11 Cantor Diagonal Argument

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Abstract.-This chapter applies Cantor's diagonal argument to a table of rational numbers proving the existence of rational antidiagonals.

Keywords: Cantor's diagonal argument, cardinal of the set of real numbers, cardinal of the set of rational numbers.

Introduction

P205 This chapter proves a result on the decimal expansion of the rational numbers in the rational open interval (0, 1), which is subsequently used to discuss on a reordering of the rows of a table T that is assumed to contain all rational numbers within (0, 1). A reordering such that the diagonal of the reordered table T could be a rational number from which different rational antidiagonals (elements of (0, 1) that cannot be in T) could be defined. If that were the case, and for the same reason as in Cantor's diagonal argument, the rational open interval (0, 1) would be non-denumerable, and we would have a contradiction in set theory, because Cantor also proved the set of the rational numbers is denumerable.

Theorem of the nth Decimal

P206 Let $\mathbb{Q}_{01}$ be the set of all rational numbers in the rational open interval $(0, 1)$ expressed in decimal notation and completed, in the cases of finitely many decimal digits, with a denumerable infinite number of 0's in the right side of their corresponding decimal expansions (numerical expressions that include all decimals digits of the number). According to the hypothesis of the actual infinity, those decimal expressions exist as complete totalities. Some infinite decimal expressions of rational numbers as, for instance, $0,30000000\ldots$ and $0,299999999\ldots$ are different when considered as strings of numerals (symbols), although they can also be considered as representing the same number. Here, we are not considering all strings of numerals that represent rational numbers in $\mathbb{Q}_{01}$ but all rational numbers in $\mathbb{Q}_{01}$ each with a unique decimal expression, the one just indicated. On the other hand, and for the reasons given in P217, the consideration of those double expressions has no consequences on the main argument of this chapter.

P207 Let d be any decimal digit, n any natural number, and q_0 any element of $\mathbb{Q}_{01}$ whose nth decimal digit is just d, for instance:

$$q_0 = 0{,}11\overset{(n-1)}{\ldots}1d000\ldots \tag{1}$$

From q_0 it is possible to define different sequences of different elements of $\mathbb{Q}_{01}$, all of them with the same nth decimal digit d. For example the sequence $\langle q_n \rangle$:

$$q_1 = 0{,}11\overset{(n-1)}{\ldots}1\mathbf{d}1000\ldots \tag{2}$$

$$q_2 = 0{,}11\overset{(n-1)}{\ldots}1\mathbf{d}11000\ldots \tag{3}$$

$$q_3 = 0{,}11\overset{(n-1)}{\ldots}1\mathbf{d}111000\ldots \tag{4}$$

$$q_4 = 0{,}11\overset{(n-1)}{\ldots}1\mathbf{d}1111000\ldots \tag{5}$$

$$q_5 = 0{,}11\overset{(n-1)}{\ldots}1\mathbf{d}11111000\ldots \tag{6}$$

$$\ldots$$

$$q_i = 0{,}11\overset{(n-1)}{\ldots}1\mathbf{d}111\overset{(i)}{\ldots}1000\ldots \tag{7}$$

$$\ldots$$

The bijection (one to one correspondence) f between the set $\mathbb{N}$ of the natural numbers and $\langle q_n \rangle$ defined by

$$\forall i \in \mathbb{N} : \ f(i) = q_i \tag{8}$$

proves the following:

Theorem P207, of the nth Decimal.-*For any given decimal digit and any given position in the decimal expansion of the elements of $\mathbb{Q}_{01}$, there exists a denumerable subset of $\mathbb{Q}_{01}$, each of whose different elements has the same given decimal digit in the same given position of its corresponding decimal expansion.*

A rational diagonal argument

P208 Let $\mathbb{Q}_{d_n}$ be the subset of $\mathbb{Q}_{01}$ each of whose elements has the same decimal digit d_n in the same nth position of its decimal expansion. According to the Theorem P207 of the nth Decimal, $\mathbb{Q}_{d_n}$ is denumerable. So, its superset $\mathbb{Q}_{01}$ will be infinite, either denumerable or non-denumerable. Let g be any injective function of $\mathbb{N}$ in $\mathbb{Q}_{01}$. This function makes it possible to define a table T whose successive rows r_1, r_2, $r_3\ldots$ are just the successive images $g(1)$, $g(2)$, $g(3)\ldots$ of the elements of $\mathbb{N}$ in $\mathbb{Q}_{01}$.

P209 Since the successive rows $\langle r_n \rangle$ of T are indexed by the whole set $\mathbb{N}$ of the natural numbers, T is ω-ordered (Theorem of the indexed collection). In addition, to assume the existence of the set of all finite natural numbers as a complete infinite totality, as Cantor did in 1883 [3, p. 103-104], means to assume the rows of T also exist as a complete infinite totality. According to this Cantor's assumption (hypothesis of the actual infinity subsumed into the Axiom of Infinity in modern set theories), *every* row r_n of T will be preceded by a finite number, $n-1$, of rows and succeeded by an infinite number, $\aleph_o$, of such rows. We will now examine a conflicting consequence of this case of ω-asymmetry.

P210 The diagonal $D = 0.d_{11}d_{22}d_{33}\dots$ of T is a real number within $(0,1)$ whose nth decimal digit d_{nn} is the nth decimal digit of the nth row r_n of T. As in Cantor's diagonal argument [2], it is possible to define another real number A, said antidiagonal, by replacing each of the infinitely many decimal digits of D with a different decimal digit. By construction A cannot be in T because it differs from each row r_i of T at least in its ith decimal digit. Since A is a real number within $(0,1)$, it will be either rational or irrational. If it were rational, and for the same reason as in Cantor's diagonal argument, g would not be a one to one correspondence

P211 A row r_i of T will be said n-modular if its nth decimal digit is $n(mod\ 10)$. This means that a row is, for instance, 2348-modular if its 2348th decimal digit is 8; or that it is 45390-modular if its 45390th decimal digit is 0. If a row r_n is n-modular (being n in n-modular the same number as n in r_n) it will be said *d-modular*. For instance, the rows:

$$
\begin{aligned}
r_1 &= 0, \mathbf{1007647464749943400034577774413}\dots \\
r_2 &= 0{,}2\mathbf{2}0004566777894300000000000000\dots \\
r_3 &= 0{,}00\mathbf{3}333333333333333333333333333\dots \\
r_7 &= 0{,}100100\mathbf{7}000111111114444444444433333\dots \\
r_{20} &= 0{,}1234567890123456789\mathbf{0}11111111111111\dots
\end{aligned}
$$

are all of them d-modular. It is clear that certain rational numbers as $0.\widehat{43}$ or $0.\overline{3353333333}$ cannot be d-modular, whatever be their corresponding rows in T. As will be seen in Chapter 30, these type of numbers pose new problems to the hypothesis of the actual infinity.

P212 Consider now the following permutation $\mathbf{D}$ of the rows $\langle r_n \rangle$ of T:

For each of the successive rows r_i of T:

- If r_i is d-modular then let it unchanged.
- If r_i is not d-modular then exchange it with any following i-modular row $r_{j,j>i}$, provided that at least one of the succeeding rows $r_{j,j>i}$ be i-modular. Otherwise let it unchanged.

	136900987838344...		**1**36900987838344...
	0**2**8282828282828...		0**2**8282828282828...
	13**3**389745600000...		13**3**389745600000...
D-exchange	032$\boxed{9}$67898354283...	→	655**4**89023467289...
	...		...
	655$\boxed{4}$89023467289...		032967898354283...
	345787352637839...		345787352637839...
	...		...

Figura 11.1 – The fourth row of T before being d-exchanged (Left); and after having been d-exchanged (right). Note that only the digits of the decimal expansions are represented, not including the initial 0 or the subsequent decimal separator.

The exchange of a non-d-modular row r_i with a *following i-modular row* will be referred to as *d-exchange* (see Figure 11.1). Thanks to the condition $j > i$ (in $r_{j,j>i}$), once a row r_i has been d-exchanged, it becomes d-modular and will remain d-modular and unaffected by the subsequent d-exchanges. On the other hand, the successive d-exchanges do not change the type of order of T but the rational numbers indexed by the same successive indexes. Or in other words, d-exchanges interchange the content of some couples of rows of T, but not its type of order.

P213 The permutation **D** could even be considered as a supertask [5]. Indeed, let $\langle t_n \rangle$ be an ω-ordered sequence of instants within a finite interval of time (t_a, t_b), being t_b the limit of the sequence. Assume that **D** is applied to each row r_i just at the precise instant t_i. The bijection $f(t_i) = r_i$ proves that at t_b the d-exchanges of the permutation **D** will have been applied to all rows of T.

P214 It can be proved that all rows of T become d-modular as a consequence of the permutation **D**. In effect, assume that a row r_n did not become d-modular as a consequence of the permutation **D**. This means that r_n is not d-modular and could not be d-exchanged with a n-modular row $r_{i,i>n}$. Now then, all n-modular rows have the same digit $n(mod\ 10)$ in the same nth position of its decimal expansion, and according to the Theorem P207 of the nth Decimal there are infinitely many rational numbers with the same digit in the same position of its decimal expansion, whatever be the digit and the position. Accordingly, since n is finite, the row r_n is preceded by a finite number k $(0 \leq k < n)$ of n-modular rows, and succeeded by an infinite number, $\aleph_o$, of n-modular rows. Any of these infinitely many n-modular rows succeeding r_n had to be d-exchanged with r_n. It is then impossible for r_n not to become d-modular as a consequence of **D**. Therefore, each and every row r_n of T becomes d-modular as a consequence of **D**.

P215 Let us remark the basic formal structure of the above argument P214 (a simple Modus Tollens). Consider the following two propositions p_1 and p_2 about the permutation **D**:

p_1: Not all rows of T becomes d-modular because of **D**.

p_2: At least one non-d-modular row r_n of T could not be d-exchanged.

It is quite clear that p_1 implies p_2: if not all rows of T becomes d-modular because of D, then at least one non-d-modular row r_n of T could not be d-exchanged. Now then, being all natural numbers finite, n is finite; and taking into account the Theorem P207 of the nth Decimal, there is a finite number, k $(0 \leq k < n)$, of n-modular rows preceding r_n and an infinite number, $\aleph_o$, of n-modular rows succeeding r_n, one of which had to be d-exchanged with r_n. In consequence proposition p_2 is false and so will be p_1. In symbols:

$$p1 \Rightarrow p2 \tag{9}$$

$$\neg p2 \tag{10}$$

$$\therefore \neg p1 \tag{11}$$

P216 The result proved in P214 is a formal consequence of both the Theorem P207 of th *nth* Decimal and the fact that *every* row r_n of T is always preceded by a finite number, k $(0 \leq k < n)$, of n-modular rows and succeeded by an infinite number, $\aleph_o$, of such n-modular rows (ω-asymmetry). Recall that this ω-asymmetry is an inevitable consequence of assuming, as Cantor did in 1883, the existence of the ω-ordered set $\mathbb{N}$ as a complete infinite totality, a hypothesis subsumed into the Axiom of Infinity.

P217 Let T_d be the table resulting from the permutation **D**. Since all of its rows are d-modular, its diagonal D will be the periodic rational number $0.\overline{1234567890}$. It is now immediate to define infinitely many rational antidiagonals from D. Indeed, let us consider periods of ten decimal digits none of which coincide in position with the ten decimal digits of the period

$\overline{1234567890}$ of the diagonal D. The number of those periods is 9^{10}. From any two of them, for instance, $q_1 = \overline{0123456789}$ and $q_2 = \overline{0321456789}$, it is possible to define different ω-ordered sequences of rational antidiagonals $\langle A_n \rangle$, for instance:

$$\forall n \in \mathbb{N}: \; A_n = 0.q_1 q_1 \overset{(n)}{\ldots} q_1 \overline{q_2} \tag{12}$$

whose elements cannot be in T_d for the same reason as in Cantor's diagonal argument. Being periodic rational numbers with a period of nine different digits, the antidiagonals $\langle A_n \rangle$ cannot be redundant decimal expressions of elements of T_d that are not in T_d just because of their redundancy with the decimal expressions that are in fact in T_d. Indeed, these redundant expressions are periodic expressions whose periods have always the same and unique digit: the digit 9. If, on the contrary, those redundant expressions were not considered redundant but representing each of them a different rational number, they would be in T_d, and the same argument above would prove they are different from the antidiagonals $\langle A_n \rangle$. In consequence, and since all those antidiagonals are rational numbers which are not in T_d, we must conclude that the injective function g between $\mathbb{N}$ and $\mathbb{Q}_{01}$ defining T, is not surjective, i.e. it is not a bijection.

P218 Since the injective function g defining T is *any injective function between* $\mathbb{N}$ *and* $\mathbb{Q}_{01}$ and it cannot be surjective, we must conclude it is impossible to define a bijection between $\mathbb{N}$ and $\mathbb{Q}_{01}$. Consequently, $\mathbb{Q}_{01}$ is non-denumerable. Although the above inference suffices to conclude that $\mathbb{Q}_{01}$ is non-denumerable, it could be (inappropriately) argued, as against Cantor's diagonal argument, that a new table T' could be defined so that $r'_1 = A$ and $r'_{i+1} = r_i, \; r_i \in T, \; \forall i \in \mathbb{N}$. The new table T' would be denumerable, but through the same diagonal argument, the same conclusion on the impossibility of a bijection between $\mathbb{N}$ and $\mathbb{Q}_{01}$ would be reached. And the same recursive argument could be applied to any table defined in terms of any other previous table and its corresponding antidiagonal, while the new table continue to be denumerable. A bijection between $\mathbb{N}$ and $\mathbb{Q}_{01}$ is impossible. So, $\mathbb{Q}_{01}$ is non-denumerable, and we have a contradiction in set theory because Cantor proved $\mathbb{Q}$ is denumerable [3, p. 123] [1].

P219 the Permutation **D** makes it possible to develop other arguments whose conclusions also point to the inconsistency of the hypothesis of the actual infinity. For instance, it is clear that certain elements of $\mathbb{Q}_{01}$ as, $0.\overline{21}$, $0.\overline{35421}$, $0.\overline{2111111111}$ and many others cannot become d-modular if they were in the table T. This problem will be analyzed in Chapter 30, although for the case of a table of natural numbers.

A final remark

P220 As with all discussions on the hypothesis of the actual infinity, the above one is a conceptual discussion unconcerned, as Cantor's diagonal argument, with the physical possibilities of carrying out all the involved operations. The formal inconsistency of a hypothesis does not depend on those possibilities, but on the fact of deducing from it a contradiction (Principle of Autonomy). And recall that from an inconsistent hypothesis anything can be deduced, from apparently reasonable assertions to any absurdity. It seems convenient to end by recalling again that an argument cannot be refuted by other different argument simply because it reaches an opposite conclusion. In W. Hodges words [4, p. 4]:

> How does anybody get into a state of mind where they persuade themselves that you can criticize an argument by suggesting a different argument which doesn't reach the same conclusion?

This inadmissible strategy is frequently used in the discussions related to the actual infinity hypothesis (and in general in any discussion involving a "main stream" of thought). But to refute an argument means to indicate where and why that argument fails. If two correct arguments based on the same set of hypotheses lead to contradictory conclusions, they are simply proving the existence of a contradiction. And, therefore, the inconsistency of at least one of the assumed hypotheses. In our case, the only hypothesis is the hypothesis of the actual infinity, according to which the infinite sets and sequences exist as complete totalities. The alternative is the hypothesis of the potential infinity, according to which only finite sets and sequences can be considered as *complete totalities*, unlimited and as large as wished, but always finite if they have to be considered as complete totalities. From this finitist perspective it is not possible to deduce the above contradictions because every row is preceded and succeeded by a finite number of rows.

Chapter References

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