_n)$. Let $\vec{a}, \vec{b} \in \mathbb{R}^l$, then we can write

$$\begin{aligned}
 \mathcal{A}_{\vec{a}, \vec{b}}(\omega \mathcal{S}_1 + \mu \mathcal{S}_2) &= \overrightarrow{O\Delta_{\vec{a}}^{\vec{b}} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}} \right] (\omega \mathcal{S}_1 + \mu \mathcal{S}_2)} \cdot \overrightarrow{OS_e} \\
 &= \sum_{j=1}^n \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} (\omega f_j + \mu g_j) dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \\
 &= \omega \sum_{j=1}^n \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} f_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \\
 &\quad + \mu \sum_{j=1}^n \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \\
 &= \omega \mathcal{A}_{\vec{a}, \vec{b}}(\mathcal{S}_1) + \mu \mathcal{A}_{\vec{a}, \vec{b}}(\mathcal{S}_2).
 \end{aligned}$$

This proves that the area of an expansion between points in space is a linear map. \square

Remark 2.3. Next we examine some immediate applications of the notion of the area of a multivariate expansion of tuples of polynomials with entries belonging to the ring $\mathbb{R}[x_1, x_2, \dots, x_l]$. The offshoot of this notion is an inequality controlling the 2-norm of any l -th fold integration by the l -th fold integration of the 2-norm of the corresponding integrand.

Theorem 2.4. *Let $g_j \in \mathbb{R}[x_1, x_2, \dots, x_l]$ for $1 \leq j \leq n$. Then there exists some constant $C := C(l) > 0$ such that the inequality holds*

$$\begin{aligned}
 &\int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \sqrt{\left(\sum_{j=1}^n g_j^2 \right)} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \\
 &\geq C(l) \sqrt{\sum_{j=1}^n \left(\int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right)^2}.
 \end{aligned}$$

for $b_i > a_i$ for each $1 \leq i \leq l$.

Proof. Using the notion of the area of an expansion we obtain an equivalent expression

$$\begin{aligned}
 \left| \mathcal{A}_{\vec{a}, \vec{b}}(\mathcal{S}) \right| &:= \left| \overrightarrow{O\Delta_{\vec{a}}^{\vec{b}} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}} \right] (\mathcal{S})} \cdot \overrightarrow{OS_e} \right| \\
 &= \sqrt{n} |\cos(\alpha)| \sqrt{\sum_{j=1}^n \left(\int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right)^2}
 \end{aligned}$$

On the other hand, we observe that by changing the order of summation we can write, using the notion of the area of an expansion, the following

$$\begin{aligned}
|\mathcal{A}_{\vec{a}, \vec{b}}(\mathcal{S})| &:= \left| \overrightarrow{O\Delta_{\vec{a}}^{\vec{b}}} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S}) \right] \cdot \overrightarrow{OS_e} \right| \\
&= \left| \sum_{j=1}^n \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right| \\
&= \left| \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \sum_{j=1}^n g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right| \\
&= \left| \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \overrightarrow{O(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S})} \cdot \overrightarrow{OS_e} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right| \\
&\leq \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \|\overrightarrow{O(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S})}\| \|\overrightarrow{OS_e}\| dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \\
&= \sqrt{n} \int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} \sqrt{\left(\sum_{j=1}^n g_j^2 \right)} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}
\end{aligned}$$

and the inequality follows by combining this upper bound with the area of expansion. \square

Remark 2.5. It is important to notice that Theorem 2.4 is a generalization of the usual integral inequality for bounded functions h ; in particular, polynomials. That is to say, if we take $\left(\sum_{j=1}^n g_j^2 \right) = h^2$ then

$$\sqrt{\left(\sum_{j=1}^n g_j^2 \right)} = h$$

and the result is the usual integral inequality with h now on the finite supports $[a_1, b_1], [a_2, b_2], \dots, [a_l, b_l]$. Next we obtain an inequality relating the minimum gap between the limits of integration to their corresponding l -th fold integration of the function.

Corollary 2.6. Let $g_j \in \mathbb{R}[x_1, x_2, \dots, x_l]$ for $1 \leq j \leq n$ such that $g_j \neq 1$ with

$$\sum_{j=1}^n g_j^2 \leq 1$$

on $\cup_{i=1}^l [a_i, b_i]$. Then there exist some constant $C := C(l) > 0$ such that the inequality holds

$$\frac{1}{C(l)} \prod_{i=1}^l |b_{\sigma(i)} - a_{\sigma(i)}| \geq \sqrt{\sum_{j=1}^n \left(\int_{a_{\sigma(l)}}^{b_{\sigma(l)}} \int_{a_{\sigma(l-1)}}^{b_{\sigma(l-1)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} g_j dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right)^2}.$$

There is no obvious reason why an analogous inequality should not hold for functions continuous on the supports $\cup_{i=1}^l [a_i, b_i]$, by replacing the space of real multivariate polynomials $\mathbb{R}[x_1, x_2, \dots, x_n]$ with the general space of real multivariate functions $\mathbb{F}[x_1, x_2, \dots, x_l]$ continuous on the interval $\cup_{i=1}^l [a_i, b_i]$.

3. The volume of an expansion

In this section we introduce the notion of the **volume** induced by n points in space. We launch the following formal language and exploit some applications.

Definition 3.1. Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials with entries $f_k \in \mathbb{C}[x_1, x_2, \dots, x_n]$. Then by the **volume** induced by the expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S}),$$

denoted $\mathcal{V}_{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n}(\mathcal{S})$, at the linearly independent **spots** $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ such that $\vec{a}_i \neq \vec{0}$ for $1 \leq i \leq n$, we mean the sum

$$\begin{aligned} \mathcal{V}_{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n}(\mathcal{S}) := & \sum_{\substack{s, t \in [1, n] \\ s \leq t \\ k \neq s, t \\ k \in [1, n]}} \overrightarrow{O\Delta_{\vec{a}_s}^{\vec{a}_t}} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S}) \right] \cdot \overrightarrow{OS_e} \\ & \times \left\| \vec{a}_s \diamond \vec{a}_t \diamond \cdots \diamond \vec{a}_k \diamond \vec{a}_v \right\| \end{aligned}$$

where $\vec{a}_s \diamond \vec{a}_t \diamond \cdots \diamond \vec{a}_k \diamond \vec{a}_v$ is the cross product of any of $n - 1$ spots including the spots \vec{a}_s, \vec{a}_t and $\sigma : [1, l] \rightarrow [1, l]$ is a permutation.

Proposition 3.2. *The volume of an expansion between spots is a linear operator.*

Proof. This is an easy consequence of Proposition 2.2. \square

Remark 3.3. As an immediate application we deduce an average version of the inequality in Theorem 2.4.

Theorem 3.4. *Let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ be any linearly independent vectors in the space \mathbb{R}^l and $\sigma : \{1, 2, \dots, l\} \rightarrow \{1, 2, \dots, l\}$ be any permutation. If for each $g_k \in \mathbb{R}[x_1, x_2, \dots, x_n]$ with $1 \leq k \leq n$ and*

$$\int_{a_{i_{\sigma(l)}}}^{a_{j_{\sigma(l)}}} \int_{a_{i_{\sigma(l-1)}}}^{a_{j_{\sigma(l-1)}}} \cdots \int_{a_{i_{\sigma(1)}}}^{a_{j_{\sigma(1)}}} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} > 0$$

then there exist a constant $C > 0$ such that

$$\begin{aligned} & \sum_{\substack{i,j \in [1,n] \\ a_{i\sigma(s)} < a_{j\sigma(s)} \\ s \in [1,l] \\ v \neq i,j \\ v \in [1,n]}} \left\| \vec{a}_i \diamond \vec{a}_j \diamond \cdots \diamond \vec{a}_v \right\| \left\| \sum_{k=1}^n \int_{a_{i\sigma(l)}}^{a_{j\sigma(l)}} \int_{a_{i\sigma(l-1)}}^{a_{j\sigma(l-1)}} \cdots \int_{a_{i\sigma(1)}}^{a_{j\sigma(1)}} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right\| \\ & \leq 2C \times \binom{n}{2} \times \sqrt{n} \times \\ & \times \int_{a_{i\sigma(l)}}^{a_{j\sigma(l)}} \int_{a_{i\sigma(l-1)}}^{a_{j\sigma(l-1)}} \cdots \int_{a_{i\sigma(1)}}^{a_{j\sigma(1)}} \sqrt{\left(\sum_{k=1}^n (\max(g_k))^2 \right)} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \end{aligned}$$

Proof. First we let

$$\int_{a_{i\sigma(l)}}^{a_{j\sigma(l)}} \int_{a_{i\sigma(l-1)}}^{a_{j\sigma(l-1)}} \cdots \int_{a_{i\sigma(1)}}^{a_{j\sigma(1)}} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)}$$

be the r^{th} entry of the vector $\overrightarrow{O\Delta_{\vec{a}_s}^{\vec{a}_t} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}}(\mathcal{S}) \right]} \cdot \overrightarrow{OS_e}$ for $1 \leq r \leq n$. Then unpacking the definition of a volume induced by an expansion at n spots yields the left-hand side expression. On the other hand we observe that each outer sum is determined by their spots and in each of these we maintain two distinct spots so that we have $\binom{n}{2}$ for the number of possible distinct sums. Under the main assumption the maximum function taken is then absorbed by the l -fold integral and the result follows via the interpolation

$$\begin{aligned} \left| \overrightarrow{O\Delta_{\vec{a}_i}^{\vec{a}_j} \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}}(\mathcal{S}) \right]} \cdot \overrightarrow{OS_e} \right| &= \left| \sum_{k=1}^n \int_{a_{i\sigma(l)}}^{a_{j\sigma(l)}} \int_{a_{i\sigma(l-1)}}^{a_{j\sigma(l-1)}} \cdots \int_{a_{i\sigma(1)}}^{a_{j\sigma(1)}} g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right| \\ &= \left| \int_{a_{i\sigma(l)}}^{a_{j\sigma(l)}} \int_{a_{i\sigma(l-1)}}^{a_{j\sigma(l-1)}} \cdots \int_{a_{i\sigma(1)}}^{a_{j\sigma(1)}} \sum_{k=1}^n g_k dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right| \\ &= \left| \int_{a_{i\sigma(l)}}^{a_{j\sigma(l)}} \int_{a_{i\sigma(l-1)}}^{a_{j\sigma(l-1)}} \cdots \int_{a_{i\sigma(1)}}^{a_{j\sigma(1)}} \overrightarrow{O \left[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}}(\mathcal{S}) \right]} \right. \\ & \quad \left. \cdot \overrightarrow{OS_e} dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(l)} \right|. \end{aligned}$$

□

REFERENCES

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