Division by Zero Calculus and Euclidean Geometry – Revolution in Euclidean Geometry –

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Abstract: In this paper, we will discuss Euclidean geometry from the viewpoint of the division by zero calculus with typical examples. Where is the point at infinity? It seems that the point is vague in Euclidean geometry in a sense. Certainly we can see the point at infinity with the classical Riemann sphere. However, by the division by zero and division by zero calculus, we found that the Riemann sphere is not suitable, but Däumler's horn torus model is suitable that shows the coincidence of the zero point and the point at infinity. Therefore, Euclidean geometry is extended globally to the point at infinity. This will give a great revolution of Euclidean geometry. The impacts are wide and therefore, we will show their essence with several typical examples.

Recall that David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Key Words: Division by zero, division by zero calculus, Euclidean geometry, Sangaku, point at infinity, horn torus, Riemann sphere, isolated singular point, analytic function, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, [(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$.

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1 Introduction

For the long history of division by zero, see [1, 23]. S. K. Sen and R. P. Agarwal [36] quite recently referred to our paper [4] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book in the following way:

"Thou shalt not divide by zero" remains valid eternally.

However, we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense ([26]).

The common sense on the division by zero with the long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on differential coefficients we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In a new and definite sense, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$. In particular, we introduced several fundamental concepts in calculus, Euclidean geometry, analytic geometry, complex analysis and differential equations. We saw new properties on the Laurent expansion, singularity, derivative, extension of solutions of differential equations beyond analytical and isolated singularities, and reduction problems of differential equations. On Euclidean geometry and analytic geometry, we found new fields by the concept of the division by zero. We gave many concrete properties in mathematical sciences from the viewpoint of the division by zero. We will know that the division by zero is our elementary and fundamental mathematics. See ([30]) for the fundamental results.

In this paper, we will discuss Euclidean geometry from the viewpoint of the division by zero calculus with typical examples. Where is the point at infinity? It seems that the point is vague in Euclidean geometry in a sense. Certainly we can see the point at infinity with the classical Riemann sphere. However, by the division by zero and division by zero calculus, we found that the Riemann sphere is not suitable, but Däumler's horn torus model is suitable that shows the coincidence of the zero point and the point at infinity. Therefore, Euclidean geometry is extended globally to the point at infinity. This will give a great revolution of Euclidean geometry. The impacts are wide and therefore, we will show their essence with several typical examples.

2 Division by zero calculus – definition

We would like to consider some values for isolated singular points for analytic functions. The very typical problem is to consider some value of the fundamental function $W = 1/z$ at the origin. We found that its value is zero. When the result is written as

$$\frac{1}{0} = 0,$$

it will have a serious sense, because it looks like the division by zero that has a mysteriously long history ([1, 3, 23, 36, 37, 38]). However, note that $0 \times 0 \neq 1$. We showed that our result gave great impacts widely with over 1100 items. For example, look the papers cited in the reference.

The essence is stated as follows:

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{2.1}$$

we will define

$$f(a) = C_0. \tag{2.2}$$

For the correspondence (2.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (2.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

With this assumption, we can obtain many new results and new concepts.

As we refer to, we found a beautiful and important circle by this division by zero calculus ([16] and [21]).

However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problem. – In this viewpoint, **the division by zero calculus may be considered as an axiom.**

3 Examples

3.1 Broken phenomena of figures by area and volume:

The strong discontinuity of the division by zero around the point at infinity will appear as the destruction of various figures. These phenomena may be looked in many situations as the universal one. However, the simplest cases are the disc and sphere (ball) with their radius $1/\kappa$. When $\kappa \to +0$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $\kappa = 0$, they are zero, because they become the half-plane and half-space, respectively. These facts were also looked by analytic geometry. However, the results are clear already from the definition of the division by zero.

For a function

$$S(x, y) = a(x^2 + y^2) + 2gx + 2fy + c, \tag{3.1}$$

the radius R of the circle $S(x, y) = 0$ is given by

$$R = \sqrt{\frac{g^2 + f^2 - ac}{a^2}}.$$

If $a = 0$, then the area πR^2 of the disc is zero, by the division by zero calculus. In this case, the circle is a line (degenerated).

The center of the circle (3.1) is given by

$$\left(-\frac{g}{a}, -\frac{f}{a}\right).$$

Therefore, the center of a general line

$$2gx + 2fy + c = 0$$

may be considered as the origin $(0, 0)$, by the division by zero.

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$. We thus see that

$$\lim_{z \to \infty} z = \infty, \tag{3.2}$$

however,

$$[z]_{z=\infty} = 0, \tag{3.3}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (3.2) and (3.3) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x \to +\infty} x = +\infty, \quad \lim_{x \to -\infty} x = -\infty,$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0.$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [8], we gave many examples for this property.

In particular, in $z \to \infty$ in (3.2), ∞ represents the topological point on the Riemann sphere, meanwhile ∞ in the left hand side in (3.2) represents the limit by means of the ϵ - δ logic. That is, for any large number M, when we take for some large number N, we have, for $|z| > N$, $|z| > M$.

3.2 Parallel lines:

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, \quad k = 1, 2.$$

The common point is given by, if $a_1 b_2 - a_2 b_1 \neq 0$; that is, the lines are not parallel

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1}\right).$$

By the division by zero, we can understand that if $a_1 b_2 - a_2 b_1 = 0$, then the common point is always given by

$$(0, 0),$$

even two lines are the same. This fact shows that our new image for the Euclidean space is right.

In particular, note that the concept of parallel lines is very important in the Euclidean plane and non-Euclidean geometry. With our sense, there are no parallel lines and all lines pass the origin. This will be our world on the Euclidean plane. However, this property is not geometrical and has a strong discontinuity. This surprising property may be looked also clearly by the polar representation of a line.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)},$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle of OP ($P = (r, \theta)$ on the line) from the positive x axis. Then, if $\theta - \alpha = \pi/2$; that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

This will mean simply that any line arrives at the point at infinity and the point is represented by zero and so, for the line we can add the point at the origin. In this sense, we can add the origin to any line as the point of the compactification of the line. This surprising new property may be looked in our mathematics globally.

The distance d from the origin to the line determined by the two planes

$$\Pi_k : a_k x + b_k y + c_k z = 1, \quad k = 1, 2,$$

is given by

$$d = \sqrt{\frac{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}.$$

If the two lines are coincident, then, of course, $d = 0$. However, if two planes are parallel, by the division by zero, $d = 0$. This will mean that any plane contains the origin as in a line.

3.3 An example by Okumura ([14]):

On the sector

$$\Delta_\alpha = \left\{ |\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2} \right\},$$

we shall change the angle and we consider a circle $C_a, a > 0$ with its fixed radius a inscribed in the sectors (see Figure 1). We see that when the circle tends to $+\infty$, the angles α tend to zero (see Figure 2). How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with its center at the origin 0 (see Figure 3). This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}.$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive y axis is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a.

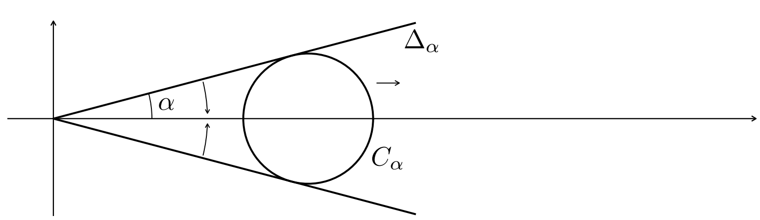

Figure 1.

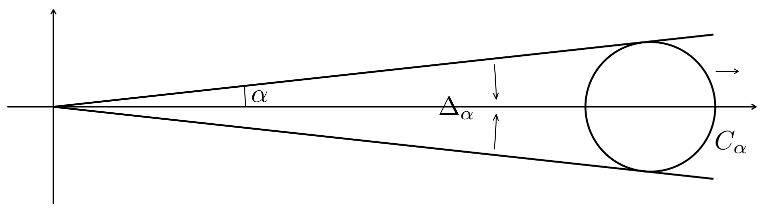

Figure 2: α approaching to 0.

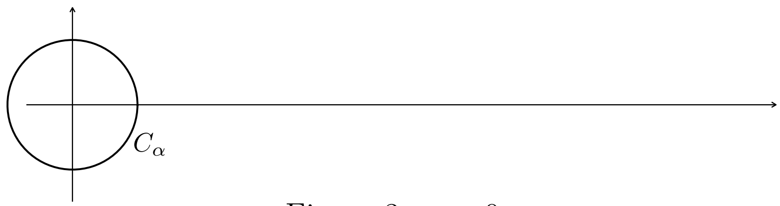

Figure 3: $\alpha = 0$.

Here note the expansion:

$$k = \frac{a}{\sin\alpha} = \frac{a}{\alpha} + \frac{a\alpha}{6} + \frac{7a\alpha^3}{369} + \frac{31a\alpha^5}{15120} + \cdots.$$

See [32] for some very interesting property of the circles on the horn torus model.

3.4 Okumura's results:

First, we recall Okumura's results from ([20]). Let C be a point on the segment AB such that $|BC| = 2a$ and $|CA| = 2b$. We consider the three circles α, β and γ with diameters CB, AC and AB, respectively. We use a rectangular coordinates system with origin C such that the point B has coordinates $(2a, 0)$. We call the line AB the baseline. Let $c = a + b$ and $d = \sqrt{ab}/c$. Then, we have:

A circle γ_z touches the circles α and β if and only if it has radius r_z^γ and center of coordinates (x_z^γ, y_z^γ) given by

$$r_z^\gamma = |q_z^\gamma| \quad \text{and} \quad (x_z^\gamma, y_z^\gamma) = \left(\frac{b-a}{c} q_z^\gamma, 2zq_z^\gamma\right), \ \text{where } q_z^\gamma = \frac{abc}{c^2z^2 - ab}$$

for a real number $z \neq \pm d$.

The circle γ_z touches α and β internally (resp. externally) if and only if $|z| < d$ (resp. $|z| > d$). The external common tangents of α and β have following equations:

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0,$$

which are denoted by $\gamma_{\pm d}$.

The distance between the center of the circle γ_z and the baseline equals $2|z|r_z^\gamma$.

The ratio of the distance from the center of γ_z to the perpendicular to the baseline at C to the radius of γ_z is constant and equals to $|a-b|/c$ for $z \neq \pm d$.

Let $g_z(x,y) = (x - x_z^\gamma)^2 + (y - y_z^\gamma)^2 - (r_z^\gamma)^2$. Then $g_z(x,y) = 0$ is an equation of the circle γ_z for $z \neq \pm d$. Let

$$g_z(x,y) = \cdots + C_{-2}(z-d)^{-2} + C_{-1}(z-d)^{-1} + C_0 + C_1(z-d) + \cdots$$

be the Laurent expansion of $g_z(x,y)$ around $z = d$, then we have

$$\cdots = C_{-4} = C_{-3} = C_{-2} = 0,$$

$$C_{-1} = d((a-b)x - 2\sqrt{ab}y + 2ab),$$

$$C_0 = \left(x - \frac{a-b}{4}\right)^2 + \left(y - \frac{\sqrt{ab}}{2}\right)^2 - \left(\frac{\sqrt{a^2+18ab+b^2}}{4}\right)^2,$$

$$C_n = -\frac{1}{2}\left(\frac{-1}{2d}\right)^n((a-b)x + 2\sqrt{ab}y + 2ab), \ for \ n = 1,2,3,\cdots.$$

Therefore $C_{-1} = 0$ gives an equation of the line γ_d. Also $C_n = 0$ gives an equation of the line γ_{-d} for $n = 1,2,3,\cdots$.

Let ε be the circle given by the equation $C_0 = 0$. Then, it has the following beautiful properties that were given in [13] (see Figure 4):

(i) The points, where γ_d touches α and β, lie on ε.
(ii) The radical center of the three circles α, β and ε has coordinates $(0, -\sqrt{ab})$, and lies on the line γ_{-d}.
(iii) The radical axis of the circles ε and γ passes though the points of coordinates $(0, 3\sqrt{ab})$ and $(2ab/(b-a), 0)$, where the latter coincides with the point of intersection of γ_d and γ_{-d}.

The y-axis meets γ and $\gamma_{\pm d}$ in the points of coordinates $(0, \pm 2\sqrt{ab})$ and $(0, \pm\sqrt{ab})$, respectively. Hence the six points, where the y-axis meets γ, $\gamma_{\pm d}$, the baseline, the radical axis of γ and ε, are evenly spaced. Reflecting the figure in the baseline, we also get similar results for the Laurent expansion of $g_z(x, y)$ around $z = -d$.

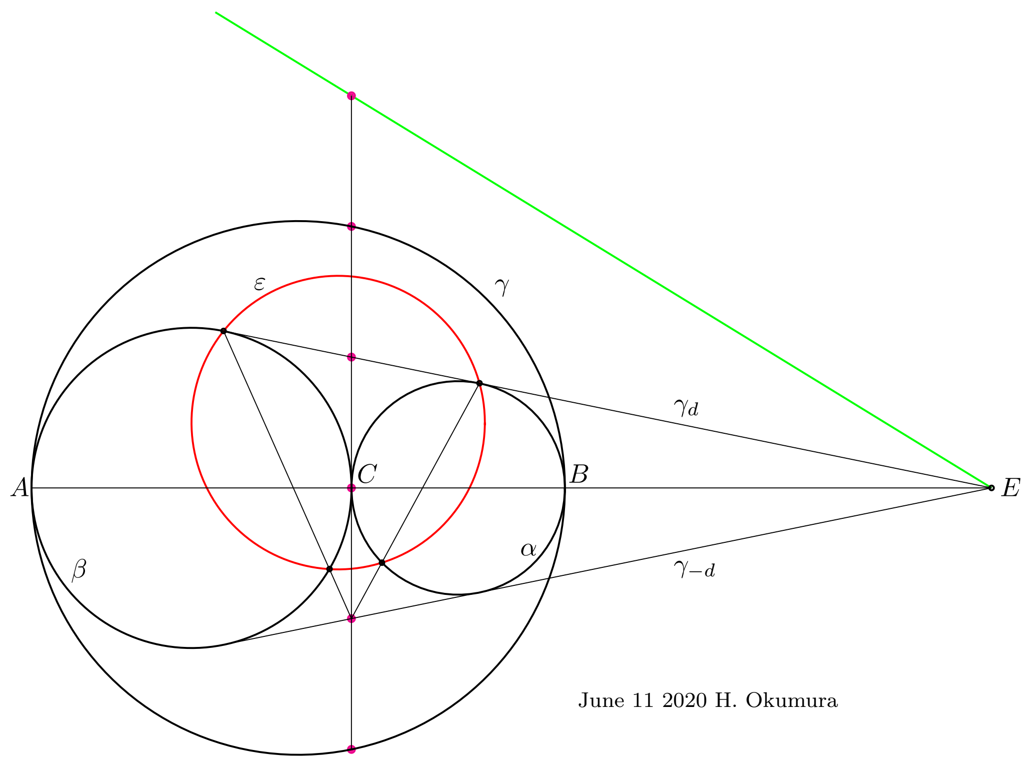

Figure 4.

For more beautiful properties and Figures, see [21].

Look 4 points on the red circle and 6 points on the y axis; they are very beautiful. The circle was discovered by the division by zero calculus.

From the division by zero calculus:

In the equation $g_z(x, y) = 0$, when we apply the division by zero calculus at $z = d$ we obtain the equation of ε. Meanwhile, in the equation $(z - d)g_z(x, y) = 0$, when we apply the division by zero calculus at $z = d$ we obtain the equation of γ_d ([16]).

Meanwhile, in the equation $g_z(x, y) = 0$ by letting $z \to \pm\infty$, we obtain the point of C.

Here, in particular note the very interesting fact that around $\pm d$ the equation $g_z(x, y) = 0$ represents the circle touching both circles α, β even near $\pm d$, however, the function $g_z(x, y)$ has poles of order one at $\pm d$, it looks like ∞ that is a contradiction with $g_z(x, y) = 0$. This fact will show some naturality of the division by zero calculus at $\pm d$.

For many differential equations with analytic and isolated singularities, this property is similar and we have interesting and general problems.

Of course, we can derive many and many examples. See the papers in the reference.

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