On the general solutions of Lienard equations with strong nonlinearities

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Abstract

We derive the Lienard equation with strong nonlinearities from the solution of the Ermakov-Pinney equation. As a result, we give the exact and explicit general solution of this equation for the first time, so that the solution of a lot of differential equations like the cubic-quintic Duffing equation may be easily obtained.

Keywords: Lienard equations, strong nonlinearity, Ermakov-Pinney equation, traveling wave solutions, general solutions.

Introduction

The Lienard differential equation

$$\ddot{x} + f(x) = 0 \tag{1}$$

where the dot means differentiation with respect to the independent variable t, and $f(x)$ is a nonlinear function of x, is of a high importance since it often arises in mathematical modeling in physics and engineering. For example, the nonlinear evolution equations of mathematical physics may be reduced to (1) for finding traveling wave solutions. In many cases where $f(x)$ contains strong high-order nonlinear terms the equation (1) has no known exact solutions. This shows the need to investigate the analytical properties of the equation (1). In this context Cheng-Shi [1] investigated the Lienard equation (1) with the $(2m+1)$-order nonlinear term in the form

$$\ddot{x} + ax + bx^{m+1} + cx^{2m+1} = 0 \tag{2}$$

According to [1] the Kundu equation, Rangwala-Rao equation, generalized BBM equation, and the generalized Pochhamer-Chree equation without dispersion term, for example, may be reduced to the equation (2) by using

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traveling wave transformation. Althought the author [1] has succeed to determine the so-called exact solutions of (2), no explicit general solutions have been calculated for the parameter scope considered. It is remarkable to notice that the equation (2) includes several well- known nonlinear differential equations as special cases. For instance, substituting $m=2$, into (2) yields

$$\ddot{x}+ax+bx^3+cx^5=0 \tag{3}$$

The equation (3) is known in the literature as the cubic-quintic Duffing equation. For $m=-2$, and $b=0$, the equation (2) becomes the celebrated Ermakov-Pinney equation

$$\ddot{x}+ax+\frac{c}{x^3}=0 \tag{4}$$

This equation may also obtained for $m=-4$, and $c=0$. The cubic-quintic Duffing equation [2] as well as the famous Ermakov-Pinney equation [3] have been widely studied in the literature [4-6]. In [6] new exact and explicit general solutions are determined for the equation (4). The problem to solve in this paper is the integration of the equation (2). This means the determination of its exact and explicit general solution. Precisely we consider the following question: what is the solution of the equation (2) expressible as $g(x)$ where x is the solution of the equation (3) or (4) that it generalizes? In this question, that is to say (2) and (3), or (2) and (4) are of the same type (1). With this restriction the question stated becomes very interesting to be solved since the transformation of equation by change of variable leads in general to another type of equation. For example, it is well-known that the Ermakov-Pinney equation turns into a quadratic Lienard type equation under point transformation [7-9]. The question under consideration has the importance to highlight the connection between several nonlinear differential equations of the same type and to show the parameter choice leading to explicit general solutions. To solve the above question, we establish the corresponding general theory (section2) which will be used to derive the equation (2) and exhibit new exact and explicit general solutions (section 3). Finally a discussion (section 4) and a conclusion of the work are carried out.

2- General theory

Let x be the solution of a nonlinear differential equation and $u=g(x)$ the solution of its generalized equation. We may write

$$\frac{du}{dt} = g'(x)\frac{dx}{dt} \tag{5}$$

and

$$\frac{d^2u}{dt^2} = g''(x)\left(\frac{dx}{dt}\right)^2 + g'(x)\frac{d^2x}{dt^2} \tag{6}$$

where the prime indicates the differentiation with respect to x. The general equation may be rewritten in the form

$$\frac{d^2u}{dt^2} - g''(x)\left(\frac{dx}{dt}\right)^2 - g'(x)\frac{d^2x}{dt^2} = 0 \tag{7}$$

where $x = g^{-1}(u)$, assuming that the function $g(x)$ is invertible.

The equation (7) is the general equation corresponding to the question to be solved.

3- Exact and explicit general solution of (2)

Let us consider the hyperlogistic function

$$x(t) = K_1\left(1 + K_2 e^{-qt}\right)^{\frac{1}{2}} \tag{8}$$

so that $u(x)$ takes the form

$$u(x) = x^{-2/n} = K_1\left(1 + K_2 e^{-qt}\right)^{-1/n} \tag{9}$$

where K_1, K_2, $n \neq 0$, and q, are arbitrary parameters.

From (8), one may obtain

$$\frac{dx}{dt} = \frac{1}{2}K_1\left(-qK_2 e^{-qt}\right)\left(1 + K_2 e^{-qt}\right)^{\frac{-1}{2}} \tag{10}$$

which may be rearranged as

$$\frac{dx}{dt} = -\frac{1}{2}q\frac{K_1^2}{x}\left(\frac{x^2}{K_1^2} - 1\right) \tag{11}$$

and the second derivative takes the form

$$\frac{d^2x}{dt^2} = \frac{q^2}{4}x - \frac{q^2}{4}\frac{K_1^4}{x^3} \tag{12}$$

From (12) one may see that the function $x(t)$ given by (8) is a new exact and explicit general solution of the Ermakov-Pinney equation (4) by setting $a=-\frac{q^2}{4}$, and $c=\frac{q^2K_1^4}{4}$. The first and second derivatives of $g(x)$ in (7), using (9), become

$$g'(x)=-\frac{2}{n}x^{-\frac{(n+2)}{n}} \tag{13}$$

and

$$g''(x)=\frac{2(n+2)}{n^2}x^{-\frac{(2n+2)}{n}} \tag{14}$$

In this way the general equation (7) may be expressed as

$$\frac{d^2u}{dt^2}-\frac{q^2}{n^2}x^{-\frac{2}{n}}+\frac{(n+2)}{n^2}q^2K_1^2x^{-\frac{(2n+2)}{n}}-\frac{(n+1)}{n^2}q^2K_1^4x^{-\frac{2(2n+1)}{n}}=0 \tag{15}$$

which reduces, using $x=u^{-\frac{n}{2}}$, to

$$\ddot{u}-\frac{q^2}{n^2}u+\frac{(n+2)}{n^2}q^2K_1^2u^{n+1}-\frac{(n+1)}{n^2}q^2K_1^4u^{2n+1}=0 \tag{16}$$

The equation (16) is identical to the Lienard differential equation (2) with ($2m+1$)-order nonlinear term, for $m=n$, $a=-\frac{q^2}{n^2}$, $b=\frac{(n+2)}{n^2}q^2K_1^2$, and $c=-\frac{(n+1)}{n^2}q^2K_1^4$. Accordingly, the exact and explicit general solution of (2) or (16) is the function given by (9). In this context, the exact and explicit general solution of several Lienard equations with strong nonlinear terms may be easily obtained. Illustrative examples are given in the discussion.

4- Discussion

It is easy to see that the substitution of $n=2$, yields immediately the cubic-quintic Duffing equation.

$$\ddot{u}-\frac{q^2}{4}u+q^2K_1^2u^3-\frac{3}{4}q^2K_1^4u^5=0 \tag{17}$$

which is identical to (3) for $m=n$, $a=-\frac{q^2}{4}$, $b=q^2K_1^2$, and $c=-\frac{3}{4}q^2K_1^4$, where its exact and explicit general solution is

$$u(t)=K_1\left(1+K_2e^{-qt}\right)^{-\frac{1}{2}} \tag{18}$$

The Helmholtz-Duffing equation may be obtained as

$$\ddot{u}-q^2u+3q^2K_1^2u^2-2q^2K_1^4u^3=0 \tag{19}$$

when $n=1$,with the exact and explicit general solution

$$u(t)=K_1\left(1+K_2e^{-qt}\right)^{-1} \tag{20}$$

Applying $n=-\frac{2}{3}$, the equation (16) yields

$$\ddot{u}-\frac{9}{4}q^2u+3q^2K_1^2u^{\frac{1}{3}}-\frac{3}{4}q^2K_1^4u^{-\frac{1}{3}}=0 \tag{21}$$

with the exact and general solution

$$u(t)=K_1\left(1+K_2e^{-qt}\right)^{\frac{3}{2}} \tag{22}$$

Using the point transformation $u=x^\ell$, the related quadratic Lienard type equation to the equation (3) may read

$$\ddot{u}+\frac{1-\ell}{\ell}\frac{\dot{u}^2}{u}+a\ell u+b\ell u^{\frac{\ell+2}{\ell}}+c\ell u^{\frac{\ell+4}{\ell}}=0 \tag{23}$$

with the exact and explicit general solution

$$u(t)=\left[K_1\left(1+K_2e^{-qt}\right)\right]^{-\frac{\ell}{2}} \tag{24}$$

where $a=-\frac{q^2}{4}$, $b=q^2K_1^2$, and $c=-\frac{3}{4}q^2K_1^4$.

The related quadratic Lienard type equation to (4), using the transformation $u=x^\ell$, takes the form

$$\ddot{u}+\frac{1-\ell}{\ell}\frac{\dot{u}^2}{u}+a\ell u+c\ell u^{\frac{\ell-4}{\ell}}=0 \tag{25}$$

with the exact and explicit general solution

$$u(t) = \left[K_1\left(1 + K_2 e^{-qt}\right)\right]^{\frac{\ell}{2}} \tag{26}$$

where $a = -\dfrac{q^2}{4}$, and $c = \dfrac{q^2 K_1^4}{4}$.

Conclusion

The Lienard equation with strong high-order nonlinear terms has been for a long time with no known exact and explicit general solutions. For the first time, in this paper, the exact and explicit general solution of this equation has been exhibited, easily. By doing so, the exact and explicit general solutions of a great number of second-order Lienard nonlinear differential equations may be immediately obtained. It is worth to notice that the current work shows how to select parameters to obtain easily the exact and explicit general solution of (2).

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